Anais da Academia Brasileira de Ciências (2006) 78(3): 391–404
(Annals of the Brazilian Academy of Sciences)
ISSN 0001-3765
www.scielo.br/aabc

Fundamental tone estimates for elliptic operators in divergence form and geometric applications

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Manuscript received on October 31, 2005; accepted for publication on April 17, 2006; presented by MANFREDO DO CARMO

ABSTRACT

We establish a method for giving lower bounds for the fundamental tone of elliptic operators in divergence form in terms of the divergence of vector fields. We then apply this method to the L_r operator associated to immersed hypersurfaces with locally bounded $(r+1)$-th mean curvature H_{r+1} of the space forms $\mathbb{N}^{n+1}(c)$ of constant sectional curvature c. As a corollary we give lower bounds for the extrinsic radius of closed hypersurfaces of $\mathbb{N}^{n+1}(c)$ with $H_{r+1} > 0$ in terms of the r-th and $(r+1)$-th mean curvatures. Finally we observe that bounds for the Laplace eigenvalues essentially bound the eigenvalues of a self-adjoint elliptic differential operator in divergence form. This allows us to show that Cheeger's constant gives a lower bounds for the first nonzero L_r-eigenvalue of a closed hypersurface of $\mathbb{N}^{n+1}(c)$.

Key words: fundamental tone, L_r operator, r-th mean curvature, extrinsic radius, Cheeger's constant.

INTRODUCTION

Let Ω be a domain in a smooth Riemannian manifold M and let $\Phi : \Omega \to \mathrm{End}\,(T\Omega)$ be a smooth symmetric and positive definite section of the bundle of all endomorphisms of $T\Omega$. Each section Φ is associated to a second order self-adjoint elliptic operator $L_\Phi(f) = \mathrm{div}\,(\Phi \text{ grad } f)$, $f \in C^2(\Omega)$. Observe that when Φ is the identity section then $L_\Phi = \triangle$, the Laplace operator. Recall that the

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L_Φ-fundamental tone of Ω is given by

$$\lambda^{L_\Phi}(\Omega) = \inf \left\{ \frac{\int_\Omega |\Phi^{1/2}\mathrm{grad}\, f|^2}{\int_\Omega f^2} \; ; \; f \in C_0^2(\Omega) \setminus \{0\} \right\}. \tag{1}$$

If Ω is bounded with smooth boundary $\partial\Omega \neq \emptyset$, the L_Φ-fundamental tone of Ω coincides with the first eigenvalue $\lambda_1^{L_\Phi}(\Omega)$ of the Dirichlet eigenvalue problem $L_\Phi\, u + \lambda\, u = 0$ on Ω, with $u|\partial\Omega = 0$, $u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \setminus \{0\}$. If Ω is bounded with empty boundary $\partial\Omega = \emptyset$ then $\lambda^{L_\Phi}(\Omega) = 0$.

A basic problem in Riemannian geometry is what lower and upper bounds for $\lambda^{L_\Phi}(\Omega)$ can be obtained in terms of Riemannian invariants. In this paper we show that the method established by Bessa and Montenegro (Bessa and Montenegro 2004) that gives lower bounds for the $\triangle$-fundamental tone can be extended for self-adjoint elliptic operators L_Φ, (Theorem 2.1). Then we consider immersed hypersurfaces $\varphi : M \to \mathbb{N}^{n+1}(c)$ of the $(n+1)$-dimensional simply connected space form $\mathbb{N}^{n+1}(c)$ of constant sectional curvature $c \in \{1, 0, -1\}$ with locally bounded $(r+1)$-th mean curvature such that the differential operators L_r, $r \in \{0, 1, \ldots, n\}$ are elliptic. We give lower bounds for the L_r-fundamental tone of domains $\Omega \subset \varphi^{-1}(B_{\mathbb{N}^{n+1}(c)}(p, R))$, in terms of the r-th and $(r+1)$-th mean curvatures H_r, H_{r+1}, (Theorem 3.2), where $B_{\mathbb{N}^{n+1}(c)}(p, R)$ is the geodesic ball of $\mathbb{N}^{n+1}(c)$ centered at p with radius R. From these estimates we derive three geometric corollaries 3.4, 3.5 and 3.8 that should be viewed as an extension of a result of Jorge and Xavier (Jorge and Xavier 1981). It should be mentioned that these corollaries are related to results due to Vlachos (Vlachos 1997) and to Fontenele and Silva (Fontenele and Silva 2001), see Remark 3.7. In Theorem 3.10 we consider immersed hypersurfaces M of $\mathbb{N}^{n+1}(c)$ such that the operators L_r and L_s, $0 \leq r, s \leq n$ are elliptic and we compare the L_r and L_s fundamental tones $\lambda^{L_r}(\Omega)$, $\lambda^{L_s}(\Omega)$ of domains $\Omega \subset M \subset \mathbb{N}^{n+1}(c)$. In section 4 we observe (Theorem 4.1) that in order to get bounds for the eigenvalues of a self-adjoint elliptic differential operator L_Φ we essentially need bounds for the Laplace operator eigenvalues. This allows us to use Cheeger's constant to give lower bounds for the first nonzero L_r-eigenvalue of a closed hypersurface of $\mathbb{N}^{n+1}(c)$. The results are stated and discussed in Sections 2, 3 and 4 and the proofs are given in Section 5.

L_Φ-FUNDAMENTAL TONE ESTIMATES

Our main estimate is the following method for giving lower bounds for L_Φ-fundamental tone of arbitrary domains of Riemannian manifolds. It extends the version of Barta's theorem (Barta 1937) proved by Cheng-Yau in (Cheng and Yau 1977). It is the same proof (with proper modifications) of a generalization of Barta's theorem proved in (Bessa and Montenegro 2004).

THEOREM 2.1. *Let Ω be a domain in a Riemannian manifold and let $\Phi : \Omega \to \mathrm{End}\,(T\Omega)$ be a smooth symmetric and positive definite section of* End $(T\Omega)$. *Then the L_Φ-fundamental tone of Ω has the following lower bound*

$$\lambda^{L_\Phi}(\Omega) \;\geq\; \sup_{\mathcal{X}(\Omega)} \inf_{\Omega} \left[\mathrm{div}\,(\Phi X) - |\Phi^{1/2} X|^2\right]. \tag{2}$$

If Ω is bounded and with piecewise smooth boundary $\partial\Omega \neq \emptyset$ then we have equality in (2).

$$\lambda^{L_\Phi}(\Omega) \;=\; \sup_{\mathcal{X}(\Omega)} \inf_{\Omega} \left[\operatorname{div}(\Phi X) - |\Phi^{1/2} X|^2\right]. \tag{3}$$

Where $\mathcal{X}(\Omega)$ is the set of all smooth vector fields on Ω.

GEOMETRIC APPLICATIONS

Consider the linearized operator L_r of the $(r+1)$-mean curvature

$$H_{r+1} = \frac{S_{r+1}}{\binom{n}{r+1}}$$

arising from normal variations of a hypersurface M immersed into the $(n+1)$-dimensional simply connected space form $\mathbb{N}^{n+1}(c)$ of constant sectional curvature $c \in \{1, 0, -1\}$, where S_{r+1} is the $(r+1)$-th elementary symmetric function of the principal curvatures $k_1, k_2, \ldots, k_n$, see (Reilly 1973) and (Rosenberg 1993) for details. Recall that the elementary symmetric function of the principal curvatures are given by

$$S_0 = 1, \quad S_r = \sum_{i_1 < \cdots < i_r} k_{i_1} \cdots k_{i_r}, \quad 1 \leq r \leq n. \tag{4}$$

Letting $A = -(\overline{\nabla}\eta)$ be the shape operator of M, where $\overline{\nabla}$ is the Levi-Civita connection of $\mathbb{N}^{n+1}(c)$ and η a globally defined unit vector field normal to M, we can recursively define smooth symmetric sections $P_r : M \to \operatorname{End}(TM)$, for $r = 0, 1, \ldots, n$, called the Newton operators, setting $P_0 = I$ and $P_r = S_r Id - AP_{r-1}$ so that $P_r(x) : T_xM \to T_xM$ is a self-adjoint linear operator with the same eigenvectors as the shape operator A. The operator L_r is the second order self-adjoint differential operator

$$L_{P_r}(f) = \operatorname{div}(P_r \operatorname{grad} f) \tag{5}$$

associated to the section P_r. However, the sections P_r may be not positive definite and then the operators L_r may not be elliptic. However, there are geometric hypothesis that imply the ellipticity of L_r, see for instance, Reilly 1973, Caffarelli et al. 1985, Korevaar 1988 or Barbosa and Colares 1997. Here we will not impose geometric conditions to guarantee ellipticity of the L_r, except in corollary 3.5. Instead we will ask the ellipticity on the set of hypothesis. It is known the ordered eigenvalues $\{\mu_1^r(x) \leq \ldots \leq \mu_n^r(x)\}$ of $P_r(x)$ depend continuously on $x \in M$. (Kato 1976 pages 106–109). In fact, this proof can be pushed to prove that they are Lipschitz thus differentiable almost everywhere. In addition, the respective eigenvectors $\{e_1(x), \ldots, e_n(x)\}$ form a smooth orthonormal frame in a neighborhood of every point. Set $\nu(P_r) = \sup_{x\in M}\{\mu_n^r(x)\}$ and $\mu(P_r) = \inf_{x\in M}\{\mu_1^r(x)\}$. Observe that if $\mu(P_r) > 0$ then P_r is positive definite, thus L_r is elliptic.

We need the following definition of locally bounded $(r+1)$-th mean curvature hypersurface in order to state our next result.

DEFINITION 3.1. *An oriented immersed hypersurface $\varphi : M \hookrightarrow N$ of a Riemannian manifold N is said to have locally bounded $(r+1)$-th mean curvature H_{r+1} if for any $p \in N$ and $R > 0$, the number*

$$h_{r+1}(p, R) = \sup\left\{ |S_{r+1}(x)| = \begin{pmatrix} n \\ r+1 \end{pmatrix} \cdot |H_{r+1}(x)|\,;\ x \in \varphi(M) \cap B_N(p, R) \right\}$$

is finite. Here $B_N(p, R) \subset N$ is the geodesic ball of radius R with center at $p \in N$.

Our next result generalizes in some aspects the main application of (Bessa and Montenegro 2003). There the first and fourth authors give lower bounds for $\triangle$-fundamental tone of domains in submanifolds with locally bounded mean curvature in complete Riemannian manifolds.

THEOREM 3.2. *Let $\varphi : M \hookrightarrow \mathbb{N}^{n+1}(c)$ be an oriented hypersurface immersed with locally bounded $(r+1)$-th mean curvature H_{r+1} for some $r \leq n-1$ and with $\mu(P_r) > 0$. Let $B_{\mathbb{N}^{n+1}(c)}(p, R)$ be the geodesic ball centered at $p \in \mathbb{N}^{n+1}(c)$ with radius R and $\Omega \subset \varphi^{-1}(\overline{B_{\mathbb{N}^{n+1}(c)}(p, R)})$ be a connected component. Then the L_r-fundamental tone $\lambda^{L_r}(\Omega)$ of Ω has the following lower bounds.*

i. *For $c = 1$ and $0 < R < \cot^{-1}\left[\dfrac{(r+1)\cdot h_{r+1}(p, R)}{(n-r)\cdot \inf_\Omega S_r}\right]$ we have that*

$$\lambda^{L_r}(\Omega) \geq 2 \cdot \frac{1}{R}\left[(n-r)\cdot \cot[R] \cdot \inf_{\Omega} S_r - (r+1)\cdot h_{r+1}(p, R)\right]. \tag{6}$$

ii. *For $c \leq 0$, $h_{r+1}(p, R) \neq 0$ and $0 < R < \dfrac{(n-r)\cdot \inf_\Omega S_r}{(r+1)\cdot h_{r+1}(p, R)}$ we have that*

$$\lambda^{L_r}(\Omega) \geq 2 \cdot \frac{1}{R^2}\left[(n-r)\cdot \inf_{\Omega} S_r - (r+1)\cdot R \cdot h_{r+1}(p, R)\right]. \tag{7}$$

iii. *If $c \leq 0$, $h_{r+1}(p, R) = 0$ and $R > 0$ we have that*

$$\lambda^{L_r}(\Omega) \geq \frac{2(n-r)\inf_\Omega S_r}{R^2} \tag{8}$$

DEFINITION 3.3. *Let $\varphi : M \hookrightarrow N$ be an isometric immersion of a closed Riemannian manifold into a complete Riemannian manifold N. For each $x \in N$, let $r(x) = \sup_{y\in M} \mathrm{dist}_N(x, \varphi(y))$. The extrinsic radius $R_e(M)$ of M is defined by*

$$R_e(M) = \inf_{x\in N} r(x).$$

Moreover, there is a point $x_0 \in N$ called the barycenter of $\varphi(M)$ in N such that $R_e(M) = r(x_0)$.

COROLLARY 3.4. *Let* $\varphi : M \hookrightarrow B_{\mathbb{N}^{n+1}(c)}(R) \subset \mathbb{N}^{n+1}(c)$ *be a complete oriented hypersurface with bounded* $(r+1)$*-th mean curvature* H_{r+1} *for some* $r \leq n-1$*,* R *chosen as in Theorem (3.2). Suppose that* $\mu(P_r) > 0$ *so that the* L_r *operator is elliptic. Then* M *is not closed.*

COROLLARY 3.5. *Let* $\varphi : M \hookrightarrow \mathbb{N}^{n+1}(c)^1$, $c \in \{1, 0, -1\}$ *be an oriented closed hypersurface with* $H_{r+1} > 0$*. Then there is an explicit constant* $\Lambda_r = \Lambda_r\left(c, \inf_M S_r, \sup_M S_{r+1}\right) > 0$ *such that the extrinsic radius* $R_e(M) \geq \Lambda_r$*.*

i. *For* $c = 1$, $\Lambda_r = \cot^{-1}\left[\dfrac{(r+1)\cdot \sup_M S_{r+1}}{(n-r)\cdot \inf_M S_r}\right]$.

ii. *For* $c \in \{0, -1\}$, $\Lambda_r = \dfrac{(n-r)\cdot \inf_M S_r}{(r+1)\cdot \sup_M S_{r+1}}$.

REMARK 3.6. The hypothesis $H_{r+1} > 0$ implies that $H_j > 0$ and L_j are elliptic for $j = 0, 1, \ldots r$, see Barbosa and Colares 1997, Caffarelli et al. 1985 or Korevaar 1988. Thus in fact have that $R_e \geq \max\{\Lambda_0, \cdots, \Lambda_r\}$.

REMARK 3.7. Jorge and Xavier, (Jorge and Xavier 1981) proved the inequalities of Corollary 3.5 when $r = 0$ for complete submanifolds with scalar curvature bounded from below contained in a compact ball of a complete Riemannian manifold. Moreover, for $c = -1$ their inequality is slightly better. It is possible to give sharp estimates for the extrinsic radius of a closed hypersurface of $\mathbb{N}^{n+1}(c)$ in terms of $\sup_M |H_r|$ alone. Vlachos (Vlachos 1997) proved a result that implies that, for each

$$
\begin{aligned}
1 \leq r \leq n,\ R_e(M) &\geq (\sup |H_r|)^{-1/r}, \\
R_e(M) &\geq \cot^{-1}(\sup |H_r|)^{1/r}, \\
R_e(M) &\geq \coth^{-1}(\sup |H_r|)^{1/r}
\end{aligned}
$$

if $c = 0$, $c = 1$ or $c = -1$ respectively, and that in any case the equality holds if and only if M is a geodesic sphere of the ambient space. The result of Vlachos was extended to any ambient space by Fontenele and Silva (Fontenele and Silva 2001).

REMARK 3.8. An interesting question is: Is it true that any closed oriented hypersurface with $\mu_1^r(M) > 0$ and $H_{r+1} = 0$ intersect every great circle? For $r = 0$ it is true and it was proved by T. Frankel, (Frankel 1966).

We now consider immersed hypersurfaces $\varphi : M \hookrightarrow \mathbb{N}^{n+1}(c)$ with L_r and L_s elliptic. We can compare the L_r and L_s fundamental tones of a domain $\Omega \subset M$. In particular we can compare with its L_0-fundamental tone.

THEOREM 3.9. *Let* $\varphi : M \hookrightarrow \mathbb{N}^{n+1}(c)$ *be an oriented n-dimensional hypersurface* M *immersed into the* $(n+1)$*-dimensional simply connected space form of constant sectional curvature* c *and*

[1] If $c = 1$ suppose that $\mathbb{N}^{n+1}(c)$ is the open hemisphere of $\mathbb{S}^{n+1}_{+}$.

$\mu(P_r) > 0$ and $\mu(P_s) > 0$, $0 \leq s,\, r \leq n-1$. Let $\Omega \subset M$ be a domain with compact closure and piecewise smooth nonempty boundary. Then the L_r and L_s fundamental tones satisfies the following inequalities

$$\lambda^{L_r}(\Omega) \geq \frac{\mu(P_r)}{\nu(P_s)} \cdot \lambda^{L_s}(\Omega) \tag{9}$$

From (9) *we have in particular that*

$$\nu(P_r) \cdot \lambda^{\triangle}(\Omega) \geq \lambda^{L_r}(\Omega) \geq \mu(P_r) \cdot \lambda^{\triangle}(\Omega) \tag{10}$$

CLOSED EIGENVALUE PROBLEM

Let M be a closed hypersurface of a simply connected space form $\mathbb{N}^{n+1}(c)$. The interesting problem is what bounds can one obtain for the first nonzero L_r-eigenvalue $\lambda_1^{L_r}(M)$ in terms of the geometries of M and of the ambient space. Upper bounds for the first nonzero $\triangle$-eigenvalue or even for the first nonzero L_r-eigenvalue, $r \geq 1$ have been obtained by many authors in contrast with lower bounds that are rare. For instance, Reilly (Reilly 1977) extending earlier result of Bleecker and Weiner (Bleecker and Weiner 1976) obtained upper bounds for $\lambda_1^{\triangle}(M)$ of a closed submanifold M of $\mathbb{R}^m$ in terms of the total mean curvature of M. Reilly's result applied to compact submanifolds of the sphere $M \subset \mathbb{S}^{m+1}(1)$, this latter viewed as a hypersurface of the Euclidean space $\mathbb{S}^{m+1}(1) \subset \mathbb{R}^{m+2}$ obtains upper bounds for $\lambda_1^{\triangle}(M)$, see Alencar, Do Carmo and Rosenberg in Alencar et al. 1993. Heintze (Heintze 1988) extended Reilly's result to compact manifolds and Hadamard manifolds $\overline{M}$. In particular for the hyperbolic space $\mathbb{H}^{n+1}$. The best upper bounds for the first nonzero $\triangle$-eigenvalue of closed hypersurfaces M of $\mathbb{H}^{n+1}$ in terms of the total mean curvature of M was obtained by El Soufi and Ilias (Soufi and Ilias 1992). Regarding the L_r operators, Alencar, Do Carmo and Rosenberg (Alencar et al. 1993) obtained sharp (extrinsic) upper bound the first nonzero eigenvalue $\lambda_1^{L_r}(M)$ of the linearized operator L_r of compact hypersurfaces M of $\mathbb{R}^{m+1}$ with $S_{r+1} > 0$. Upper bounds for $\lambda_1^{L_r}(M)$ of compact hypersurfaces of $\mathbb{S}^{n+1}$, $\mathbb{H}^{n+1}$ under the hypothesis that L_r is elliptic were obtained by Alencar, Do Carmo, Marques in (Alencar et al. 2001) and by Alias and Malacarne in (Alias and Malacarne 2004) see also the work of Veeravalli (Veeravalli 2001). On the other hand, lower bounds for $\lambda_1^{L_r}(M)$ of closed hypersurfaces $M \subset \mathbb{N}^{n+1}(c)$ are not so well studied as the upper bounds, except for $r = 0$ in which case $L_0 = \triangle$. In this paper we make a simple observation (Theorem 4.1) that to obtain lower and upper bounds for the L_Φ-eigenvalues (Dirichlet or Closed eigenvalue problem) it is enough to obtain lower and upper bounds for the eigenvalues of Φ and for the eigenvalues for the Laplacian in the respective problem. When applied to the L_r operators (supposing them elliptic) we obtain lower bounds for closed hypersurfaces of the space forms via Cheeger's lower bounds for the first $\triangle$-eigenvalue of closed manifolds. Let $\{\mu_1(x) \leq \ldots \leq \mu_n(x)\}$ be the ordered eigenvalues of $\Phi(x)$. Setting $\nu(\Phi) = \sup_{x\in\Omega}\{\mu_n(x)\}$ and $\mu(\Phi) = \inf_{x\in\Omega}\{\mu_1(x)\}$ we have the following theorem.

THEOREM 4.1. *Let $\lambda^{L_\Phi}(\Omega)$ denote the L_Φ-fundamental tone of Ω if Ω is unbounded or $\partial\Omega \neq \emptyset$ and the first nonzero L_Φ-eigenvalue $\lambda_1^{L_\Phi}(\Omega)$ if Ω is a closed manifold. Then $\lambda^{L_\Phi}(\Omega)$ satisfies the following inequalities,*

$$\nu(\Phi) \cdot \lambda^{\triangle}(\Omega) \geq \lambda^{L_\Phi}(\Omega) \geq \mu(\Phi) \cdot \lambda^{\triangle}(\Omega), \tag{11}$$

where $\lambda^{\triangle}(\Omega)$ is the $\triangle$-fundamental tone of Ω or the first nonzero $\triangle$-eigenvalue of Ω.

Let M be a closed n-dimensional Riemannian manifold, Cheeger (Cheeger 1970) defined the following constant given by

$$h(M) = \inf_S \frac{vol_{n-1}(S)}{\min\{vol_n(\Omega_1),\ vol_n(\Omega_2)\}}, \tag{12}$$

where $S \subset M$ ranges over all connected closed hypersurfaces dividing M in two connected components, i.e. $M = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \emptyset$ such that $S = \partial\Omega_1 = \partial\Omega_2$ and he proved that the first nonzero $\triangle$-eigenvalue $\lambda_1^{\triangle}(M) \geq h(M)^2/4$.

COROLLARY 4.2. *Let $\varphi : M \hookrightarrow \mathbb{N}^{n+1}(c)$, $c \in \{1, 0, -1\}$[2] be an oriented closed hypersurface with $H_{r+1} > 0$. Then the first nonzero L_r-eigenvalue of M has the following lower bound*

$$\lambda_1^{L_r}(M) \geq \mu(P_r) \cdot \frac{h^2(M)}{4}$$

PROOF OF THE RESULTS

PROOF OF THEOREM 2.1.

Let Ω be an arbitrary domain, X be a smooth vector field on Ω and $f \in C_0^\infty(\Omega)$. The vector field $f^2\Phi X$ has compact support $\operatorname{supp}(f^2\Phi X) \subset \operatorname{supp}(f) \subset \Omega$. Let $\mathcal{S}$ be a regular domain containing the support of f. We have by the divergence theorem that

$$\begin{aligned} 0 = \int_{\mathcal{S}} \operatorname{div}(f^2\Phi X) &= \int_\Omega \operatorname{div}(f^2\Phi X) \\ &= \int_\Omega \left[\langle \operatorname{grad} f^2, \Phi X\rangle + f^2 \operatorname{div}(\Phi X)\right] \\ &\geq \int_\Omega \left[-2 \cdot |f| \cdot |\Phi^{1/2}\operatorname{grad} f| \cdot |\Phi^{1/2}X| + \operatorname{div}(\Phi X) \cdot f^2\right] \\ &\geq \int_\Omega \left[-|\Phi^{1/2}\operatorname{grad} f|^2 - f^2 \cdot |\Phi^{1/2}X|^2 + \operatorname{div}(\Phi X) \cdot f^2\right]. \end{aligned} \tag{13}$$

Therefore

$$\begin{aligned} \int_\Omega |\Phi^{1/2}\operatorname{grad} f|^2 &\geq \int_\Omega \left[\operatorname{div}(\Phi X) - |\Phi^{1/2}X|^2\right] f^2 \\ &\geq \inf_\Omega \left[\operatorname{div}(\Phi X) - |\Phi^{1/2}X|^2\right] \int_\Omega f^2 \end{aligned} \tag{14}$$

[2] If $c = 1$ suppose that $\mathbb{N}^{n+1}(c)$ is the open hemisphere of $\mathbb{S}_+^{n+1}$.

By the variational formulation (1) of $\lambda^{L_\Phi}(\Omega)$ this inequality above implies that

$$\lambda^{L_\Phi}(\Omega) \;\geq\; \inf_{\Omega}\left[\operatorname{div}(\Phi X) - |\Phi^{1/2}X|^2\right]. \tag{15}$$

When Ω is a bounded domain with smooth boundary $\partial\Omega \neq \emptyset$ then $\lambda^{L_\Phi}(\Omega) = \lambda_1^{L_\Phi}(\Omega)$. This proof above shows that

$$\lambda_1^{L_\Phi}(\Omega) \;\geq\; \inf_{\Omega}\left[\operatorname{div}(\Phi X) - |\Phi^{1/2}X|^2\right].$$

Let $v \in C^2(\Omega)\cap C^0(\overline{\Omega})$ be a positive first L_Φ-eigenfunction[3] of Ω and if we set $X_0 = -\operatorname{grad}\,\log(v)$ we have that

$$\begin{aligned}\operatorname{div}(\Phi X_0) - |\Phi^{1/2}X_0|^2 &= -\operatorname{div}\,((1/v)\,\Phi\operatorname{grad} v) - (1/v^2)\,|\Phi^{1/2}\operatorname{grad} v|^2\\ &= (1/v^2)\langle\operatorname{grad} v, \Phi\operatorname{grad} v\rangle - (1/v)\operatorname{div}(\Phi\operatorname{grad} v) - (1/v^2)\,|\Phi^{1/2}\operatorname{grad} v|^2\\ &= -(1/v)\operatorname{div}(\Phi\operatorname{grad} v) = -L_\Phi(v)/v = \lambda_1^{L_\Phi}(\Omega).\end{aligned} \tag{16}$$

This proves (3).

PROOF OF THEOREM 3.2 AND COROLLARIES 3.4, 3.5 AND 3.8

We start this section stating few lemmas necessary to construct the proof of Theorem 3.2. The first lemma was proved in (Jorge and Koutrofiotis 1980) for the Laplace operator and for the L_r operator in (Lima 2000). We reproduce its proof to make the exposition complete.

LEMMA 5.1. *Let $\varphi : M \hookrightarrow \mathbb{N}^{n+1}(c)$ be a hypersurface immersed in $(n+1)$-dimensional simply connected space form $\mathbb{N}^{n+1}(c)$ of constant sectional curvature c. Let $g : \mathbb{N}^{n+1}(c) \to \mathbb{R}$ be a smooth function and set $f = g\circ\varphi$. Identify $X \in T_pM$ with $d\varphi(p)X \in T_{\varphi(p)}\,\varphi(M)$ then we have that*

$$L_r f(p) \;=\; \sum_{i=1}^{n}\mu_i^r \operatorname{Hess} g(\varphi(p))\,(e_i,e_i) \;+\; \operatorname{Trace}(AP_r)\langle\operatorname{grad} g,\eta\rangle \tag{17}$$

PROOF. Each P_r is also associated to a second order self-adjoint differential operator defined by $\Box f = \operatorname{Trace}(P_r \operatorname{Hess}(f))$ see (Cheng and Yau 1977, Hartmann 1978). We have that

$$\Box f = \operatorname{Trace}(P_r \operatorname{Hess}(f)) = \operatorname{div}(P_r \operatorname{grad} f) - \langle\operatorname{Trace}\,(\nabla P_r), \operatorname{grad} f\rangle. \tag{18}$$

Rosenberg (Rosenberg 1993) proved that when the ambient manifold is the simply connected space form $\mathbb{N}^{n+1}(c)$ then $\operatorname{Trace}\,(\nabla P_r)\operatorname{grad} \equiv 0$, see also (Reilly 1973). Thus one has that $L_r f = \operatorname{Trace}(P_r \operatorname{Hess}(f))$. Using Gauss equation to compute $\operatorname{Hess}(f)$ we obtain

$$\operatorname{Hess} f(p)(X,Y) = \operatorname{Hess} g(\varphi(p))(X,Y) + \langle\operatorname{grad} g, \alpha(X,Y)\rangle_{\varphi(p)}, \tag{19}$$

where $\langle\alpha(X,Y),\eta\rangle = \langle A(X),Y\rangle$. Let $\{e_i\}$ be an orthonormal frame around p that diagonalize the section P_r so that $P_r(x)(e_i) = \mu_i^r(x)e_i$. Thus

$$L_r f = \sum_{i=1}^{n}\langle P_r \operatorname{Hess} f(e_i), e_i\rangle = \sum_{i=1}^{n}\langle\operatorname{Hess} f(e_i), \mu_i^r e_i\rangle = \sum_{i=1}^{n}\mu_i^r \operatorname{Hess} f(e_i,e_i) \tag{20}$$

[3] $v \in C^2(\Omega)\cap H_1^0(\Omega)$ if $\partial\Omega$ is not smooth.

Substituting (19) into (20) we have that

$$\begin{aligned} L_r f &= \textstyle\sum_{i=1}^n \mu_i^r \operatorname{Hess} g\,(e_i, e_i) + \big\langle \operatorname{grad} g, \sum_{i=1}^n \mu_i^r \alpha(e_i, e_i)\big\rangle \\ &= \textstyle\sum_{i=1}^n \mu_i^r \operatorname{Hess} g\,(e_i, e_i) + \big\langle \operatorname{grad} g, \alpha\big(\sum_{i=1}^n P_r(e_i), e_i\big)\big\rangle \\ &= \textstyle\sum_{i=1}^n \mu_i^r \operatorname{Hess} g\,(e_i, e_i) + \operatorname{Trace}(AP_r)\langle \operatorname{grad} g, \eta\rangle \end{aligned} \tag{21}$$

Here Hess $f(X) = \nabla_X \operatorname{grad} f$ and Hess $f(X, Y) = \langle \nabla_X \operatorname{grad} f, Y\rangle$. The next two lemmas we are gong to present are well known and their proofs are easily found in the literature thus we will omit them here.

LEMMA 5.2 [Hessian Comparison Theorem]. *Let M be a complete Riemannian manifold and $x_0, x_1 \in M$. Let $\gamma : [0, \rho(x_1)] \to M$ be a minimizing geodesic joining x_0 and x_1 where $\rho(x)$ is the distance function* $\operatorname{dist}_M(x_0, x)$. *Let K be the sectional curvatures of M and $\upsilon(\rho)$, defined below.*

$$\upsilon(\rho) = \begin{cases} k_1 \cdot \coth(k_1 \cdot \rho(x)), & if \quad \sup_\gamma K = -k_1^2 \\ \dfrac{1}{\rho(x)}, & if \quad \sup_\gamma K = 0 \\ k_1 \cdot \cot(k_1 \cdot \rho(x)), & if \quad \sup_\gamma K = k_1^2 \text{ and } \rho < \pi/2k_1. \end{cases} \tag{22}$$

Let $X = X^\perp + X^T \in T_x M$, $X^T = \langle X, \gamma'\rangle\gamma'$ and $\langle X^\perp, \gamma'\rangle = 0$. Then

$$\operatorname{Hess}\ \rho(x)(X, X) = \operatorname{Hess}\ \rho(x)(X^\perp, X^\perp) \geq \upsilon(\rho(x)) \cdot \|X^\perp\|^2 \tag{23}$$

See (Schoen and Yau 1994) for a proof.

LEMMA 5.3. *Let $p \in M$ and $1 \leq r \leq n-1$, let $\{e_i\}$ be an orthonormal basis of T_pM such that $P_r(e_i) = \mu_i^r e_i$ and $A(e_i) = k_i e_i$. Then*

i. Trace $(P_r) = \sum_{i=1}^n \mu_i^r = (n-r)S_r$

ii. Trace $(AP_r) = \sum_{i=1}^n k_i \mu_i^r = (r+1)S_{r+1}$

In particular, if the Newton operator P_r is positive definite then $S_r > 0$.

To prove Theorem (3.2) set $g : B(p, R) \subset \mathbb{N}^{n+1}(c) \to \mathbb{R}$ given by $g = R^2 - \rho^2$, where ρ is the distance function ($\rho(x) = \operatorname{dist}(x, p)$) of $\mathbb{N}^{n+1}(c)$. Setting $f = g \circ \varphi$ we obtain by (17) that

$$L_r f = \sum_{i=1}^n \mu_i^r \cdot \operatorname{Hess} g\,(e_i, e_i) + (r+1) \cdot S_{r+1} \cdot \langle \operatorname{grad} g, \eta\rangle, \tag{24}$$

since Trace $(AP_r) = (r+1) \cdot S_{r+1}$. Letting $X = -\operatorname{grad} \log f$ we have that

$$\operatorname{div} P_r X - |P_r^{1/2} X|^2 = -L_r(f)/f$$

then by Theorem (2.1) we have that

$$\begin{aligned}\lambda^{L_r}(\Omega) &\geq \inf_{\Omega}(-L_r f/f)\\ &= \inf_{\Omega}\left\{-\frac{1}{g}\left[\sum_{i=1}^{n}\mu_i^r \cdot \text{Hess}\, g\,(e_i, e_i) + (r+1)\cdot S_{r+1}\cdot\langle \text{grad}\, g, \eta\rangle\right]\right\}.\end{aligned} \tag{25}$$

Computing the Hessian of g we have that

$$\begin{aligned}\text{Hess}\, g\,(e_i, e_i) &= \langle \nabla_{e_i}\text{grad}\, g, e_i\rangle\\ &= -2\langle \nabla_{e_i}\rho\,\text{grad}\,\rho, e_i\rangle\\ &= -2\langle \text{grad}\,\rho, e_i\rangle^2 - 2\rho\,\langle \nabla_{e_i}\text{grad}\,\rho, e_i\rangle\\ &= -2\langle \text{grad}\,\rho, e_i\rangle^2 - 2\rho\,\text{Hess}\,\rho(e_i, e_i).\end{aligned} \tag{26}$$

Therefore we have that

$$-\frac{L_r f}{f} = \frac{2}{R^2-\rho^2}\left[\sum_{i=1}^{n}\mu_i^r\,[\langle \text{grad}\,\rho, e_i\rangle^2 + \rho\,\text{Hess}\,\rho(e_i,e_i)] + (r+1)\cdot S_{r+1}\cdot\rho\cdot\langle \text{grad}\,\rho, \eta\rangle\right] \tag{27}$$

Setting $e_i^T = \langle \text{grad}\,\rho, e_i\rangle \text{grad}\,\rho$ and $e_i^{\perp} = e_i - e_i^T$, by the Hessian Comparison Theorem we have that

$$\sum_{i=1}^{n}\mu_i^r[\langle \text{grad}\,\rho, e_i\rangle^2 + \rho\text{Hess}\,\rho(e_i,e_i)] \geq \sum_{i=1}^{n}\mu_i^r\left[\|e_i^T\|^2 + \rho\cdot\upsilon(\rho)\|e_i^{\perp}\|^2\right] \tag{28}$$

and

$$(r+1)\cdot S_{r+1}\cdot\rho\cdot\langle \text{grad}\,\rho, \eta\rangle \leq (r+1)\,R\,\cdot h_{r+1}(p,R) \tag{29}$$

From (28) and (29) we have that

$$\begin{aligned}\lambda^{L_r}(\Omega) &\geq \inf_{\Omega}(-L_r f/f)\\ &\geq 2\cdot\inf_{\Omega}\left\{\frac{1}{R^2-\rho^2}\left[\sum_{i=1}^{n}\mu_i^r\left[\|e_i^T\|^2 + \rho\cdot\upsilon(\rho)\|e_i^{\perp}\|^2\right] - (r+1)\cdot R\,\cdot h_{r+1}(p,R)\right]\right\}\end{aligned} \tag{30}$$

If $c \leq 0$ then $\rho\cdot\upsilon(\rho) \geq 1$ thus from (30) we have that

$$\begin{aligned}\lambda^{L_r}(\Omega) &\geq 2\cdot\frac{1}{R^2}\left[\inf_{\Omega}\left\{\sum_{i=1}^{n}\mu_i^r\left[\|e_i^T\|^2 + \|e_i^{\perp}\|^2\right]\right\} - (r+1)\cdot R\cdot h_{r+1}(p,R)\right]\\ &= 2\cdot\frac{1}{R^2}\left[\inf_{\Omega}\sum_{i=1}^{n}\mu_i^r - (r+1)\cdot R\cdot h_{r+1}(p,R)\right]\\ &= 2\cdot\frac{1}{R^2}\left[(n-r)\inf_{\Omega} S_r - (r+1)\cdot R\cdot h_{r+1}(p,R)\right].\end{aligned} \tag{31}$$

If $c > 0$ then $\rho \cdot \upsilon(\rho) = \rho \cdot \sqrt{c} \cdot \cot[\sqrt{c}\,\rho] \leq 1$ thus from (30) we have that

$$
\begin{aligned}
\lambda^{L_r}(\Omega) &\geq 2 \cdot \frac{1}{R^2}\left[\inf_{\Omega}\left\{\sum_{i=1}^{n}\mu_i^r\left[\|e_i^T\|^2 + \|e_i^\perp\|^2\right]\rho \cdot \sqrt{c} \cdot \cot[\sqrt{c}\,\rho]\right\} - (r+1)\cdot R \cdot h_{r+1}(p, R)\right] \\
&= 2 \cdot \frac{1}{R^2}\left[\inf_{\Omega}\left\{\sum_{i=1}^{n}\mu_i^r\,\rho\sqrt{c}\cot[\sqrt{c}\rho]\right\} - (r+1)\cdot R \cdot h_{r+1}(p, R)\right] \qquad (32)\\
&= 2 \cdot \frac{1}{R^2}\left[(n-r)\cdot R \cdot \sqrt{c}\cdot \cot[\sqrt{c}R]\cdot \inf_{\Omega} S_r - (r+1)\cdot R \cdot h_{r+1}(p, R)\right].
\end{aligned}
$$

To prove the Corollaries (3.4) and (3.5) observe that the hypotheses $\mu(P_r)(M) > 0$ (in Corollary 3.4) and $H_{r+1} > 0$ (in Corollary 3.5) imply that the L_r is elliptic. If the immersion is bounded (contained in a ball of radius R, for those choices of R) and M is closed we would have by one hand that the L_r-fundamental tone would be zero and by Theorem (3.2) that it would be positive. Then M can not be closed if the immersion is bounded. On the other hand if M is closed a ball of radius R centered at the barycenter of M could not contain M because the fundamental tone estimates for any connected component $\Omega \subset \varphi^{-1}(\varphi(M) \cap B_{\mathbb{N}^{n+1}(c)}(p, R)$ is positive. Showing that $M \neq \Omega$.

PROOF OF THEOREM 3.9.

Let $\varphi : W \hookrightarrow \mathbb{N}^{n+1}(c)$ be an isometric immersion of an oriented n-dimensional Riemannian manifold W into a $(n+1)$-dimensional simply connected space form of sectional curvature c. Let $M \subset W$ be a domain with compact closure and piecewise smooth nonempty boundary and suppose that the Newton operators P_r and P_s, $0 \leq s,\ r \leq n-1$ are positive definite when restricted to M. Given a vector field X on M we can find a vector field Y on M such that $P_r X = \kappa \cdot P_s Y$, κ constant. Now

$$
\begin{aligned}
\operatorname{div}(P_r X) - |P_r^{1/2}X|^2 &= \kappa \cdot \operatorname{div}(P_s Y) - \langle P_r X, X\rangle \\
&= \kappa \cdot \operatorname{div}(P_s Y) - \kappa^2\langle P_s Y, P_r^{-1}P_s Y\rangle \qquad (33)\\
&= \kappa \cdot \left[\operatorname{div}(P_s Y) - |P_s^{1/2}Y|^2 + |P_s^{1/2}Y|^2 - \kappa \cdot |P_r^{-1/2}P_s Y|^2\right]
\end{aligned}
$$

Consider $\{e_i\}$ be an orthonormal basis such that $P_r e_i = \mu_i^r e_i$ and $P_s e_i = \mu_i^s e_i$. Letting $Y = \sum_{i=1}^n y_i e_i$ then

$$
\begin{aligned}
|P_s^{1/2}Y|^2 - \kappa \cdot |P_r^{-1/2}P_s Y|^2 &= \sum_{i=1}^{n}\mu_i^s\,y_i^2 - \kappa\sum_{i=1}^{n}\frac{(\mu_i^s)^2}{\mu_i^r}\,y_i^2 \\
&= \sum_{i=1}^{n}\mu_i^s\,y_i^2\left[1 - \kappa \cdot \frac{\mu_i^s}{\mu_i^r}\right] \qquad (34)\\
&\geq 0, \quad if \quad \kappa \leq \frac{\mu(P_r)}{\nu(P_s)}
\end{aligned}
$$

Combining (33) with (34) and by Theorem (2.1) we have that

$$\begin{aligned} \lambda^{L_r}(M) &= \sup_X \inf_M \operatorname{div}(P_r X) - |P_r^{1/2} X|^2 \\ &\geq \kappa \cdot \sup_Y \inf_M \operatorname{div}(P_s Y) - |P_s^{1/2} Y|^2 \\ &= \kappa \cdot \lambda^{L_s}(M), \end{aligned} \tag{35}$$

for every $0 < \kappa \leq \dfrac{\mu(P_r)}{\nu(P_s)}$. This proves (9).

PROOF OF THEOREM 4.1.

Recall that for any smooth symmetric section $\Phi : \Omega \to \operatorname{End}(T\Omega)$ there is an open and dense subset $U \subset \Omega$ where the ordered eigenvalues $\{\mu_1(x) \leq \ldots \leq \mu_n(x)\}$ of $\Phi(x)$ depend continuously in all Ω. In addition, the respective eigenvectors $\{e_1(x), \ldots, e_n(x)\}$ form a smooth orthonormal frame in a neighborhood of every point of Ω, see (Kato 1976). Let $f \in C_0^2(\Omega) \setminus \{0\}$ ($f \in C^2(\Omega)$ with $\int_\Omega f = 0$) be an admissible function for (the closed L_Φ-eigenvalue problem if Ω is a closed manifold) the Dirichlet L_Φ-eigenvalue problem. It is clear that f is an admissible function for the respective $\triangle$-eigenvalue problem. Writing $\operatorname{grad} f(x) = \sum_{i=1}^n e_i(f) e_i(x)$ we have that

$$\begin{aligned} |\Phi^{1/2} \operatorname{grad} f|^2(x) &= \langle \Phi \operatorname{grad} f, \operatorname{grad} f\rangle(x) \\ &= \left\langle \sum_{i=1}^n \mu_i(x) e_i(f) e_i, \sum_{i=1}^n e_i(f) e_i \right\rangle \\ &= \sum_{i=1}^n \mu_i(x) e_i(f)^2(x). \end{aligned} \tag{36}$$

From (36) we have that

$$\nu(\Phi) \cdot |\operatorname{grad} f|^2(x) \geq |\Phi^{1/2} \operatorname{grad} f|^2(x) \geq \mu(\Phi) \cdot |\operatorname{grad} f|^2(x) \tag{37}$$

and

$$\nu(\Phi) \cdot \frac{\int_M |\operatorname{grad} f|^2}{\int_M f^2} \geq \frac{\int_M |\Phi^{1/2} \operatorname{grad} f|^2}{\int_M f^2} \geq \mu(\Phi) \cdot \frac{\int_M |\operatorname{grad} f|^2}{\int_M f^2} \tag{38}$$

Taking the infimum over all admissible functions in (38) we obtain (11).

RESUMO

Estabelecemos um método para obter limites inferiores para o tom fundamental de operadores elípticos em forma divergente em termos do divergente de campos de vetores. Aplicamos esse método para os operadores L_r associados a hipersuperfícies imersas nas formas espaciais $\mathbb{N}^n(c)$ de curvatura seccional constante c com $(r+1)$-curvatura média H_{r+1} localmente limitada. Obtemos como corolário limites inferiores para o raio extrínseco de hipersuperfícies compactas das formas espaciais $\mathbb{N}^n(c)$ com $H_{r+1} > 0$ em termos das r-ésima e $r+1$-ésima curvatura médias. Finalmente, observamos que limites para os autovalores do Laplaciano essencialmente limitam os autovalores dos operadores elípticos em forma divergente. Isso permite mostrar que a constante de Cheeger limita inferiormente o primeiro autovalor não-nulo dos operadores L_r em hypersuperfícies compactas de $\mathbb{N}^n(c)$.

Palavras-chave: tom fundamental, operador L_r, r-curvatura média, raio extrínseco, constante de Cheeger.

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