Anais da Academia Brasileira de Ciências (2013) 85(2): 457-471
(Annals of the Brazilian Academy of Sciences)
Printed version ISSN 0001-3765 / Online version ISSN 1678-2690
www.scielo.br/aabc

L^2 harmonic forms and finiteness of ends

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Manuscript received on February 24, 2011; accepted for publication on October 1, 2012

ABSTRACT

In this paper, we obtain vanishing theorems and finitely many ends theorems of complete Riemannian manifolds with weighted Poincaré inequality, applying them to minimal hypersurfaces.

Key words: end, minimal hypersurface, weighted Poincaré inequality, L^2 harmonic forms, complete manifolds.

INTRODUCTION

Li and Wang (2001) proved a vanishing theorem for L^2 harmonic one-forms on M under the assumption of the Ricci curvature bounded from below in terms of $\lambda_1(M)$, which is the greatest lower bound of the spectrum of the Laplacian acting on L^2 functions. Lam (2010) generalized this result to the manifold satisfying a weighted Poincaré inequality. Recalling that in Li and Wang (2006), a complete Riemannian manifold (M, ds^2) is said to satisfy a weighted Poincaré inequality with a non-negative weighted function ρ if the inequality

$$\int_M \rho\phi^2 \leq \int_M |\nabla\phi|^2 \tag{1.1}$$

holds for all compactly supported piecewise smooth function $\phi \in C_0^\infty(M)$. (M, ds^2) is is said to satisfy the property $(\mathcal{P}_\rho)$ if M satisfies a weighted Poincaré inequality with ρ and ρds^2 is complete. If M is a complete manifold with property $(\mathcal{P}_\rho)$, Li and Wang (2006) -Theorem 5.2- gave a rigidity theorem under the assumption of the Ricci curvature bounded from below in terms of ρ. Cheng and Zhou (2009) generalized this result and applied to minimal hypersurfaces. Lam (2010) also obtained a result similar to Theorem 5.2 in Li and Wang (2006) by relaxing Ricci curvature to be only satisfied outside a compact set of M.

In this paper, on the one hand, we obtain vanishing theorems of two classes of complete manifolds that satisfy weighted Poincaré inequalities (Theorems 2.2 and 2.7). At the same time, we apply Theorem 2.2 to minimal hypersurfaces (Corollaries 2.4 and 2.5). On the other hand, we also obtain two results on finitely many ends of complete manifolds (Theorems 3.1 and 3.6). As an application, we obtain that finitely many ends of minimal hypersurfaces in non-positively curved manifolds (Corollaries 3.7 and 3.9).

AMS Classication: 53C21, 54C42.
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2 VANISHING THEOREMS

In this section, we study $H^1(L^2(M))$, the space of L^2 integrable harmonic 1-forms. The following lemma is due to Lam (Lam 2010):

Lemma 2.1. *Suppose that* $\omega \in H^1(L^2(M))$ *and* $h = |\omega|$ *satisfies that*

$$h\Delta h \geq -ah^2 + b|\nabla h|^2,$$

for a function a on M and some positive constant b. Then, for any $\phi \in C_0^\infty(M)$ *and* $0 < \epsilon < 1 + b^{-1}$, *the following inequality holds*

$$(b(1-\epsilon)+1)\int_M |\nabla(\phi h)|^2 \leq (b(\epsilon^{-1}-1)+1)\int_M h^2|\nabla\phi|^2 + \int_M a\phi^2 h^2.$$

Theorem 2.2. *Suppose that M is an n-dimensional complete non-compact Riemannian manifold satisfying the weighted Poincaré inequality with a non-negative weight function ρ (n ≥ 2). Assume the Ricci curvature satisfies*

$$Ric_M(x) \geq -\frac{n}{n-1}\rho(x) + \sigma(x), \tag{2.1}$$

for a non-negative continuous function σ (σ ≠ 0). If $\rho(x) = O(r^{2-\alpha})$, *where* $r_p(x)$ *is the distance function from x to some fixed point p, for some* $0 < \alpha < 2$, *then* $H^1(L^2(M)) = \{0\}$.

Proof. If $\omega \in H^1(L^2(M))$, then $h = |\omega|$ satisfies a formula (Li and Wang 2006):

$$h\Delta h \geq Ric_M(x)\,(\omega,\omega) + (n-1)^{-1}|\Delta h|^2. \tag{2.2}$$

By Lemma 2.1 with b = $(n-1)^{-1}$ and $a = (b+1)\rho - \sigma$, we obtain that

$$\begin{aligned}(b(1-\epsilon)+1)\int_M \rho\phi^2 h^2 &\leq (b(1-\epsilon)+1)\int_M |\nabla(\phi h)|^2 \\ &\leq (b(\epsilon^{-1}-1)+1)\int_M h^2|\nabla\phi|^2 + \int_M a\phi^2 h^2,\end{aligned}$$

that is,

$$(n-1)\int_M \sigma\phi^2 h^2 \leq (n-2+\epsilon^{-1})\int_M h^2|\nabla\phi|^2 + \epsilon\int_M \rho h^2\phi^2, \tag{2.3}$$

for any $0 < \epsilon < n$ and $\phi \in C_0^\infty(M)$. Choose $\epsilon = R^{\alpha/2-2}$ (R sufficient large) and

$$\phi = \begin{cases} 1 & \text{on B(R)}, \\ 0 & \text{on M} \setminus \text{B(2R)}, \end{cases}$$

such that

$$|\nabla\phi|^2 \leq CR^{-2}$$

on $B(2R) \setminus B(R)$. By (2.3) and $\rho(x) = O(r_p^{2-\alpha})$, we obtain that

$$(n-1)\int_{B(R)} \sigma h^2 \leq C((n-2)\,R^{-2} + R^{\alpha/2})\int_{B(2R)} h^2.$$

Let $R \to +\infty$. We obtain that

$$\int_M \sigma h^2 \leq 0,$$

since $h \in L^2(M)$. By assumption, there exist $p_0 \in M$ and $r_0 > 0$ such that $\sigma > 0$ on $B(p_0, r_0)$. Thus,

$$\int_{B(p_0,\, r_0)} \sigma h^2 = 0.$$

It implies that $\omega = 0$ on $B(p_0, r_0)$. Therefore, $\omega = 0$ on M, by uniqueness of expanding theorem of harmonic forms.

Remark 2.3. *When σ is equal to a positive constant and $c\rho$ for some positive constant c and a positive weight function ρ, respectively, Theorem 2.2 is just Theorem 3.4 and Theorem 3.5 in Lam (2010), respectively. We should also point out that Carron (2002) gave a very nice survey about L^2 harmonic k-forms on non-compact Riemannian manifolds.*

Let M^n be a minimal hypersurface of $\mathbb{R}^{n+1}$. ν denotes the unit normal vector field of M. $|A|$ is the normal of the second fundamental form A. A minimal hypersurface $M^n \subset N^n$+1 is called δ-stable if, for each $\phi \in C_0^\infty(M)$,

$$\delta\int_M |A|^2\, \phi^2 \leq \int_M |\nabla \phi|^2. \tag{2.4}$$

Corollary 2.4. Suppose that M^n $(n \geq 2)$ *is a complete minimal δ-stable* $(\delta > \frac{(n-1)^2}{n^2})$ *hypersurface in* $\mathbb{R}^{n+1}$. *If* $|A| = O(r_p^{2-\alpha})$*, for some* $0 < \alpha < 2$, *then* $H^1(L^2(M)) = \{0\}$.

Proof. First, a complete minimal hypersurface in $\mathbb{R}^{n+1}$ is non-compact. For any point $p \in M$ and any unit tangent vector $v \in T_pM$, we can choose an orthonormal frame $\{e_1, e_2, \ldots, e_n\}$ on M at p such that $e_1 = v$. Since M is a minimal hypersurface, there has the following inequality:

$$|A|^2 \geq h_{11}^2 + \frac{(\sum_{i=2}^{n} h_{ii})^2)^2}{n-1} + 2\sum_{i=2}^{n} h_{1i}^2 \geq \frac{n}{n-1}\sum_{i=1}^{n} h_{1i}^2. \tag{2.5}$$

The Gauss equation implies that

$$Ric_M(v, v) = \sum_{i=1}^{n} (h_{11}h_{ii} - h_{1i}^2) = -\sum_{i=1}^{n} h_{1i}^2. \tag{2.6}$$

From (2.5) and (2.6), we obtain that (2.1) holds, where

$$\rho(x) = \delta|A|^2$$

and

$$\sigma(x) = \frac{n}{n-1}\left(\left(\delta - \frac{(n-1)^2}{n^2}\right)|A|^2\right).$$

Since M is δ-stable, (1.1) holds. If $|A| = 0$ then $H^1(L^2(M)) = \{0\}$. Otherwise, by Theorem 2.2, we have $H^1(L^2(M)) = \{0\}$.

If M^n is a minimal hypersurface in $\mathbb{R}^{n+1}$ ($n \geq 3$), then each end of M is non-parabolic and the number of non-parabolic end of M is bounded from above by dim $H^1(L^2(M))$ +1 (Li and Wang 2002). Thus, Corollary 2.4 implies that:

Corollary 2.5. *Suppose that* M^n ($n \geq 3$) *is a complete minimal* δ*-stable* ($\delta > \frac{(n-1)^2}{n^2}$) *hypersurface in* $\mathbb{R}^{n+1}$. *If it has the bounded norm of the second fundamental form, then M has only one end.*

Remark 2.6. *Cheng and Zhou (2009) proved that: let M be an* $\frac{n-2}{n}$ *-stable complete minimal hypersurface* $\mathbb{R}^{n+1}$ (n $\geq$ 3). *If it has bounded norm of its second fundamental form, then M either has only one end or is a catenoid. In Cheng et al. (2008), it was proved that a complete oriented weakly minimal hypersurface in* $\mathbb{R}^{n+1}$ ($n \geq 3$) *must have only one end. It was shown that in Cheng (2008) one end theorem holds for complete noncompact stable minimal hypersurfaces in general manifolds and the method is different from above.*

If M is a quaternionic manifold and $\omega \in H^1(L^2(M))$, then $h = |\omega|$ satisfies (Kong et al. 2008):

$$h\Delta h \geq Ric_M(\omega,\omega) + \tfrac{1}{3}|\nabla h|^2.$$

Using the similar method as proof of Theorem 2.2, we also can obtain:

Theorem 2.7. *Suppose that M is a 4n-dimensional complete non-compact quaternionic manifold satisfying the weighted Poincaré inequality with a nonnegative weight function* $\rho(x)$. *Assume the Ricci curvature satisfies*

$$Ric_M(x) \geq -\tfrac{4}{3}\rho(x) + \sigma(x),$$

for a positive function σ. If

$$\rho(x) = O(r_p^{2-\alpha}),$$

for some $0 < \alpha < 2$, *then* $H^1(L^2(M)) = \{0\}$.

Remark 2.8. *If we choose σ is equal to positive constant and* $\rho = \lambda_1(M) > 0$, *Theorems 2.7 is Theorem 4.3 in Lam (2010).*

3 FINITENESS OF ENDS

In this section, we study complete Riemannian manifolds with a weighted Poincaré inequality, giving two results on finitely many ends of complete manifolds. Then, we apply one of the results to hypersurfaces in manifolds with non-positive sectional curvature. First, following the idea of Cheng and Zhou (2009) with some changes of technique, we obtain that:

Theorem 3.1. *Let M be a complete n-dimensional manifold with property* (P_ρ) $(n \geq 3)$. *Suppose that the Ricci curvature of M satisfies that*

$$Ric_{M\setminus\Omega}(x) \geq -(n-1)\tau(x) + \varepsilon,\ x \in M,$$

for a constant $\varepsilon > 0$ *and* Ω *is a compact subset of M, where the non-negative function* $\tau(x)$ *satisfies Poincaré inequality*

$$(n-2)\int_M \tau\phi^2 \leq \int_M |\nabla\phi|^2,$$

for all $\phi \in C_0^\infty(M \setminus \Omega)$. *Set*

$$S(R) = \sup_{x \in B_\rho(R)} (\sqrt{\rho(x)}, \sqrt{\tau(x)}).$$

If ρ *and* τ *satisfy the growth estimates*

$$\liminf_{R\to\infty} S(R)\exp(-((n-3)/(n-2))R) = 0, \qquad \text{for } n \geq 4,$$

$$\liminf_{R\to\infty} S(R)R^{-1} = 0, \qquad \text{for } n = 3,$$

then M has only finitely many non-parabolic ends.

Remark 3.2. *If* $\tau = \frac{\rho}{n-2}$ *then Theorem 3.1 is just Theorem 0.2 in Lam (2010).*

Now we prove Theorem 3.1. A theory of Li-Tam (Li and Tam 1991) gives the number of non-parabolic ends of M. More precisely, suppose that M has at least two non-parabolic ends E_1 and E_2. There exists a non-constant harmonic function f_1 with finite Dirichlet integral by taking a convergent subsequence of harmonic functions f_R as $R \to +\infty$, satisfying $\Delta f_R = 0$ in $B(R)$ with boundary conditions that $f_R = 1$ on $\partial B(R) \cap E_1$ and $f_R = 0$ on $\partial B(R) \setminus E_1$. It follows from the Maximum Principle that $0 \leq f_R \leq 1$ for all R and hence $0 \leq f_1 \leq 1$. For each non-parabolic ends E_i, we can construct a corresponding fi by the above method. Let $\mathcal{K}^0(M)$ be the linear space containing all f_i's constructed as above. Thus, the number of non-parabolic ends of M is dim $\mathcal{K}^0(M)$. The following lemma is due to Li (Li 1993):

Lemma 3.3. *Suppose that* $\mathcal{H}$ *is a finite dimensional subspace of* L^2 1*-forms defined over a set* $D \subseteq M^n$. *Then there exists* $\omega_0 \in \mathcal{H}$ *such that*

$$\dim \mathcal{H}\int_D |\omega_0|^2 \leq nV(D) \sup_D |\omega_0|^2,$$

where $V(D)$ *is the volume of the set D.*

Let

$$\mathcal{K} = \{df : f \in \mathcal{K}^0(M)\}.$$

Obviously,

$$\dim \mathcal{K}^0(M) = \dim \mathcal{K} + 1.$$

In order to estimate dim $\mathcal{K}$, by Lemma 3.3, it is suffice to choose a suitable subset $D = B(R_0)$ of M and prove that

$$\sup_{B(R_0)} |\nabla f|^2 \leq C \int_{B(R_0)} |\nabla f|^2, \tag{3.1}$$

for each non-constant bounded harmonic function $f \in \mathcal{K}^0(M)$. Next, we will prove (3.1) by Lemmas 3.4 and 3.5. For convenience, we replace ε by $\frac{n-1}{n-2}\varepsilon$. By (2.2) and the assumption of the Ricci curvature in Theorem 3.1, we obtain that, on $M \setminus \Omega$,

$$\Delta|\nabla f| \geq -(n-1)|\nabla f|\,(\tau - \frac{\varepsilon}{n-2}) + \frac{1}{n-1}\frac{|\nabla|\nabla f||^2}{|\nabla f|}. \tag{3.2}$$

Set $h = |\nabla f|^{\frac{n-2}{n-1}}$. Inequality (3.2) becomes

$$\Delta h + (n-2)\tau h \geq \varepsilon h \tag{3.3}$$

on $M \setminus \Omega$. For each $\phi \in C_0^\infty(M \setminus \Omega)$, we have

$$(n-2)\int_M \tau(\phi h)^2 \leq \int_M |\nabla(\phi h)|^2 = \int_M |\nabla \phi h|^2 h^2 - \int_M \phi^2 h \Delta h.$$

Combining with (3.3), we obtain that

$$\varepsilon \int_M \phi^2 h^2 \leq \int_M \phi^2 h(\Delta h + (n-2)\,\tau h) \leq \int_M |\nabla \phi h|^2 h^2. \tag{3.4}$$

Let $B(R)$ be the geodesic ball of radius R for some fixed point p. Since Ω is compact, we can choose a positive R_0 (to be fixed in Lemma 3.4) such that

$$\Omega \subseteq \cup_{x\in\Omega} B_\rho(x, 1) \subseteq B(R_0 - 1).$$

Let $R > 0$, such that

$$B(R_0) \subseteq B_\rho(R - 1).$$

Lemma 3.4. *Under the assumption of Theorem 3.1, there exists a constant C depending on n and* $\sup_{B(R_0)} \rho$ *such that*

$$\varepsilon \int_{M \setminus B(R_0)} h^2 \leq C \int_{B(R_0)\setminus B(R_0-1)} h^2.$$

Proof. From (3.4), we have

$$\varepsilon \int_{B_\rho(R)\setminus B(R_0-1)} \phi^2 h^2 \leq \int_{B_\rho(R)\setminus B(R_0-1)} |\nabla \phi|^2 h^2, \tag{3.5}$$

for all $\phi \in C_0^\infty(B_\rho(R) \setminus B(R_0 - 1))$. Choose $\phi = \psi\chi$, where ψ and χ will be chosen later. Then

$$\int_{B_\rho(R)\setminus B(R_0-1)} |\nabla\phi|^2 h^2 \leq 2\mathcal{A} + 2\mathcal{B}, \tag{3.6}$$

where

$$\mathcal{A} = \int_{B_\rho(R)\setminus B(R_0-1)} |\nabla\psi|^2\chi^2 h^2$$

and

$$\mathcal{B} = \int_{B_\rho(R)\setminus B(R_0-1)} |\nabla\chi|^2\psi^2 h^2.$$

For $n \geq 4$. Choose ψ, χ as follows,

$$\psi(x) = \begin{cases} 0 & \text{on } B(R_0-1), \\ 1 & \text{on } B_\rho(R-1) \setminus B(R_0), \\ R - r_\rho & \text{on } B_\rho(R) \setminus B_\rho(R-1), \\ 0 & \text{on } M \setminus B_\rho(R). \end{cases}$$

For $\sigma \in (0, 1)$ and $\epsilon \in (0, \frac{1}{2})$, we define χ on the levels sets of f:

$$\chi(x) = \begin{cases} 0 & \text{on } \mathcal{L}(0, \sigma\epsilon) \cup \mathcal{L}(1-\sigma\epsilon, 1), \\ (\epsilon - \sigma\epsilon)^{-1}(f - \sigma\epsilon) & \text{on } \mathcal{L}(\sigma\epsilon, \epsilon) \cap M \setminus E_1, \\ (\epsilon - \sigma\epsilon)^{-1}(1 - f - \sigma\epsilon) & \text{on } \mathcal{L}(1-\epsilon, 1-\sigma\epsilon) \cap E_1, \\ 1 & \text{otherwise,} \end{cases}$$

where,

$$\mathcal{L}(a, b) = \{x \in M \,|\, a < f(x) < b\}.$$

From the definition of ψ, we obtain

$$\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2, \tag{3.7}$$

where

$$\mathcal{A}_1 = \int_{B_\rho(R)\setminus B_\rho(R-1)} |\nabla\psi|^2\chi^2 h^2$$

and

$$\mathcal{A}_2 = \int_{B(R_0)\setminus B(R_0-1)} |\nabla\psi|^2\chi^2 h^2.$$

First, consider

$$\mathcal{A}_1 = \mathcal{A}_{11} + \mathcal{A}_{12},$$

where

$$\mathcal{A}_{11} = \int_{(B_\rho(R)\setminus B_\rho(R-1))\cap E_1} |\nabla\psi|^2\chi^2 h^2$$

and

$$\mathcal{A}_{12} = \int_{(B_\rho(R)\setminus B_\rho(R-1))\cap(M\setminus E_1)} |\nabla\psi|^2\chi^2 h^2.$$

Denote the set

$$\Omega_1 = E_1 \cap (B_\rho(R) \setminus B_\rho(R-1)) \cap (\mathcal{L}(\sigma\epsilon, 1-\sigma\epsilon)).$$

Note that

$$|\nabla r_\rho|^2(x) = \rho(x).$$

So

$$|\nabla\psi|^2 \leq \rho$$

on $B_\rho(R) \setminus B_\rho(R-1)$. Thus, by the definition of χ and the Hölder inequality, we obtain that

$$\mathcal{A}_{11} \leq \int_{\Omega_1} \rho|\nabla f|^{\frac{2(n-2)}{n-1}} \leq \left(\int_{\Omega_1} |\nabla f|^2\right)^{\frac{(n-2)}{n-1}} \left(\int_{\Omega_1} \rho^{n-1}\right)^{\frac{1}{n-1}}. \tag{3.8}$$

Recall that, under the assumption of complete manifold M with the property (P_ρ), the growth estimate for $|\nabla f|$ (Li and Wang 2006, Corollary 2.3):

$$\int_{B_\rho(R+1)\setminus B_\rho(R)} |\nabla f|^2 \leq C\exp(-2R) \tag{3.9}$$

and the decay estimate for f (Li and Wang 2006, (2.10)):

$$\int_{(B_\rho(R+1)\setminus B_\rho(R))\cap E_1} \rho\,(1-f)^2 \leq C\exp(-2R). \tag{3.10}$$

From the definition of $S(R)$, we obtain that

$$\begin{aligned}\int_{\Omega_1} \rho^{n-1} &\leq (S(R))^{2(n-2)} \int_{\Omega_1} \rho \leq (\sigma\epsilon)^{-2}(S(R))^{2(n-2)} \int_{\Omega_1} \rho(1-f)^2 \\ &\leq C(\sigma\epsilon)^{-2}(S(R))^{2(n-2)}\exp(-2R),\end{aligned} \tag{3.11}$$

where the second inequality holds because $(\sigma\epsilon)^{-2}(1-f)^2 \geq 1$ on Ω_1 and the last one holds because of (3.10). From (3.8), (3.9) and (3.11), we obtain

$$\mathcal{A}_{11} \leq C\,(\sigma\epsilon)^{-2/(n-1)}(S(R))^{2(n-2)/(n-1)} \exp(-2R). \tag{3.12}$$

Now, we consider $\mathcal{B}$. Let

$$\mathcal{B}_1 = \int_{E_1} |\nabla\chi|^2 \psi^2 h^2,$$

and

$$\mathcal{B}_2 = \int_{M\setminus E_1} |\nabla\chi|^2 \psi^2 h^2.$$

then

$$\mathcal{B} \leq \mathcal{B}_1 + \mathcal{B}_2,$$

Now, we only consider $\mathcal{B}_1$. Suppose that

$$Ric_M(x) \geq -(n-1)l(x),$$

where l is a non-negative function on M. Set

$$\bar{\rho}(x) = \frac{1}{2}\,(\rho(x) + (n-2)l(x)),$$

$x \in M$. Then

$$\sqrt{l} \leq \sqrt{\frac{2}{n-2}\bar{\rho}}.$$

Under the assumption of

$$Ric_M(x) \geq -(n-1)l(x),$$

Cheng and Yau (1975) gives a local gradient estimate for positive harmonic functions:

$$|\nabla f|\,(x) \leq C\left(\sup_{y\in B(x,R)} \sqrt{\bar{\rho}} + R^{-1}\right)|f(x)|, \tag{3.13}$$

for any $R > 0$. Fix $x \in M$ and consider the function

$$\eta(r) = \sqrt{2}r - (\sup_{B(x,r)} \sqrt{\bar{\rho}})^{-1}.$$

Obviously, $\eta(r)$ tends to a negative number as $r \to 0$ and tends to $+\infty$ as $r \to +\infty$. Thus, there exists a r_0 depending on x such that $\sqrt{2}r_0 = (\sup_{B(x,r_0)} \sqrt{\bar{\rho}})^{-1}$. Combining with (3.13), we obtain that

$$|\nabla f|(x) \leq C\left(\sup_{B(x,r_0)} \sqrt{\bar{\rho}}\right)|f(x)|. \tag{3.14}$$

For any $y \in B(x, r_0)$, let $\gamma(s)$ $s \in [0, b]$ be a minimizing geodesic with respect to the background metric ds^2 jointing x and y. Then

$$d_\rho\,(x, y) \leq \int_0^b \sqrt{\rho\,(\gamma(t))}dt \leq \sqrt{2}\int_0^b \sqrt{\bar{\rho}\,(\gamma(t))}dt \leq \sup_{B(x,r_0)} \sqrt{\bar{\rho}}\sqrt{2}r_0 = 1,$$

which implies that $B(x, r_0) \subseteq B_\rho(x, 1)$. Hence, for any $x \in M$, there is

$$|\nabla f|(x) \leq C \left(\sup_{B_\rho(x,1)} \sqrt{\rho} \right) |f(x)|. \tag{3.15}$$

Since the Ricci cuvature has low bound only on $M \setminus \Omega$, we can choose R_0 such that, for every $x \in B_\rho(R) \setminus B(R_0 - 1)$, $B_\rho(x, 1) \cap \Omega = \emptyset$. Therefore,

$$|\nabla f|(x) \leq C\,(S(R+1))\,|f(x)|,$$

for $x \in B_\rho(R) \setminus B(R_0 - 1)$. Replacing f by $1 - f$, we obtain that

$$|\nabla f|(x) \leq C\,(S(R+1))\,|\,1 - f(x)|, \tag{3.16}$$

for $x \in B_\rho(R) \setminus B(R_0 - 1)$. Thus, from the definition of χ, we have

$$\begin{aligned}
\mathcal{B}_1 &\leq \int_{E_1 \cap (B_\rho(R) \setminus B(R_0-1))} |\nabla\chi|^2 |\nabla f|^{2(n-2)/(n-1)} \\
&\leq (\epsilon - \sigma\epsilon)^{-2} \int_{E_1 \cap (B_\rho(R) \setminus B(R_0-1)) \cap \mathcal{L}(1-\epsilon, 1-\epsilon\sigma)} |\nabla f|^{2(n-2)/(n-1)+2} \\
&\leq C(\epsilon - \sigma\epsilon)^{-2} S(R+1)^{2(n-2)/(n-1)} \\
&\quad \cdot \int_{E_1 \cap (B_\rho(R) \setminus B(R_0-1)) \cap \mathcal{L}(1-\epsilon, 1-\epsilon\sigma)} |\nabla f|^2 (1-f)^{2(n-2)/(n-1)},
\end{aligned}$$

where the second inequality holds because, on $E_1 \cap \mathcal{L}(1 - \epsilon, 1 - \epsilon\sigma)$, $|\nabla\chi| \leq (\epsilon - \sigma\epsilon)^{-1} |\nabla f|$ and the last one holds because of (3.16). Lemma 5.1 in Li and Wang (2006) implies the integral of ∇f on the level set $\mathcal{L}(t) = \{x \in M | f(x) = t\}$, $0 \leq t \leq 1$ is independent of t and bounded. By the co-area formula, we obtain

$$\begin{aligned}
&\int_{E_1 \cap (B_\rho(R) \setminus B(R_0-1)) \cap \mathcal{L}(1-\epsilon, 1-\epsilon\sigma)} |\nabla f|^2 (1-f)^{2(n-2)/(n-1)} \\
&\leq \int_{1-\epsilon}^{1-\sigma\epsilon} (1-t)^{2(n-2)/(n-1)} \int_{\mathcal{L}(t) \cap E_1 \cap (B_\rho(R) \setminus B(R_0-1))} |\nabla f| \, dA\, dt \\
&\leq C \int_{\mathcal{L}(b)} |\nabla f| \, dA \cdot (1 - \sigma^{2(n-2)/(n-1)+1})\, \epsilon^{2(n-2)/(n-1)+1},
\end{aligned}$$

for any level b. Thus,

$$\mathcal{B}_1 \leq C\,(1-\sigma)^{-2} \epsilon^{(n-3)/(n-1)} (1 - \sigma^{2(n-2)/(n-1)+1}) S(R+1)^{2(n-2)/(n-1)}.$$

Replacing f by $1 - f$, similar to the above argument, we have that

$$\mathcal{A}_{12} \leq C\,(\sigma\epsilon)^{-2/(n-1)}(S(R))^{2(n-2)/(n-1)}\exp(-2R)$$
$$\mathcal{B}_2 \leq C\,(1-\sigma)^{-2}\epsilon^{(n-3)/(n-1)}(1-\sigma^{2(n-2)/(n-1)+1})S(R+1)^{2(n-2)/(n-1)}.$$

Set $\sigma = \frac{1}{2}$ and $\epsilon = \exp(-2R)$. We obtain that $\mathcal{A}_1 \to 0$ and $\mathcal{B} \to 0$, as $R \to +\infty$. Combining with (3.5)-(3.7), we have

$$\varepsilon \int_{M\setminus B(R_0)} h^2 \leq C \int_{B(R_0)\setminus B(R_0-1)} h^2.$$

For $n = 3$, we may choose ψ as above and χ to be

$$\chi(x) = \begin{cases} 0 & \text{on } \mathcal{L}(0, \sigma\epsilon) \cup \mathcal{L}(1-\sigma\epsilon, 1), \\ (-\log\sigma)^{-1}(\log f - \log(\sigma\epsilon)) & \text{on } \mathcal{L}(\sigma\epsilon, \epsilon) \cap M \setminus E_1, \\ (-\log\sigma)^{-1}(\log(1-f) - \log(\sigma(1-\epsilon))) & \text{on } \mathcal{L}(1-\epsilon, 1-\sigma\epsilon) \cap E_1, \\ 1 & \text{otherwise.} \end{cases}$$

By an argument similar to the above one for $n \geq 4$ and the corresponding estimate in reference (Li and Wang 2006), we have

$$\mathcal{A}_{11}, \mathcal{A}_{12} \leq CS\,(R)^{\frac{2(n-2)}{n-1}}\,(\sigma\epsilon)^{\frac{-2}{n-1}}\exp(-2R)$$

and

$$\mathcal{B}_1, \mathcal{B}_2 \leq CS\,(R+1)(-\log\sigma),$$

where C is independent of R. Choose

$$\sigma = \epsilon = \exp(-R\sqrt{q(R)})$$

with

$$q(R) = \sqrt{S(R+1)/R}$$

As $R \to +\infty$, we also have $\mathcal{A}_1, \mathcal{B} \to 0$. Thus, for $n \geq 3$, the desired result is obtained.

Lemma 3.5. $\sup_{B(R_0)} |\nabla f|^2 \leq C\,(n, \beta, \varepsilon, v, R_0) \int_{B(R_0)} |\nabla f|^2$.

Proof. Since the function h satisfies the differential inequality

$$\Delta h \geq \beta h,$$

where

$$\beta = \frac{n-2}{n-1} \inf_{B(p,2R_0)} Ric_M.$$

By the mean value inequality of Li and Tam (1991), for any $x \in B(p,R_0)$, there is

$$h^2(x) \leq C(n, \beta, v) \int_{B(x,\,R_0)} h^2 \leq C(n, \beta, v) \int_{B(p,\,2R_0)} h^2,$$

where $v = \inf_{x\in B(p,\,R_0)} V_x(R_0)$. From Lemma 3.4, we obtain

$$\int_{B(2R_0)} h^2 \leq C(n, \varepsilon) \int_{B(R_0)} h^2. \tag{3.17}$$

Thus,

$$\sup_{B(R_0)} h^2 \leq C\,(n, \beta, \varepsilon, v, R_0) \int_{B(R_0)} h^2. \tag{3.18}$$

By the Schwarz's inequality, we have

$$\int_{B(R_0)} h^2 \leq \left(\int_{B(R_0)} |\nabla f|^2 \right)^{\frac{n-2}{n-1}} V_p(R_0)^{\frac{1}{n-1}}.$$

Together with (3.18), we obtain the desired result.

Now, we give finite ends theorem when a weighted Poincaré inequality and the Sobolev inequality hold outside a compact subset.

Theorem 3.6. Let M^n $(n \geq 3)$ *be a complete Riemannian manifold. Suppose that*

$$Ric(x) \geq -\tau(x)$$

and τ is a non-negative function satisfying

$$\int_{M\setminus\Omega} \tau\phi^2 \leq \int_{M\setminus\Omega} |\nabla\phi|^2, \tag{3.19}$$

for all $\phi \in C_0^\infty(M \setminus \Omega)$, where Ω is a compact subset of M. If M satisfies the Sobolev inequality

$$||\phi||_{L^{\frac{2n}{n-2}}(M)} \leq C||\nabla\phi||_{L^2(M)}, \tag{3.20}$$

for all $\phi \in C_0^\infty(M \setminus \Omega)$, then M has finitely many non-parabolic ends.

Proof. Suppose that $\Omega \subset B(R_0)$, where $B(R_0)$ is a geodesic ball center at $p \in M$ of radius R_0. Let $\omega \in H^1(L^2(M))$. Then $h = |\omega|$ satisfies Li and Wang (2006):

$$\Delta h \geq -\tau h + \frac{|\nabla h|^2}{(n-1)h}. \tag{3.21}$$

Thus, for $\phi \in C_0^\infty\,(M\backslash B(R_0))$,

$$\int_M \tau\phi^2\, h^2 \leq \int_M |\nabla(\phi h)|^2 = \int_M h^2|\nabla\phi|^2 - \int_M \phi^2\, h\Delta h$$

$$\leq \int_M h^2|\nabla\phi|^2 + \int_M (\tau\phi^2\, h^2 - \frac{1}{n-1}\,\phi^2|\nabla h|^2). \tag{3.22}$$

From (3.20) and (3.22), we have

$$||\phi h||^2_{L^{\frac{2n}{n-2}}(M)} \leq C||\nabla(\phi h)||^2_{L^2(M)} \leq 2nC\int_M h^2|\nabla\phi|^2. \tag{3.23}$$

Choose ϕ as follows ($R > R_0 + 1$):

$$\phi = \begin{cases} 0, & \text{on } B(R_0), \\ 1, & \text{on } B(R) \setminus B(R_0+1), \\ 0, & \text{on } M \setminus B(2R), \end{cases}$$

such that

$$|\nabla\phi| \leq C$$

on $B(R_0 + 1) \setminus B(R_0)$ and

$$|\nabla\phi| \leq CR^{-1}$$

on $B(2R) \setminus B(R)$, where $C > 0$. Thus, by (3.23), we have

$$\begin{aligned} ||h||^2_{L^{\frac{2n}{n-2}}(B(R)\setminus B(R_0+1))} \leq ||\phi h||^2_{L^{\frac{2n}{n-2}}(M)} &\leq 2nC\int_M h^2|\nabla\phi|^2 \\ &\leq C\int_{B(R_0-1)\setminus B(R_0)} h^2 + CR^{-2}\int_{B(2R)\setminus B(R)} h^2. \end{aligned}$$

Let $R \to +\infty$. Then

$$||h||^2_{L^{\frac{2n}{n-2}}(M\setminus B(R_0+1))} \leq C\int_{B(R_0+1)\setminus B(R_0)} h^2. \tag{3.24}$$

The Schwarz inequality implies that

$$\int_{B(R_0+2)\setminus B(R_0+1)} h^2 \leq V^{\frac{2}{n}}(B(R_0+2))\left(\int_{B(R_0+2)\setminus B(R_0+1)} h^{\frac{2n}{n-2}}\right)^{\frac{n-2}{2n}}.$$

Combining with (3.24), we obtain that

$$\int_{B(R_0+2)} h^2 \leq C\,(R_0)\int_{B(R_0+1)} h^2. \tag{3.25}$$

By (3.21) and Li (1993) -Lemma 11.1-, we obtain that

$$h^2(x) \leq C\,(n, \sup_{B_x(1)} \tau)\int_{B_x(1)} h^2,$$

For each $x \in B(R_0 + 1)$. Thus, Combining with (3.25), we obtain that

$$\sup_{B(R_0+1)} h^2 \leq C\,(n, \sup_{B(R_0+2)} \tau) \int_{B(R_0+2)} h^2 \leq C(n, R_0, \sup_{B(R_0+2)} \tau) \int_{B(R_0+1)} h^2.$$

Choose $D = B(R_0 + 1)$, by Lemma 3.3, we obtain the desired result.

Let N^{n+1} be a Riemannian manifold and X, Y two orthonormal tangent vectors. Then the bi-Ricci curvature in the directions X, Y is defined as Cheng (2000)

$$b - \overline{Ric}(X, Y) = \overline{Ric}(X) + \overline{Ric}(Y) - \overline{K}(X, Y).$$

As an application of Theorem 3.6, we obtain that

Corollary 3.7. *Let N^{n+1} be a complete simply connected manifold with non-positive sectional curvature. Let M^n be a complete minimal finite index hypersurface in N^{n+1} ($n \geq 3$). If*

$$b - \overline{Ric}(X, Y) + \frac{1}{n}|A|^2 \geq 0,$$

for all orthonormal tangent vectors X, Y in T_pN, $p \in M$, then M must has finitely many ends.

Proof. Fix a point $p \in M$ and a local orthonormal frame field $\{e_1, e_2, \cdots, e_n, v\}$ such that $\{e_1, e_2, \cdots, e_n\}$ are tangent fields and v is unit normal vector at p on M. The Gauss equation implies that

$$Ric(e_1, e_1) = \overline{Ric}(e_1, e_1) - \overline{K}(e_1, v) + \sum_{i=1}^{n} \frac{1}{2}\, h_{11}\, h_{ii} - h_{1i}^2)$$

$$\geq \overline{Ric}(e_1, e_1) - \overline{K}(e_1, v) + \frac{n-1}{n}|A|^2.$$

Since M has finite index, it implies that there exists a compact set Ω such that

$$\int_M (\overline{Ric}(v,v) + |A|^2)\phi^2 \leq \int_M |\nabla\phi|^2,$$

for all $\phi\ C_0^\infty\ (M \setminus \Omega)$. Since N is a complete simply connected manifold with non-positive sectional curvature, there is the following Sobolev inequality (Hoftman and Spruck 1974):

$$||\phi||_{L^{\frac{2n}{n-2}}(M)} \leq C||\nabla\phi||_{L^2(M)},$$

for all $\phi\ C_0^\infty\ (M)$. Set

$$\tau = -\,\overline{Ric}\,(e_1, e_1)\,\overline{K}\,(e_1, v) + \frac{n-1}{n}|A|^2.$$

Obviously, $\tau \geq 0$. Thus,

$$\int_{M\setminus\Omega} \tau\phi^2 \leq \int_{M\setminus\Omega} (\overline{Ric}\,(v,v) + |A|^2)\phi^2 \leq \int_{M\setminus\Omega} |\nabla\phi|^2,$$

Combining Theorem 3.6 with Proposition 2.3 in Cheng et al. (2008), we obtain the result.

From Corollary 3.7, we obtain the following results directly:

Corollary 3.8. *Li and Wang (2002) Let M^n be a complete minimal finite index hypersurface in* $\mathbb{R}^{n+1}$ $(n \geq 3)$. *Then M must has finitely many ends.*

Corollary 3.9. *Let M^n be a complete minimal finite index hypersurface in* $\mathbb{H}^{n+1}(-1)$ $(n \geq 3)$. *If* $|A|^2 \geq 2n^2 - n$, *then M must has finitely many ends.*

ACKNOWLEDGMENTS

The author would like to express his sincere gratitude toward the referee for having pointed out several improvements in the writing the paper. This work was partially supported by NSF Grant (China) 11101352 and New Century Talent Project of Yangzhou University.

RESUMO

Neste artigo, obtemos teoremas de anulamento e de número finito de extremidades sobre variedades Riemannianas completas com desigualdade de Poincaré ponderada, aplicando-os a superfícies mínimas.

Palavras-chave: extremidade, superfície mínima, desigualdade de Poincaré ponderada, formas L^2 harmônicas, variedades completas.

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