Anais da Academia Brasileira de Ciências (2008) 80(3): 411–418
(Annals of the Brazilian Academy of Sciences)
ISSN 0001-3765
www.scielo.br/aabc

On the other law of the iterated logarithm for self-normalized sums

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Manuscript received on March 17, 2008; accepted for publication on May 15, 2008;
presented by ARON SIMIS

ABSTRACT

In this note, we obtain a Chung's integral test for self-normalized sums of i.i.d. random variables. Furthermore, we obtain a convergence rate of Chung law of the iterated logarithm for self-normalized sums.

Key words: Chung's integral test, self-normalized sums, convergence rate.

1 INTRODUCTION

Let $X, X_1, X_2, \ldots$ be i.i.d. random variables with mean zero and variance one, and set

$$S_n = \sum_{k=1}^{n} X_k, \quad M_n = \max_{1\le k\le n} |S_k| \quad \text{and} \quad V_n^2 = \sum_{k=1}^{n} X_k^2, \quad n \ge 1.$$

Also let $\log x = \ln(x \vee e)$, $\log_2 x = \log(\log x)$. Then by the so-called Chung's law of the iterated logarithm we have

$$\liminf_{n\to\infty} \sqrt{\log_2 n/n}\, M_n = \pi/\sqrt{8} \quad a.s. \tag{1.1}$$

This result was first proved by Chung (1948) under $\mathsf{E}|X|^3 < \infty$, and by Jain and Pruitt (1975) under the sole assumption of a finite second moment. Einmahl (1989) obtained the Darling Erdös theorem for sums of i.i.d. random variables. Griffin and Kuelbs (1989) got Self-normalized laws of the iterated logarithm. Griffin and Kuelbs (1991) obtained some extensions of the laws of the iterated logarithm via self-normalized. Lin (1996) got a self-normalized Chung-type law of iterated logarithm. Einmahl (1993) obtained the following integral test refining (1.1) under the minimal conditions.

THEOREM A. *Let $\{X, X_n; n \ge 1\}$ be a sequence of i.i.d. random variables with $\mathsf{E}X = 0$, $\mathsf{E}X^2 = 1$ and*

$$\mathsf{E}X^2 I\big\{|X| \ge t\big\} = O\big((\log_2 t)^{-1}\big) \ \ as \ \ t \to \infty. \tag{1.2}$$

AMS Classification: 60F15, 62F05
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Then for any eventually non-decreasing function $\phi : [1, \infty) \to (0, \infty)$*,*

$$\begin{aligned} &\mathsf{P}(M_n \le \sqrt{n}/\phi(n) \quad i.o.) = 0 \ \ or \ \ 1 \\ &according\ as \ \ J(\phi) := \int_1^\infty \frac{\phi(t)^2}{t} \exp\big(-\pi^2\phi(t)^2/8\big)dt < \infty \ \ or \ \ = \infty. \end{aligned} \tag{1.3}$$

Einmahl (1993) showed that if (1.2) is not true, Theorem A is false. We thus see that condition (1.2) is sharp. However, if we use V_n to replace $\sqrt{n}$, we can eliminate the condition (1.2) in Theorem A. Explicitly, we get the following theorem.

THEOREM 1.1. *Let* $\{X, X_n; n \ge 1\}$ *be a sequence of i.i.d. random variables with* $EX = 0$*,* $EX^2 = 1$*. Then for any eventually non-decreasing function* $\phi : [1, \infty) \to (0, \infty)$*,*

$$\begin{aligned} &P(M_n \le V_n/\phi(n) \quad i.o.) = 0 \ \ or \ \ 1 \\ &according\ as \ \ J(\phi) := \int_1^\infty \frac{\phi(t)^2}{t} \exp\big(-\pi^2\phi(t)^2/8\big)dt < \infty \ \ or \ \ = \infty. \end{aligned} \tag{1.4}$$

Our next theorem gives a result on a convergence rate of (1.1).

THEOREM 1.2. *Let* $\{X, X_n; n \ge 1\}$ *be a sequence of i.i.d. random variables with* $EX = 0$*,* $EX^2 = 1$*. Then for any* $b > -1$*, we have*

$$\sum_{n=1}^{\infty} \frac{(\log_2 n)^b}{n \log n} \mathsf{P}\left(M_n \le \varepsilon\sqrt{\pi^2 V_n^2/(8\log_2 n)}\right) < \infty, \quad \forall \varepsilon > 0. \tag{1.5}$$

Throughout this note, let C denote a positive constant, whose values can differ in different places.

2 PROOF

PROOF OF THEOREM 1.1. It is enough to prove the result for eventually non-decreasing function ϕ: $[1, \infty) \to (0, \infty)$ satisfying

$$\frac{1}{2}(\log_2 t)^{1/2} \le \phi(t) \le (\log_2 t)^{1/2}, \qquad t \ge 1 \tag{2.1}$$

(See Einmahl 1993). Let

$$X_{1j} = X_j I\left\{|X_j| \le \sqrt{j}/(\log_2 j)^2\right\}, \quad j \ge 1$$

$$B_n^2 = \sum_{i=1}^{n} \mathsf{E}X_{1i}^2 \quad \text{and} \quad \triangle_n = \left|\frac{M_n}{B_n} - \frac{M_n}{V_n}\right|, \quad n \ge 1.$$

Observe that by (2.1)

$$\begin{aligned} &\mathsf{P}(M_n \le V_n/\phi(n) \ \ i.o.) \\ &\le \mathsf{P}(M_n \le V_n/\phi(n), \triangle_n \ge (\log_2 n)^{-3/2} \ \ i.o.) + \mathsf{P}(M_n \le V_n/\phi(n), \triangle_n \le (\log_2 n)^{-3/2} \ \ i.o.) \\ &\le \mathsf{P}(M_n \le V_n/\phi(n), \triangle_n \ge (\log_2 n)^{-3/2} \ \ i.o.) + \mathsf{P}(M_n/B_n \le \phi(n)^{-1} + (\log_2 n)^{-3/2} \ \ i.o.) \\ &\le \mathsf{P}(M_n \le V_n/\phi(n), \triangle_n \ge (\log_2 n)^{-3/2} \ \ i.o.) + \mathsf{P}(M_n \le B_n/\Psi(n) \ \ i.o.), \end{aligned} \tag{2.2}$$

where $\Psi(t) = \phi(t)^3/(1+\phi(t)^2)$, $t \geq 1$, and similarly,

$$\begin{aligned} &\mathsf{P}(M_n \leq V_n/\phi(n) \;\; i.o.) \\ &\geq \mathsf{P}(M_n/B_n \leq \phi(n)^{-1} - (\log_2 n)^{-3/2} \;\; i.o.) - \mathsf{P}(M_n \leq B_n/\phi(n), \triangle_n \geq (\log_2 n)^{-3/2} \;\; i.o.) \\ &\geq \mathsf{P}(M_n \leq B_n/\Psi^{'}(n) \;\; i.o.) - \mathsf{P}(M_n \leq B_n/\phi(n), \triangle_n \geq (\log_2 n)^{-3/2} \;\; i.o.), \end{aligned} \tag{2.3}$$

where $\Psi^{'}(t) = \phi(t)^3/(\phi(t)^2 - 1)$, $t \geq 1$. It is easily checked that $J(\phi) < \infty$ implies $J(\Psi) < \infty$ and $J(\phi) = \infty$ implies $J(\Psi^{'}) = \infty$, and by Theorem 1 of Einmahl (1993), we have

$$J(\Psi) < \infty \Longrightarrow \mathsf{P}(M_n \leq B_n/\Psi(n) \;\; i.o.) = 0$$

and

$$J(\Psi^{'}) = \infty \Longrightarrow \mathsf{P}(M_n \leq B_n/\Psi^{'}(n) \;\; i.o.) = 1.$$

Now by Lemma 2.2 below,

$$\mathsf{P}(M_n \leq V_n/\phi(n), \triangle_n \geq (\log_2 n)^{-3/2} \;\; i.o.) = 0$$

and

$$\mathsf{P}(M_n \leq B_n/\phi(n), \triangle_n \geq (\log_2 n)^{-3/2} \;\; i.o.) = 0.$$

From these equations and (2.2), (2.3), hence we see that Theorem 1.1 holds true. □

We now present two lemmas used in the main proof of Theorem 1.1.

LEMMA 2.1. *For any $x > 0$ there exist positive constants $\eta = \eta(x)$ and $A = A(x)$ such that*

$$\mathsf{P}\left(M_n \leq x\sqrt{n/\log_2 n}\right) \leq A(\log n)^{-\eta}.$$

PROOF. See the Lemma 2(b) of Einmahl (1993). □

LEMMA 2.2. *We have*

$$\mathsf{P}(M_n \leq V_n/\phi(n), \triangle_n \geq (\log_2 n)^{-3/2} \;\; i.o.) = 0 \tag{2.4}$$

and

$$\mathsf{P}(M_n \leq B_n/\phi(n), \triangle_n \geq (\log_2 n)^{-3/2} \;\; i.o.) = 0. \tag{2.5}$$

PROOF. Let

$$X_{2j} = X_j I\left\{\sqrt{j}/(\log_2 j)^2 < |X_j| \leq \sqrt{j}\right\}, X_{3j} = X_j - X_{1j} - X_{2j}, \;\; j \geq 1$$

and

$$V_{1n} = \sum_{k=1}^{n} \left(X_{1k}^2 - \mathsf{E}X_{1k}^2\right), V_{2n} = \sum_{k=1}^{n} X_{3k}^2, \;\; n \geq 1.$$

First using $\mathsf{E}X^2 = 1$, we have

$$\sum_{n=1}^{\infty} \frac{(\log_2 n)^4}{n^2} \mathsf{E}\big(X_{1n}^2 - \mathsf{E}X_{1n}^2\big)^2 \le 2\sum_{n=1}^{\infty} \frac{(\log_2 n)^4}{n^2} \mathsf{E}X^4 I\big\{|X| \le \sqrt{n}/(\log_2 n)^2\big\}$$
$$= 2\sum_{k=1}^{\infty} \mathsf{E}X^4 I\big\{\sqrt{k-1}/\big(\log_2(k-1)\big)^2 < |X| \le \sqrt{k}/(\log_2 k)^2\big\} \sum_{n=k}^{\infty} \frac{(\log_2 n)^4}{n^2}$$
$$\le C\sum_{k=1}^{\infty} \mathsf{E}X^2 I\big\{\sqrt{k-1}/\big(\log_2(k-1)\big)^2 < |X| \le \sqrt{k}/(\log_2 k)^2\big\}$$
$$\le C\mathsf{E}X^2 < \infty$$

and

$$\sum_{n=1}^{\infty} \mathsf{P}(X_{3n} \ne 0) = \sum_{n=1}^{\infty} \mathsf{P}(|X| > \sqrt{n}) \le C\mathsf{E}X^2 < \infty.$$

Thus, it follows by applying Corollary 3.1 of Lin et al. (1999, P.95) and Borel-Cantelli lemma that

$$\frac{(\log_2 n)^2}{n} V_{1n} \to 0 \quad a.s. \quad \text{and} \quad V_{2n} = O(1) \quad a.s. \tag{2.6}$$

Using strong law of large numbers and Hartman-Wintner LIL, we have

$$\lim_{n\to\infty} \frac{V_n^2}{n} = 1 \quad a.s. \quad \text{and} \quad \limsup_{n\to\infty} \frac{M_n}{\sqrt{2n\log_2 n}} \le 1 \quad a.s.$$

Thus, by $\mathsf{E}X^2 = 1$, we obtain that for large n,

$$\begin{aligned} \triangle_n &= \left| \frac{M_n(V_n^2 - B_n^2)}{B_n V_n (B_n + V_n)} \right| \\ &\le \frac{3\sqrt{\log_2 n}}{n} \Big(|V_{1n}| + V_{2n} + \sum_{j=1}^{n} X_{2j}^2 \Big) \\ &\le \frac{3\sqrt{\log_2 n}}{n} \Big(|V_{1n}| + V_{2n} + \sqrt{n} \sum_{j=1}^{n} |X_{2j}| \Big) \quad a.s. \end{aligned} \tag{2.7}$$

Recalling that $B_n^2 \le n,\ n \ge 1$ and $\lim_{n\to\infty} V_n^2/n = 1$ a.s., in order to prove (2.4) and (2.5), by (2.1), (2.6) and (2.7), it suffices to show that

$$\mathsf{P}\Big(M_n \le 2\sqrt{n/\log_2 n}, \sum_{j=1}^{n} |X_{2j}| \ge \frac{1}{4}\sqrt{n}/(\log_2 n)^2 \ \ i.o. \Big) = 0. \tag{2.8}$$

Now, set $m(n) := \big[n/(\log_2 n)^9\big]$, $n \ge 1$. By $\mathsf{E}X^2 = 1$, we have

$$\sum_{j=1}^{n} \mathsf{E}|X_{2j}| \le \sum_{j=1}^{n} \mathsf{E}|X_j| I\big\{|X_j > \sqrt{j}/(\log_2 j)^2|\big\} \le \sum_{j=1}^{n} \frac{(\log_2 j)^2}{\sqrt{j}} \le C\sqrt{n}(\log_2 n)^2.$$

Applying Kolmogorov's LIL and $\mathsf{E}X^2 = 1$, we have

$$\limsup_{n\to\infty} \frac{\sum_{j=1}^n (|X_{2j}| - \mathsf{E}|X_{2j}|)}{\sqrt{2n\log_2 n}} \le 2 \quad a.s.$$

it easily follows from above inequalities that

$$\sum_{j=1}^{m(n)} |X_{2j}| = o\big(\sqrt{n}/(\log_2 n)^2\big) \quad a.s. \tag{2.9}$$

Hence observe that on account of (2.9) it is enough to show that

$$\mathsf{P}\bigg(M_n \le 2\sqrt{n/\log_2 n},\ \sum_{j=m(n)+1}^{n} |X_{2j}| \ge \frac{1}{5}\sqrt{n}/(\log_2 n)^2 \ \ i.o.\bigg) = 0. \tag{2.10}$$

Let $n_k = 2^k$ and $m_k = \big[2^k/(\log k)^{10}\big]$, $k \ge 0$, for large enough k,

$$\bigcup_{n=n_{k-1}+1}^{n_k} \bigg\{M_n \le 2\sqrt{n/\log_2 n},\ \sum_{j=m(n)+1}^{n} |X_{2j}| \ge \frac{1}{5}\sqrt{n}/(\log_2 n)^2\bigg\}$$

$$\subseteq \bigg\{M_{n_{k-1}} \le 2\sqrt{2n_{k-1}/\log_2 n_{k-1}},\ \sum_{j=m_k+1}^{n_k} |X_{2j}| \ge \frac{1}{10}\sqrt{n_k}/(\log_2 n_k)^2\bigg\}.$$

Thus, in order to prove (2.10), it suffices to show that

$$\mathsf{P}\bigg(M_{n_{k-1}} \le 2\sqrt{2n_{k-1}/\log_2 n_{k-1}},\ \sum_{j=m_k+1}^{n_k} |X_{2j}| \ge \frac{1}{10}\sqrt{n_k}/(\log_2 n_k)^2 \ \ i.o.\bigg) = 0. \tag{2.11}$$

Let

$$M_{n_{k-1},j} \;=\; M_{j-1} \vee \max_{j<n\le n_{k-1}} |S_n - X_j|, \quad 1 \le j \le n_{k-1} \quad \text{and} \quad n_k^{'} \;=\; n_{k-1} - 1.$$

Notice that

$$M_{n_{k-1},j} \le M_{n_{k-1}} + |X_j| \le 3M_{n_{k-1}}, \quad 1 \le j \le n_{k-1}.$$

Using the independence and Lemma 2.1, it is clear that for some constant $\eta > 0$ and large enough k,

$$\begin{aligned}
&\mathsf{P}\bigg(M_{n_{k-1}} \le 2\sqrt{2n_{k-1}/\log_2 n_{k-1}},\ \sum_{j=m_k+1}^{n_k} |X_{2j}| \ge \frac{1}{10}\sqrt{n_k}/(\log_2 n_k)^2\bigg) \\
&\le \mathsf{P}\bigg(\bigcup_{j=m_k+1}^{n_k} \{|X_j| > \sqrt{j}/(\log_2 j)^2,\ M_{n_{k-1}} \le 2\sqrt{2n_{k-1}/\log_2 n_{k-1}}\}\bigg) \\
&\le \sum_{j=m_k+1}^{n_{k-1}} \mathsf{P}\big(|X_j| > \sqrt{j}/(\log_2 j)^2,\ M_{n_{k-1}} \le 2\sqrt{2n_{k-1}/\log_2 n_{k-1}}\big) \\
&\quad + \sum_{j=n_{k-1}+1}^{n_k} \mathsf{P}\big(|X_j| > \sqrt{j}/(\log_2 j)^2,\ M_{n_{k-1}} \le 2\sqrt{2n_{k-1}/\log_2 n_{k-1}}\big)
\end{aligned}$$

$$
\begin{aligned}
&\le \sum_{j=m_k+1}^{n_{k-1}} \mathsf{P}\big(|X_j| > \sqrt{j}/(\log_2 j)^2,\, M_{n_{k-1},j} \le 6\sqrt{2n_{k-1}/\log_2 n_{k-1}}\big)\\
&\quad + \sum_{j=n_{k-1}+1}^{n_k} \mathsf{P}\big(|X_j| > \sqrt{j}/(\log_2 j)^2\big)\mathsf{P}\big(M_{n_{k-1}} \le 2\sqrt{2n_{k-1}/\log_2 n_{k-1}}\big)\\
&\le \sum_{j=m_k+1}^{n_{k-1}} \mathsf{P}\big(|X_j| > \sqrt{j}/(\log_2 j)^2\big)\mathsf{P}\big(M_{n_k'} \le 9\sqrt{n_k'/\log_2 n_k'}\big)\\
&\quad + \sum_{j=n_{k-1}+1}^{n_k} \mathsf{P}\big(|X_j| > \sqrt{j}/(\log_2 j)^2\big)\mathsf{P}\big(M_{n_{k-1}} \le 3\sqrt{n_{k-1}/\log_2 n_{k-1}}\big)\\
&\le Ck^{-\eta} \sum_{j=m_k+1}^{n_k} \mathsf{P}\big(|X| > \sqrt{j}/(\log_2 j)^2\big).
\end{aligned}
$$

Finally, By Lemma 4 of Einmahl (1993), we have

$$
\begin{aligned}
&\sum_{k=1}^{\infty} \mathsf{P}\left(M_{n_{k-1}} \le 2\sqrt{2n_{k-1}/\log_2 n_{k-1}},\ \sum_{j=m_k+1}^{n_k} |X_{2j}| \ge \frac{1}{10}\sqrt{n_k}/(\log_2 n_k)^2\right)\\
&\le C\sum_{k=1}^{\infty} k^{-\eta} \sum_{j=m_k+1}^{n_k} \mathsf{P}\big(|X| > \sqrt{j}/(\log_2 j)^2\big) < \infty
\end{aligned}
$$

and hence we obtain (2.11) from the Borel-Cantelli lemma. □

PROOF OF THEOREM 1.2. For each $n \ge 1$ and $1 \le i \le n$, we have

$$
\bar{X}_{ni} = X_i I\big\{|X_i| \le n^{1/2}(\log n)^{-1/3}\big\}, \quad \bar{V}_n^2 = \sum_{j=1}^{n} \bar{X}_{nj}^2 \quad \text{and} \quad \bar{B}_n^2 = \sum_{j=1}^{n} \mathsf{Var}(\bar{X}_{nj}).
$$

By $\mathsf{E}X^2 = 1$, it is easy to show that $\bar{B}_n^2 \le n$. Hence for some $\frac{1}{7} < \delta < 1$ and any $\varepsilon > 0$

$$
\begin{aligned}
&\mathsf{P}\big(M_n \le \varepsilon\sqrt{\pi^2 V_n^2/(8\log_2 n)}\big)\\
&\le \mathsf{P}\big(M_n \le \varepsilon\sqrt{\pi^2(1+\delta)\bar{B}_n^2/(8\log_2 n)}\big) + \mathsf{P}\big(V_n^2 \ge (1+\delta)\bar{B}_n^2\big)\\
&\le \mathsf{P}\big(M_n \le \varepsilon\sqrt{\pi^2(1+\delta)/8}\sqrt{n/\log_2 n}\big) + \mathsf{P}\big(\bar{V}_n^2 \ge (1+\delta)\bar{B}_n^2\big)\\
&\quad + \mathsf{P}\Big(\bigcup_{j=1}^{n}\big\{|X_j| > n^{1/2}(\log n)^{-1/3}\big\}\Big)\\
&:= I_1 + I_2 + I_3.
\end{aligned}
$$

In order to prove (1.5), it suffices to show that for any $b > -1$

$$
\sum_{n=1}^{\infty} \frac{(\log_2 n)^b}{n\log n} I_i < \infty, \quad \forall\varepsilon > 0, \quad i = 1, 2, 3. \tag{2.12}
$$

By Lemma 2.1, there exists a positive constant η such that

$$\sum_{n=1}^{\infty}\frac{(\log_2 n)^b}{n\log n}I_1 \le C\sum_{n=1}^{\infty}\frac{(\log_2 n)^b}{n\log n}(\log n)^{-\eta} < \infty, \quad \forall\varepsilon > 0.$$

Since $\mathsf{E}X = 0$ and $\mathsf{E}X^2 = 1$, there exists a positive integer n_0 such that for all $n \ge n_0$

$$\mathsf{E}\bar{X}_{n1}^2 \ge \frac{3}{4} \quad \text{and} \quad \mathsf{E}\bar{X}_{n1} \le \frac{1}{4}.$$

Hence using the Bernstein inequality, there exists a positive constant $\beta < 1/3000$ such that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(\log_2 n)^b}{n\log n}I_2 &\le C + \sum_{n=n_0}^{\infty}\frac{(\log_2 n)^b}{n\log n}\mathsf{P}\big(\bar{V}_n^2 \ge (1+\delta/2)n\mathsf{E}\bar{X}_{n1}^2\big)\\ &\le C + \sum_{n=n_0}^{\infty}\frac{(\log_2 n)^b}{n\log n}(\log n)^{-\beta}\\ &< \infty.\end{aligned}$$

Finally, by $\mathsf{E}X^2 = 1$, we have

$$\sum_{n=1}^{\infty}\frac{(\log_2 n)^b}{n\log n}I_3 \le \sum_{n=1}^{\infty}\frac{(\log_2 n)^b}{\log n}\mathsf{P}\big(|X| > n^{1/2}(\log n)^{-1/3}\big) \le C\mathsf{E}X^2 < \infty.$$

Thus, (2.12) holds true. □

ACKNOWLEDGMENTS

This paper was supported by National Natural Science Foundation of China and Youth Talent Foundation of Zhejiang Gongshang University (Q07-07).

RESUMO

Nesta nota, obtemos um teste integral de Chung para somas auto-normalizadas de variáveis aleatórias i.i.d. (independentes e identicamente distribuídas). Além disso, obtemos uma taxa de convergência da lei de Chung do logaritmo iterado para somas auto-normalizadas.

Palavras-chave: teste integral de Chung, somas auto-normalizadas, taxa de convergência.

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