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MAGNETOHYDRODYNAMIC AXISYMMETRIC FLOW OF A THIRD-GRADE FLUID BETWEEN TWO POROUS DISKS

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Abstract - This paper investigates magnetohydrodynamic axisymmetric flow of a third-grade fluid between two porous disks. The governing partial differential equations are transformed into ordinary differential equations by similarity transformations. The resulting non-linear problem is solved by a homotopy analysis method (HAM). The effects of dimensionless parameters on the radial and axial components of the velocity are illustrated through plots. The skin-friction coefficients at the upper and lower disks are tabulated for various values of the dimensionless physical parameters.
Keywords: Magnetohydrodynamic (MHD) fluid; Axisymmetric flow; Third-grade fluid; HAM solution; Porous disk.

INTRODUCTION

Non-Newtonian fluid flows are important because of their applications in industry and engineering. Several models of non-Newtonian fluids have been proposed in view of the diversity in the nature of fluids (Fetecau, 2011; Jamil *et al*., 2011a, 2011b; Hayat and Nawaz, 2010, 2011). The simplest subclass of differential type fluids is called second grade. Second-grade fluid models describe normal stress differences, but cannot predict shear thinning/ thickening effects. However the third-grade fluid model is capable of predicting shear thinning/ thickening effects. Despite various complexities in the constitutive equations, several researchers have investigated the flows of third-grade fluids taking into account various aspects. For instance, Sajid and Hayat (2007) discussed the two-dimensional boundary layer flow of a third-grade fluid over a stretching sheet. Sajid *et al*. (2007) considered heat transfer characteristics in an electrically conducting third-grade fluid. Hayat *et al*. (2010) examined the simultaneous effects of heat and mass transfer on an unsteady flow of third-grade fluid. Sahoo (2010) computed the numerical solutions for heat transfer in Heimenz flow of third-grade fluid. Sahoo and De (2009) analyzed slip effects in third-grade fluid flow caused by a stretching surface.

Flows over a disk or between disks are popular among researchers in view of their applications in engineering. Disk-shaped bodies are often encountered in engineering. Examples include, rotating heat

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exchangers, rotating disk reactors for bio-fluids production, gas or marine turbine and chemical and automobile industries. In view of the above mentioned application of flow over a disk/between disks, several investigators have analyzed such flows. In his pioneering work, von Karman (1921) considered hydrodynamic flow over an infinite disk. Cochran (1934) derived asymptotic solutions for steady hydrodynamic flow over a rotating infinite disk. Benton (1966) extended Cochran's work to the initial value problem governing flow induced by an impulsively started disk. Takhar *et al*. (2003) investigated the effect of a magnetic field on unsteady mixed convection flow from a rotating vertical cone. Maleque *et al*. (2005) studied fully developed laminar flow of a viscous fluid with variable properties. Stuart (1954) examined the effect of uniform suction on the steady flow due to a rotating disk. Sparrow *et al*. (1971) considered flow over a porous disk. Miklavcic and Wang (2004) studied the flow due to a rough rotating disk. Recently Hayat and Nawaz (2011) discussed unsteady stagnation point flow over a rotating disk. Rashidi and Pour (2010) examined the three dimensional problem of a condensation film inclined on a rotating disk by a differential transform method.

Here, we have considered the axisymmetric flow of an electrically conducting third-grade fluid between two permeable disks. The similarity solution is derived by a homotopy analysis method (HAM). This method is very efficient for solving nonlinear problems and has been used by many researchers (Liao, 2003, 2004; Abbasbandy and Shivanian, 2010, 2009; Rashidi *et al*., 2009; Bataineh *et al*., 2009; Hayat *et al*., 2010, 2010, 2009; Rashidi and Pour, 2010).

MATHEMATICAL FORMULATION

We consider the axisymmetric flow of an electrically conducting fluid between two porous disks at $z = \pm H$. A flow is induced by suction/ injection. A constant magnetic field $\mathbf{B}_0$ is applied perpendicular to the plane of the disks, i.e., along the z-axis. There is no external electric field and the induced magnetic field is neglected under the assumption of a small magnetic Reynolds number. The equations which govern the magnetohydrodynamic flow are:

$$\nabla.\mathbf{V} = 0, \tag{1}$$

$$\rho \frac{d\mathbf{V}}{dt} = \operatorname{div}\boldsymbol{\tau} + \mathbf{J} \times \mathbf{B}_0, \tag{2}$$

$$\mathbf{J} = \sigma\left(\mathbf{V} \times \mathbf{B}_0\right), \tag{3}$$

where $\mathbf{V}$ is the velocity field, ρ is the fluid density, p is pressure, $\mathbf{J}$ is the current density, σ is the electrical conductivity of the fluid, d/dt is the material derivative and the Cauchy stress tensor $\boldsymbol{\tau}$ in the third-grade fluid is given by:

$$\boldsymbol{\tau} = -p\mathbf{I} + \mu\mathbf{A}_1 + \alpha_1\mathbf{A}_2 + \alpha_2\mathbf{A}_1^2 + \beta_3\left(tr\mathbf{A}_1^2\right)\mathbf{A}_1, \tag{4}$$

in which $\mathbf{I}$ is the identity tensor, μ is the fluid viscosity and α_1, α_2, β_1, β_2, β_3 are material constants. These material constants satisfy the following constraints:

$$\begin{gathered} \mu \ge 0,\ \alpha_1 \ge 0,\ \beta_1 = \beta_2 = 0,\ \beta_3 \ge 0, \\ \alpha_1 + \alpha_2 \le \sqrt{24\mu\beta_3}, \end{gathered} \tag{5}$$

and Rivlin-Ericksen tensors $\mathbf{A}_1$, and $\mathbf{A}_2$ are defined by:

$$\mathbf{A}_1 = \nabla\mathbf{V} + \left(\nabla\mathbf{V}\right)^T, \tag{6}$$

$$\mathbf{A}_2 = \frac{d\mathbf{A}_1}{dt} + \mathbf{A}_1\left(\nabla\mathbf{V}\right) + \left(\nabla\mathbf{V}\right)^T\mathbf{A}_1. \tag{7}$$

The velocity field for the flow under consideration is:

$$\mathbf{V} = \left[u\left(r,z\right), 0, w\left(r,z\right)\right], \tag{8}$$

with u and w as the velocity components. By virtue of above expression, Eqs. (1) and (2) yield:

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0, \tag{9}$$

$$\rho\left(u\frac{\partial u}{\partial r}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 u}{\partial r^2}+\frac{\partial^2 u}{\partial z^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right)$$

$$+\alpha_1\left[-\frac{2u^2}{r^3}-\frac{2w}{r^2}\frac{\partial u}{\partial z}+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2+\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}+w\frac{\partial^3 u}{\partial z^3}-\frac{2u}{r^2}\frac{\partial u}{\partial r}+3\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+3\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}\right.$$

$$+\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{2w}{r}\frac{\partial^2 u}{\partial r\partial z}+5\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+4\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+u\frac{\partial^3 u}{\partial r\partial z^2}+w\frac{\partial^3 w}{\partial r\partial z^2}$$

$$+\frac{2u}{r}\frac{\partial^2 u}{\partial r^2}+10\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}+3\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+4\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+2w\frac{\partial^3 u}{\partial r^2\partial z}+u\frac{\partial^3 w}{\partial r^2\partial z}+2u\frac{\partial^3 u}{\partial r^3}$$

$$\left.+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\right]+\alpha_2\left[-\frac{4u^2}{r^3}+\frac{4}{r}\left(\frac{\partial u}{\partial r}\right)^2+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial z^2}\right.$$

$$+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}+4\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+\frac{1}{r}\left(\frac{\partial w}{\partial r}\right)^2+\frac{1}{r}\left(\frac{\partial u}{\partial z}\right)^2+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial w}{\partial z}$$

$$\left.+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+8\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r^2}+2\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\right] \qquad (10)$$

$$+\beta_3\left[-\frac{8u^3}{r^4}+\frac{8}{r}\left(\frac{\partial u}{\partial r}\right)^3-\frac{8u^2}{r^3}\frac{\partial u}{\partial r}+\frac{8}{\mathrm{r}}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2+\frac{4}{\mathrm{r}}\frac{\partial u}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2+\frac{4}{\mathrm{r}}\frac{\partial u}{\partial r}\left(\frac{\partial w}{\partial r}\right)^2\right.$$

$$+\frac{8}{\mathrm{r}}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}+24\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial r^2}+\frac{8u^2}{r^2}\frac{\partial^2 u}{\partial r^2}+\frac{16\mathrm{u}}{r^2}\frac{\partial u}{\partial r}+8\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial r^2}$$

$$+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}+4\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2}+16\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+8\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r^2}$$

$$+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+8\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2}+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z}+\frac{4u^2}{r^2}\frac{\partial^2 u}{\partial z^2}$$

$$+\frac{4u^2}{r^2}\frac{\partial^2 w}{\partial r\partial z}+\frac{4u}{r^2}\left(\frac{\partial u}{\partial z}\right)^2+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}$$

$$+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial z^2}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial z^2}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}$$

$$\left.-\frac{8u}{r^2}\left(\frac{\partial u}{\partial r}\right)^2-\frac{8u}{r^2}\left(\frac{\partial w}{\partial z}\right)^2-\frac{4u}{r^2}\left(\frac{\partial w}{\partial r}\right)^2+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial r\partial z}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial r\partial z}\right]-\sigma B_0^2 u,$$

$$
\begin{aligned}
&\rho\left(u\frac{\partial w}{\partial r}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial r^2}+\frac{\partial^2 w}{\partial z^2}+\frac{1}{r}\frac{\partial w}{\partial r}\right)\\
&+\alpha_1\left[\frac{u}{r}\frac{\partial^2 w}{\partial r^2}+\frac{w}{r}\frac{\partial^2 u}{\partial z^2}+\frac{u}{r}\frac{\partial^2 u}{\partial r\partial z}+\frac{w}{r}\frac{\partial^2 w}{\partial r\partial z}+\frac{3}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}+\frac{3}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}+\frac{1}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\right.\\
&+\frac{1}{r}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r^2}+u\frac{\partial^3 w}{\partial r^3}+3\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+w\frac{\partial^3 u}{\partial r\partial z^2}+4\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+u\frac{\partial^3 u}{\partial z\partial r^2}\\
&+4\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+w\frac{\partial^3 w}{\partial z\partial r^2}+3\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}+3\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2}+5\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}\\
&\left.+\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2}+2u\frac{\partial^3 w}{\partial r\partial z^2}+10\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}+2w\frac{\partial^3 w}{\partial z^3}+4\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}\right]+\alpha_2\left[\frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\right.\\
&+\frac{2}{r}\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}+\frac{2}{r}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+\frac{2}{r}\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}+2\frac{\partial u}{\partial r}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2}\\
&+2\frac{\partial u}{\partial r}\frac{\partial^2 w}{\partial r^2}+4\frac{\partial u}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}+2\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r^2}+8\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial z^2}\\
&\left.+2\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial z^2}+2\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r\partial z}\right]+\beta_3\left[\frac{4}{r}\frac{\partial u}{\partial z}\left(\frac{\partial u}{\partial r}\right)^2+\frac{4}{r}\frac{\partial w}{\partial r}\left(\frac{\partial u}{\partial r}\right)^2\right.\\
&+\frac{4u^3}{r^3}\frac{\partial u}{\partial z}+\frac{4u^3}{r^3}\frac{\partial w}{\partial r}+\frac{4}{r}\frac{\partial u}{\partial z}\left(\frac{\partial w}{\partial z}\right)^2+\frac{4}{r}\frac{\partial w}{\partial r}\left(\frac{\partial w}{\partial z}\right)^2+\frac{2}{r}\left(\frac{\partial u}{\partial z}\right)^3+\frac{2}{r}\left(\frac{\partial w}{\partial r}\right)^3\\
&+\frac{6}{r}\frac{\partial w}{\partial r}\left(\frac{\partial u}{\partial z}\right)^2+\frac{6}{r}\frac{\partial u}{\partial z}\left(\frac{\partial w}{\partial r}\right)^2+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 u}{\partial r\partial z}+4\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial r^2}+8\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial r^2}\\
&+8\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r^2}+16\frac{\partial u}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial r\partial z}+\frac{8u}{r^2}\frac{\partial u}{\partial r}\frac{\partial u}{\partial z}+\frac{8u}{r^2}\frac{\partial u}{\partial r}\frac{\partial w}{\partial r}+4\frac{u^2}{r^2}\frac{\partial^2 u}{\partial r\partial z}+4\frac{u^2}{r^2}\frac{\partial^2 w}{\partial r^2}\\
&+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 u}{\partial r\partial z}+4\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial r^2}+8\frac{u^2}{r^2}\frac{\partial^2 w}{\partial z^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}\\
&+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 u}{\partial r\partial z}+6\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial r^2}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 u}{\partial r\partial z}+6\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial r^2}+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 u}{\partial r\partial z}\\
&+12\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2}+8\left(\frac{\partial u}{\partial r}\right)^2\frac{\partial^2 w}{\partial z^2}+\frac{16u}{r^2}\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}+24\left(\frac{\partial w}{\partial z}\right)^2\frac{\partial^2 w}{\partial z^2}+4\left(\frac{\partial u}{\partial z}\right)^2\frac{\partial^2 w}{\partial z^2}\\
&\left.+4\left(\frac{\partial w}{\partial r}\right)^2\frac{\partial^2 w}{\partial z^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial z^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 u}{\partial z^2}+8\frac{\partial u}{\partial z}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}+8\frac{\partial w}{\partial r}+8\frac{\partial w}{\partial r}\frac{\partial w}{\partial z}\frac{\partial^2 w}{\partial r\partial z}\right]
\end{aligned}
\tag{11}
$$

The relevant boundary conditions are:

$$u(r,H)=0,\ u(r,-H)=0,\ w(r,H)= -V_0,\ w(r,-H)=V_0, \tag{12}$$

where V_0 is the constant velocity. Here $V_0>0$ corresponds to the case when fluid is being sucked but $V_0<0$ leads to the situation when there is injection.

Utilizing the following transformations:

$$u(r,z)=\frac{V_0 r}{2H}f'(\eta),\ w(r,z)=-V_0 f(\eta), \quad \eta=\frac{z}{H} \tag{13}$$

the continuity equation is identically satisfied and Eqs (10)-(12) become:

$$f^{(iv)}+\mathrm{Re}\, ff'''-\alpha\left[2f'f'''+f'f^{(iv)}+ff^{(v)}\right] -\gamma\left[2f'f'''+f'f^{(iv)}\right] +\beta\left(\left[\begin{matrix}7f''^3+24f'f''f'''+3f'^2f^{(iv)}\\ +\frac{3}{2}\delta^2 f''f'''^2+\frac{3}{4}\delta^2 f''^2 f^{(iv)}\end{matrix}\right]-\mathrm{Re}\,M^2 f''=0\right), \tag{14}$$

$$f'(1)=0,\ f'(-1)=0\ f(1)=1,\ f(-1)=-1, \tag{15}$$

where:

$$\mathrm{Re}=\frac{V_0H}{\upsilon},\ M^2=\frac{\sigma B_0^2 H}{\rho V_0},\ \upsilon=\frac{\mu}{\rho},\ \alpha=\frac{\alpha_1 V_0}{\mu H} \quad \gamma=\frac{\alpha_2 V_0}{\mu H},\ \beta=\frac{2\beta_3 V_0^2}{\mu H^2},\ \delta=\frac{r}{H} \tag{16}$$

respectively indicate the Reynolds number (Re), the Hartman number (M), the third-grade parameters (α,β,γ) and the dimensionless radial distance (δ). It is important to note that $\mathrm{Re}<0$ corresponds to the case when there is suction at the upper disk and injection at the lower disk. However, $\mathrm{Re}<0$ leads to the situation when there is injection at the upper disk and suction at the lower disk.

The skin friction coefficients C_{1f} and C_{2f} at the upper and lower disks are:

$$C_{1f}=\frac{\tau_w}{\frac{1}{2}\rho(V_0)^2}=\frac{\tau_{rz}|_{z=H}}{\frac{1}{2}\rho(V_0)^2}= \mathrm{Re}_r^{-1/2}\left[f''(1)-\alpha f'''(1)+\frac{\beta\delta^2}{4}f''^3(1)\right], \tag{17}$$

$$C_{2f}=\frac{\tau_w}{\frac{1}{2}\rho(V_0)^2}=\frac{\tau_{rz}|_{z=-H}}{\frac{1}{2}\rho(V_0)^2}= \mathrm{Re}_r^{-1/2}\left[f''(-1)+\alpha f'''(-1)+\frac{\beta\delta^2}{4}f''^3(-1)\right], \tag{18}$$

in which $\mathrm{Re}_r=V_0H^2/\nu r$ is the local Reynolds number.

SOLUTION PROCEDURE

In order to find the homotopy analysis solution, we choose the base functions:

$$\left\{\eta^{2n+1};n\geq 0\right\} \tag{19}$$

and write:

$$f(\eta)=\sum_{n=0}^{\infty}a_n\eta^{2n+1}, \tag{20}$$

where a_n are the constant coefficients. We take the initial guess and auxiliary linear operator of the following forms:

$$f_0(\eta)=\frac{3}{2}\eta-\frac{1}{2}\eta^3, \tag{21}$$

$$L_1\left[f(\eta)\right]=\frac{d^4f}{d\eta^4}. \tag{22}$$

The above linear operator preserves the following property:

$$L_1\left[\frac{C_1}{6}\eta^3+\frac{C_2}{2}\eta^2+C_3\eta+C_4\right]=0, \tag{23}$$

where $C_i\ (i=1-4)$ are the constants.

Zeroth-Order Deformation Problem

The problem at the zeroth order deformation can be expressed as:

$$(1-q)L_1\left[\hat{f}(\eta,q)-f_0(\eta)\right]=qh_f\mathbf{N}_1\left[\hat{f}(\eta,q)\right], \quad (24)$$

$$\begin{aligned}&f^{\hat{'}}(1,q)=0, \quad \hat{f}'(-1,q)=0,\\&\hat{f}(1,q)=1, \quad \hat{f}(-1,q)=-1,\end{aligned} \quad (25)$$

where $h_f \neq 0$ and $q\ \varepsilon\ [0,1]$ are respectively the auxiliary and embedding parameters. When q varies from 0 to 1, $\hat{f}\ (\eta,q)$ varies from the initial guess $f_0(\eta)$ to the final solution $f(\eta)$. The non-linear operator is given by:

$$\begin{aligned}\mathbf{N}_1\left[f(\eta,q)\right]=&\frac{\partial^4\hat{f}}{\partial\eta^4}+\operatorname{Re}\hat{f}\frac{\partial^3\hat{f}}{\partial\eta^3}\\&-\alpha\left[2\frac{\partial^2\hat{f}}{\partial\eta^2}\frac{\partial^3\hat{f}}{\partial\eta^3}+\frac{\partial\hat{f}}{\partial\eta}\frac{\partial^4\hat{f}}{\partial\eta^4}+f\frac{\partial^5\hat{f}}{\partial\eta^5}\right]\\&-\gamma\left[2\frac{\partial^2\hat{f}}{\partial\eta^2}\frac{\partial^3\hat{f}}{\partial\eta^3}+\frac{\partial\hat{f}}{\partial\eta}\frac{\partial^4\hat{f}}{\partial\eta^4}\right]\\&+\beta\begin{bmatrix}7\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^3+24\frac{\partial\hat{f}}{\partial\eta}\frac{\partial^2\hat{f}}{\partial\eta^2}\frac{\partial^3\hat{f}}{\partial\eta^3}\\+3\left(\frac{\partial\hat{f}}{\partial\eta}\right)^2\frac{\partial^4\hat{f}}{\partial\eta^4}\\+\frac{3}{2}\delta^2\frac{\partial^2\hat{f}}{\partial\eta^2}\left(\frac{\partial^3\hat{f}}{\partial\eta^3}\right)^2\\+\frac{3}{4}\delta^2\left(\frac{\partial^2\hat{f}}{\partial\eta^2}\right)^2\frac{\partial^4\hat{f}}{\partial\eta^4}\end{bmatrix}\\&-\operatorname{Re}M^2\frac{\partial^2\hat{f}}{\partial\eta^2},\end{aligned} \quad (26)$$

By Taylor series expansion we have:

$$\hat{f}(\eta,q)=f_0(\eta)+\sum_{m=1}^{\infty}f_m(\eta)q^m, \quad (27)$$

in which:

$$\hat{f}_m(\eta)=\frac{1}{m!}\left.\frac{\partial^m f(\eta,q)}{\partial q^m}\right|_{p=0}. \quad (28)$$

Setting $q=1$ in Eq. (27) one obtains

$$\hat{f}(\eta)=f_0(\eta)+\sum_{m=1}^{\infty}f_m(\eta). \quad (29)$$

mth-Order Deformation Problems

Differentiating the zeroth-order deformation problem (Eqs. (24) and (25)) m-times with respect to q and then dividing them by $m!$ and setting $q=0$ we get following the higher-order deformation problems as:

$$L_1\left[\hat{f}_m(\eta)-\chi_m\hat{f}_{m-1}(\eta)\right]=h\mathbf{R}_m^f(\eta), \quad (30)$$

$$\begin{aligned}&\left.\frac{\partial\hat{f}_m(\eta,q)}{\partial\eta}\right|_{\eta=1}=0, \quad \left.\frac{\partial\hat{f}_m(\eta,q)}{\partial\eta}\right|_{\eta=-1}=0,\\&\hat{f}_m(1,q)=0. \quad \hat{f}_m(-1,q)=0,\end{aligned} \quad (31)$$

$$\begin{aligned}\mathbf{R}_m^f(\eta)=&f_{m-1}^{(iv)}(\eta)\\&+\sum_{k=0}^{m-1}\begin{bmatrix}\operatorname{Re}f_{m-1-k}f_k'''\\-\alpha\begin{pmatrix}2f_{m-1-k}f_k'''\\+f_{m-1-k}'f_k^{(iv)}\\+f_{m-1-k}f_k^{(v)}\end{pmatrix}\\-\gamma\begin{pmatrix}2f_{m-1-k}''f_k'''\\+f_{m-1-k}f_k^{(iv)}\end{pmatrix}\\+\beta\sum_{l=0}^{k}\begin{pmatrix}7f_{m-1-k}''f_{k-l}''f_l''\\+24f_{m-1-k}'f_{k-l}''f_l'''\\+3f_{m-1-k}'f_{k-l}f_l'^{(iv)}\\+\frac{3}{2}\delta^2f_{m-1-k}''f_{k-l}'''f_l'''\\+\frac{3}{4}\delta^2f_{m-1-k}''f_{k-l}''f_l^{(iv)}\end{pmatrix}\end{bmatrix}\\&-\operatorname{Re}M^2f_{m-1}'',\end{aligned} \quad (32)$$

$$\chi_m=\begin{cases}0, & m\leq 1,\\1, & m>1.\end{cases} \quad (33)$$

The general solution of the problem consisting of Eqs. (30) and (31) is:

$$f(\eta) = f^* + \frac{1}{6}C_1^m \eta^3 + \frac{1}{2}C_2^m \eta^2 + C_3^m \eta + C_4^m, \qquad (34)$$

in which $C_i^m \quad (i = 1-4)$ are the arbitrary constants, which can be determined by the boundary conditions given in Eq. (31).

CONVERGENCE OF THE HOMOTOPY SOLUTIONS

The homotopy series solution contains the auxiliary parameter h_f Convergence of the series solution (34) strongly depends upon the suitable range of this auxiliary parameter h_f. For this purpose the h-curve is plotted in Figure 1. From this figure it is noted that a suitable range for h_f is $-0.85 \le h_f < -0.35$. Furthermore, convergence of the series solution is checked and shown in Table 1. This table shows that the series solution converges up to the 10th order of approximation.

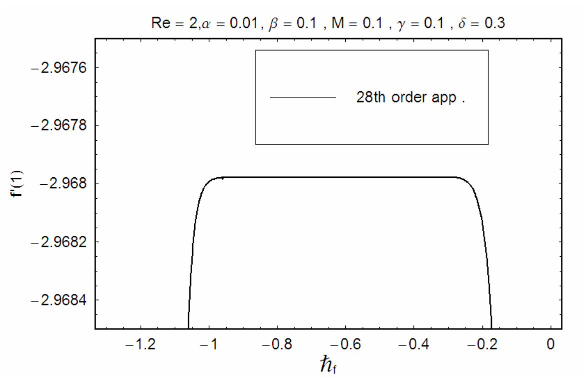

Figure 1: h_f -curve of $f''(1)$.

Table 1: Convergence of the homotopy solution when $\alpha = 0.01$, $\beta = 0.1$, $\gamma = 0.1$, $\delta = 0.3$, $M = 0.1$, Re = 2 and $h_f = -0.6$.

Order of approximation	$f''(1)$
1	-3.03822
2	-2.99133
5	-2.96884
8	-2.96795
10	-2.96798
15	-2.96798
20	-2.96798

RESULTS AND DISCUSSION

In this section, we examine the influence of physical parameters on dimensionless radial and axial velocities. Variation of the skin friction coefficients at the upper and lower disks is presented in Tables 2 and 3. Figures 2 and 3 are sketched to examine the influence of suction/injection on the radial velocity $f'(\eta)$. Here $\mathrm{Re} > 0$ corresponds to the case when fluid is being sucked from the upper disk and the lower disk is subjected to injection and vice versa for $\mathrm{Re} < 0$. From these figures it is noted that the effect of $\mathrm{Re} > 0$ on $f'(\eta)$ is opposite to that of $\mathrm{Re} < 0$ on $f'(\eta)$. Figures 4 and 5 demonstrate the effect of the third-grade parameter β on the radial velocity $f'(\eta)$ for both cases $\mathrm{Re} > 0$ and $\mathrm{Re} < 0$. Figures 6-9 reflect the influence of Re and β on the axial velocity $f(\eta)$. Figures 6 and 7 show that the magnitude of the axial velocity $f(\eta)$ is a decreasing function of Re. The magnitude of the axial velocity $f(\eta)$ is an increasing function of β for both cases $\mathrm{Re} > 0$ and $\mathrm{Re} < 0$, as shown in Figures 8 and 9. Tables 2 and 3 are constructed to see the effects of the third-grade parameter β, second-grade parameters α, γ, the Reynolds number Re and the Hartman number M on the variation of the skin friction coefficient $\mathrm{Re}_r^{1/2} C_f$. Table 2 represents the variation of skin friction coefficients $\mathrm{Re}_r^{1/2} C_{1f}$ and $\mathrm{Re}_r^{1/2} C_{2f}$ at the upper and lower disks when $\mathrm{Re} > 0$ whereas Table 3 depicts the behavior of $\mathrm{Re}_r C_{1f}$ and $\mathrm{Re}_r C_{2f}$ when $\mathrm{Re} < 0$. It can be observed from Table 2 that $\mathrm{Re}_r^{1/2} C_{1f}$ and $\mathrm{Re}_r^{1/2} C_{2f}$ are increasing functions of the third-grade parameter β and the Hartman number M, whereas $\mathrm{Re}_r^{1/2} C_{1f}$ and $\mathrm{Re}_r^{1/2} C_{2f}$ decrease when α, γ and Re are increased. Hence it can be concluded that tangential stresses at both the upper and lower disks are increasing functions of β and M. However, tangential stresses on the surface of the disks can be reduced by increasing α, γ and Re. Since Re corresponds to suction/injection phenomena. Therefore, stresses on the surface of disks can be reduced or adjusted by a suction or injection mechanism. Table 2 also demonstrates that in a second-grade fluid $(\beta = 0)$. the stresses on the surface of the disks are smaller than those in a third-grade fluid $(\beta \neq 0)$. Furthermore, an increase in strength of the external magnetic field results in an increase of the stresses at the surface of the disks.

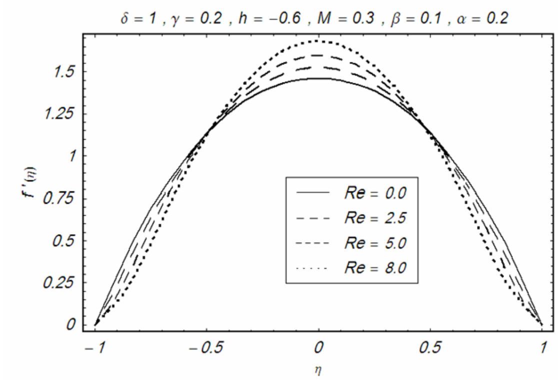

Figure 2: Influence of Re>0 on $f'(\eta)$.

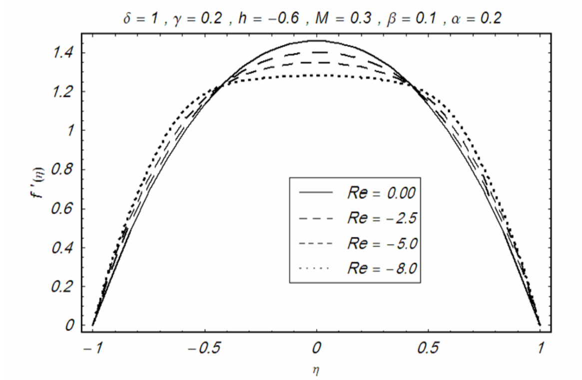

Figure 3: Influence of Re<0 on $f'(\eta)$.

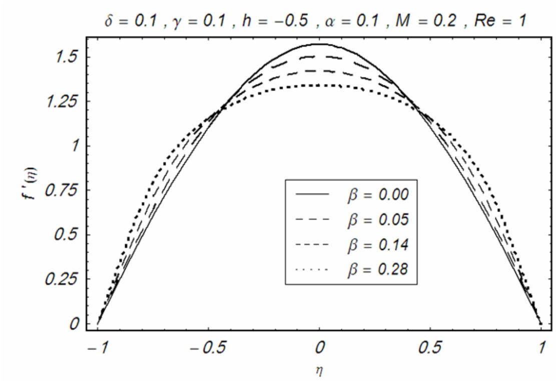

Figure 4: Influence of β on $f'(\eta)$ when Re>0.

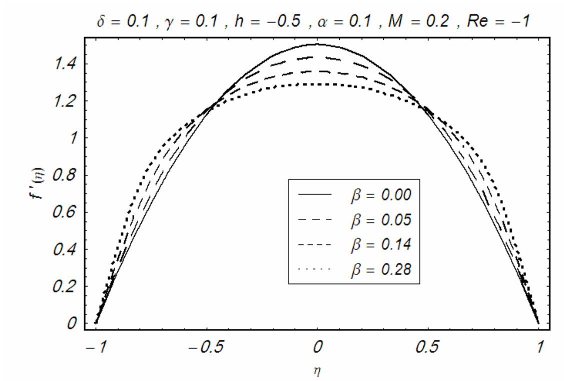

Figure 5: Influence of β on $f'(\eta)$ when Re<0.

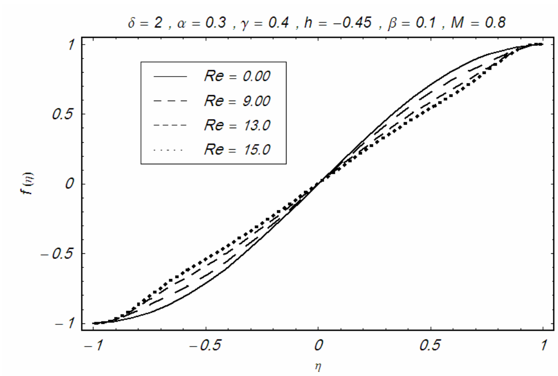

Figure 6: Influence of Re>0 on $f(\eta)$.

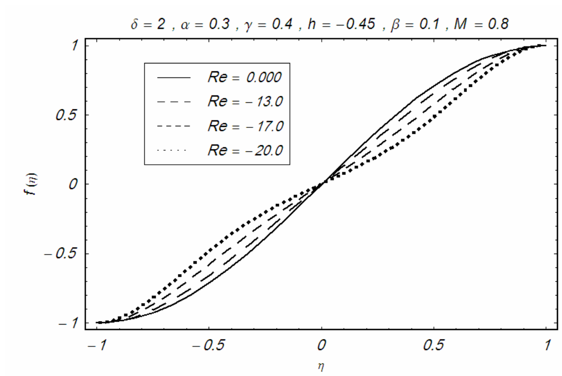

Figure 7: Influence of Re<0 on $f(\eta)$.

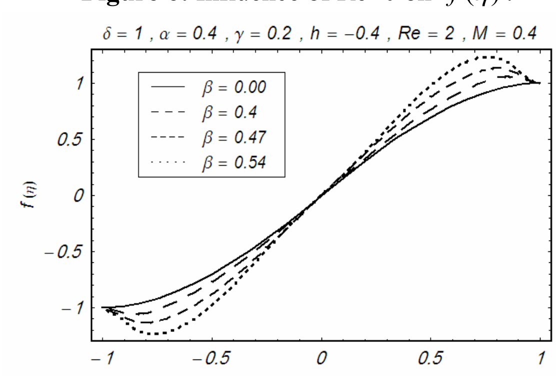

Figure 8: Influence of β on $f(\eta)$ when Re>0.

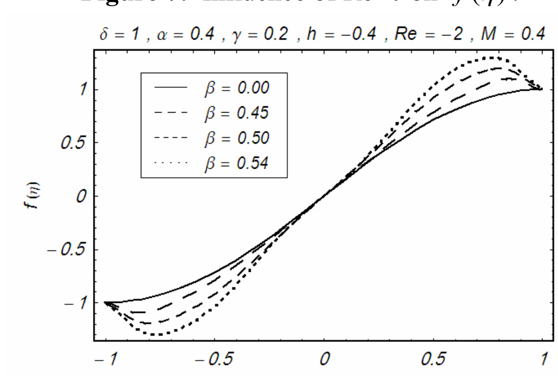

Figure 9: Influence of β on $f(\eta)$ when $\text{Re} < 0$.

Table 2: Numerical values of the skin friction coefficients $\mathrm{Re}_r^{1/2} C_{1f}$ and $\mathrm{Re}_r^{1/2} C_{2f}$ when $\mathrm{Re} > 0$

α	γ	β	R	M	$-\mathrm{Re}^{1/2} C_{1f}$	$\mathrm{Re}^{1/2} C_{2f}$
0.00	0.01	0.1	2	0.1	3.06726	3.06726
0.01					3.05566	3.05566
0.02					3.02173	3.02173
0.03					2.97896	2.97896
0.01	0.0	0.1	2	0.1	3.05325	3.05325
	0.1				2.95380	2.95380
	0.2				2.85605	2.85605
	0.3				2.75956	2.75956
0.01	0.1	0.0	2	0.1	2.50211	2.50211
		0.1			2.97327	2.97327
		0.2			3.42907	3.42907
		0.3			3.97299	3.97299
0.01	0.01	0.1	0.0	0.3	3.51253	3.51253
			0.1		3.48955	3.48955
			0.2		3.46019	3.46019
			0.3		3.43428	3.43428
0.01	0.1	0.1	2	0.0	2.96128	2.96128
				0.1	2.96470	2.96470
				0.2	2.97496	2.97496
				0.3	2.99211	2.99211

Table 3: Numerical values of the skin friction coefficients $\mathrm{Re}_r^{1/2} C_{1f}$ and $\mathrm{Re}_r^{1/2} C_{2f}$ when $\mathrm{Re} < 0$

α	γ	β	R	M	$-\mathrm{Re}^{1/2} C_{1f}$	$\mathrm{Re}^{1/2} C_{2f}$
0.00	0.01	0.1	-0.2	0.1	3.66567	3.66567
0.01					3.81843	3.81843
0.02					3.80093	3.80093
0.03					3.67106	3.67106
0.01	0.0	0.1	-0.2	0.1	3.75011	3.75011
	0.1				3.63775	3.63775
	0.2				3.52178	3.52178
	0.3				3.40317	3.40317
0.01	0.1	0.0	-0.2	0.1	2.97450	2.97450
		0.1			3.74536	3.74536
		0.2			4.32452	4.32452
		0.21			4.37740	4.37740
0.01	0.01	0.1	0.0	0.3	3.51253	3.51253
			-0.1		3.53571	3.53571
			-0.2		3.56554	3.56554
			-0.3		3.59227	3.59227
0.01	0.1	0.1	-0.2	0.0	3.70103	3.70103
				0.1	3.70057	3.70057
				0.2	3.69916	3.69916
				0.3	3.69681	3.69681

FINAL REMARKS

In this paper, we have studied the magnetohydrodynamic flow of a third-grade fluid between two permeable disks. The governing non-linear problem is solved by a homotopy analysis method. The observations noted from the present analysis are:

For $\mathrm{Re} > 0$, the behavior of M with radial velocity $f'(\eta)$ and axial velocity is opposite to that of $f'(\eta)$ when $\mathrm{Re} < 0$ This means that the effect of magnetic field on radial velocity $f'(\eta)$ when the upper disk is subjected to suction and the lower disk is subjected to injection is opposite to that on $f'(\eta)$ when there is an injection at the upper disk and suction at the lower disk. It is also noted that, in the vicinity of the disks, the radial velocity increases with an increase in Hartman number M, whereas it is a decreasing function of M in the central region between the disks. This behavior of the radial velocity $f'(\eta)$ is due to the mass conservation constraint. An increase in $f'(\eta)$ near the disks is compensated by a decrease in $f'(\eta)$ in the central region.

- The influence of the third-grade parameter β on the radial velocity $f'(\eta)$ is similar for both cases of $\mathrm{Re} > 0$ and $\mathrm{Re} < 0$.
- The effects of the second grade parameters α and γ on the radial velocity $f'(\eta)$ are similar in a qualitative sense.
- The effects of the parameters α, β and γ on the radial velocity $f'(\eta)$ are similar for both the cases (i) $\mathrm{Re} > 0$ and (ii) $\mathrm{Re} < 0$. Furthermore, it is observed that the radial velocity $f'(\eta)$ decreases in the vicinity of the disks, whereas it increases in the central region between the disks for both $\mathrm{Re} > 0$ and $\mathrm{Re} < 0$ for various values of α, β and γ. Qualitatively, α, β and γ have similar effects on the axial velocity $f(\eta)$.
- $\mathrm{Re}_r^{1/2} C_{1f}$ and $\mathrm{Re}_r^{1/2} C_{2f}$ are increasing functions of the third-grade parameter (β) and Hartman number (M), whereas $\mathrm{Re}_r^{1/2} C_{1f}$ and $\mathrm{Re}_r^{1/2} C_{2f}$ decrease when α, γ and Re increase. The increase of the third-grade fluid stresses at the surface of the disks is higher than those of Newtonian and second grade fluids.

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