Bending of Light in the Framework of $R + R^2$ Gravity

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We present a tree level approach to the issue of the deflection of photons by the gravitational field of the Sun, treated as an external field, on the basis of $R + R^2$ gravity. We show that the deflection angle of a photon grazing the surface of the Sun is exactly the same as that given by general relativity. An explanation for this strange coincidence is provided.

I. Introduction

One of the simplest and most enduring ideas for extending general relativity is to include in the Einstein's gravitational action terms involving higher powers of the curvature tensor. These extra terms neither break the invariance of the action under general coordinate transformations nor can be excluded by any known principle. If, for instance, we include only a term proportional to R^2, the resulting action is[1]

$$S = \int d^4x \sqrt{-g} \left[\frac{2R}{\kappa^2} + \frac{\alpha}{2} R^2 \right] \quad , \qquad (1)$$

where $\kappa^2 = 32\pi G$, and α is a dimensionless parameter. Since the quantity R involves two derivatives acting on the metric and in an interaction each derivative becomes a factor of the momentum transfer k, or of the inverse distance scale $k \sim \frac{1}{r}$, we may say that R is of order $\kappa^2 \left(\frac{1}{r^2} \right)$, while R^2 is of order $\kappa^4 \left(\frac{1}{r^4} \right)$, respectively. Thus, at small enough energies, or at very long distances, the quadratic term is negligible and $R + R^2$ gravity reduces to general relativity. Accordingly, for small values of the energy E of incident radiation ($E \sim 0$) we expect the measurements of the

solar gravitational deflection to be in excellent agreement with general relativity. As a matter of fact, in the measurements of the solar gravitational deflection of the radio waves ($E \sim 1 cm^{-1}$), Fomalont and Sramek [2] found a value of 1.76 ± 0.016 arcsec for the deflection at the solar limbs (the prediction of general relativity is 1.75 arcsec). Therefore, a non negligible contribution of the quadratic term to the solar gravitational deflection of starlight ($E \sim 10^5 cm^{-1}$) is to be expected. Incidentally, the mean value of the deflection of light at the solar limb obtained by averaging over the measurements made on all eclipses expeditions to date is 2.04 arcsec. These considerations lead us to conjecture that $R + R^2$ gravity should provide a value for the deflection angle of starlight which passes the Sun's limb on its way to the Earth nearer to the experimental value than that given by general relativity. We address this question here. To do that we compute the scattering of a photon by the Sun's gravitational field, treated as an external field, on the basis of $R + R^2$ gravity.

We show in Sec. II that only in the absence of tachyons in the dynamical field does $R + R^2$ gravity give an acceptable newtonian limit. In Sec. III we compute the gravitational field of the Sun, treated as a

point particle, in the weak field approximation. In Sec. IV we calculate the cross-section for the scattering of a photon by an external gravitational field and show from this result that the deflection angle of a photon grazing the surface of the Sun predicted by $R + R^2$ gravity is exactly the same as that provided by general relativity. An explanation for this strange coincidence is given in Sec. V.

Natural units are used throughout ($c = \hbar = 1$). In our notation the signature is $(+ \ - \ - \ -)$. The curvature tensor is defined by $R^{\alpha}_{\ \beta\gamma\delta} = -\partial_{\delta}, \ ^{\alpha}_{\ \beta\gamma} + \cdots$, the Ricci tensor by $R_{\mu\nu} = R^{\alpha}_{\ \mu\nu\alpha}$, and the curvature scalar by $R = g^{\mu\nu} R_{\mu\nu}$, where $g_{\mu\nu}$ is the metric tensor.

II. Absence of tachyons as a condition for $R + R^2$ gravity to give an acceptable newtonian limit

The effective nonrelativistic potential for the gravitational interaction of two identical massive bosons of zero spin (see Fig. 1) is
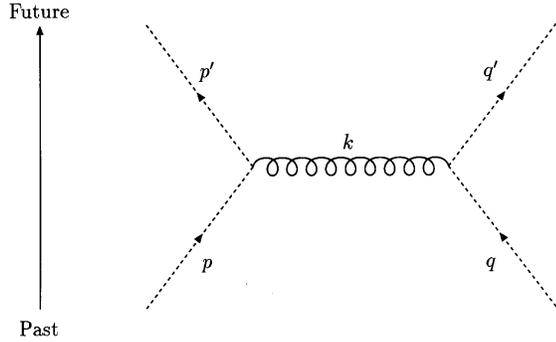

Figure 1. Feynman diagram for the gravitational interaction of two identical massive bosons of zero spin.

$$U\left(\mathbf{r}\right) = \frac{1}{4m^2} \frac{1}{(2\pi)^3} \int d^3\mathbf{k} \mathcal{F}_{NR} e^{-i\mathbf{k}\cdot\mathbf{r}} \quad , \qquad (2)$$

whereupon

$$\mathcal{F}_{NR} = i\mathcal{M}_{NR} \quad ,$$

where $\mathcal{M}_{NR}$ is the nonrelativistic limit of the Feynman amplitude for the process depicted in Fig. 1. On the other hand, the Lagrangian density for the interaction of gravity with a free, massive scalar field ϕ, is

$$\mathcal{L}_{int} = \sqrt{-g}\frac{1}{2}\left(g^{\mu\nu}\partial_{\mu}\phi\partial_{\nu}\phi - m^2\phi^2\right) \quad . \qquad (3)$$

The quantum fluctuations of the gravitational field may be expanded about a smooth background metric, which in our case is flat spacetime

$$
\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + \kappa h_{\mu\nu} \quad , \\
\eta_{\mu\nu} &= diag\left(1, -1, -1, -1\right) \quad .
\end{aligned}
\qquad (4)
$$

From (3) and (4), we obtain

$$\mathcal{L}_{int} = -\frac{\kappa}{2}h^{\mu\nu}\left[\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}\eta_{\mu\nu}\left(\partial_{\alpha}\phi\partial^{\alpha}\phi - m^2\phi^2\right)\right] \quad ,$$

where the Lagrangian density for the free scalar field has been omitted. From the previous expression the Feynman rule for the elementary vertex may readily be deduced. It is shown in Fig. 2. In momentum space, the free propagator for $R + R^2$ gravity, in the de Donder (harmonic) gauge, is (see appendix A)
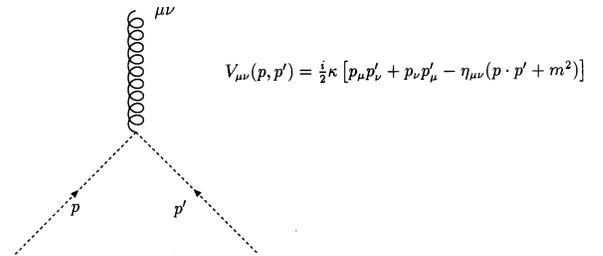

$$V_{\mu\nu}(p, p') = \frac{i}{2}\kappa\left[p_{\mu}p'_{\nu} + p_{\nu}p'_{\mu} - \eta_{\mu\nu}(p\cdot p' + m^2)\right]$$

Figure 2. The relevant Feynman rule for boson-boson interaction.

$$
\begin{aligned}
\Delta_{\mu\nu,\lambda\theta}(k) &= \frac{i}{2k^2}\left\{-\left[\frac{2 - 2\alpha k^2\kappa^2}{2 - 3\alpha k^2\kappa^2}\right]\eta_{\mu\nu}\eta_{\lambda\theta} + (\eta_{\mu\lambda}\eta_{\nu\theta} + \eta_{\nu\lambda}\eta_{\mu\theta})\right. \\
&\quad \left. - \frac{2\alpha\kappa^2}{2 - 3k^2\kappa^2\alpha}\left[\eta_{\mu\nu}k_{\lambda}k_{\theta} + \eta_{\lambda\theta}k_{\mu}k_{\nu} + \frac{2}{k^2}k_{\mu}k_{\nu}k_{\lambda}k_{\theta}\right]\right\} \quad .
\end{aligned}
$$

The Feynman amplitude for the reaction displayed in Fig. 1 is then given by

$$
\begin{aligned}
\mathcal{M} &= V_{\mu\nu}\left(p, -p'\right) \Delta^{\mu\nu,\alpha\beta}\left(k\right) V_{\alpha\beta}\left(q, -q'\right) \\
&= -\kappa^2 A\left[B\left(p \cdot p' - 2m^2\right)\left(q \cdot q' - 2m^2\right) + \left(p \cdot q\right)\left(p' \cdot q'\right) \right. \\
&\quad + \left(p \cdot q'\right)\left(p' \cdot q\right) + \left(p \cdot p'\right)\left(m^2 - q \cdot q'\right) \\
&\quad \left. + \left(q \cdot q'\right)\left(m^2 - p \cdot p'\right) + 2\left(m^2 - p \cdot p'\right)\left(m^2 - q \cdot q'\right)\right] \quad,
\end{aligned}
$$

where

$$
A \equiv \frac{i}{2\kappa^2} \quad,
$$

$$
B \equiv \frac{2 - 2\alpha\kappa^2 k^2}{-2 + 3\alpha\kappa^2 k^2} \quad;
$$

which in the nonrelativistic limit reduces to

$$
\mathcal{M}_{N.R.} = \frac{i\kappa^2 m^4}{2|\mathbf{k}|^2}\left[2 - \frac{2 + 2\alpha\kappa^2|\mathbf{k}|^2}{2 + 3\alpha\kappa^2|\mathbf{k}|^2}\right] \quad. \tag{5}
$$

Eq. (5) can be rewritten as

$$
\mathcal{M}_{N.R.} = \frac{i\kappa^2 m^4}{|\mathbf{k}|^2}\left[1 - \frac{3M_0^2 + 2|\mathbf{k}|^2}{6\left(M_0^2 + |\mathbf{k}|^2\right)}\right] \quad, \tag{6}
$$

where $M_0^2 \equiv \dfrac{2}{3\alpha\kappa^2}$.

From (6) and (2) we promptly obtain

$$
U\left(\mathbf{r}\right) = \frac{\kappa^2 m^2}{8\pi^2}\left[-\frac{\pi}{4r} - \frac{I}{12r}\right] \quad,
$$

where

$$
I \equiv \int_{-\infty}^{\infty} \frac{x \sin x\, dx}{x^2 + M_0^2 r^2} \quad. \tag{7}
$$

From a mathematical point of view integral (7) makes sense only if α is positive ($M_0^2 > 0$), which corresponds to the absence of tachyons in the dynamical field. Assuming that $\alpha > 0$, integral (7) can be easily evaluated by the method of contour integration. The result is

$$
I = \pi e^{-M_0 r} \quad.
$$

Therefore

$$
U\left(\mathbf{r}\right) = m^2 G\left[-\frac{1}{r} - \frac{1}{3}\frac{e^{-M_0 r}}{r}\right] \quad.
$$

The associated nonrelativistic gravitational potential for $R + R^2$ gravity is then given by

$$
V\left(\mathbf{r}\right) = mG\left[-\frac{1}{r} - \frac{1}{3}\frac{e^{-M_0 r}}{r}\right] \quad. \tag{8}
$$

Note that (8) gives an acceptable newtonian limit since there is only a falling exponential and not a oscillating $\dfrac{1}{r}$ term at infinity.

III. The gravitational field outside the Sun in the framework of $R + R^2$ gravity and in the weak field approximation

The action for $R + R^2$ gravity theory is given by

$$
S = \int d^4 x \sqrt{-g}\left[\frac{2R}{\kappa^2} + \frac{\alpha}{2}R^2 + \mathcal{L}_M\right] \quad, \tag{9}
$$

where $\mathcal{L}_M$ is the Lagrangian density for the usual matter. The corresponding field equation is[3]

$$
\frac{2}{\kappa^2}G_{\mu\nu} + \frac{\alpha}{2}\left[-\frac{1}{2}R^2 g_{\mu\nu} + 2RR_{\mu\nu} + 2\nabla_\mu\nabla_\nu R - 2g_{\mu\nu}\Box R\right] + \frac{1}{2}T_{\mu\nu} = 0 \quad,
$$

where

$$\delta \int d^4 x \left(\sqrt{-g} \mathcal{L}_M \right) \equiv \int d^4 x \sqrt{-g} \frac{T_{\mu\nu}}{2} \delta g^{\mu\nu} \quad .$$

In the weak field approximation this equation reduces to

$$\frac{2}{\kappa^2} \Box \gamma_{\mu\nu} + \frac{\alpha}{2} \Box \left(\partial_\mu \partial_\nu - \eta_{\mu\nu} \Box \right) h + \frac{1}{\kappa} T_{\mu\nu} = 0 \quad , \quad (10)$$

where

$$\gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h \quad .$$

In obtaining the previous equation we have used the harmonic condition $\gamma_{\mu\nu},^\nu = 0$.

Contracting (10), we get

$$\Box \left(\frac{2}{\kappa^2} + 3\alpha \Box \right) h = \frac{T}{\kappa} \quad . \tag{11}$$

Let us assume that the Sun is a point particle of mass M located at $\mathbf{r} = \mathbf{0}$. The corresponding energy-momentum tensor is given by

$$T_{\mu\nu} = M \delta^0{}_\mu \delta^0{}_\nu \delta^3(\mathbf{r}) \quad . \tag{12}$$

From (11) and (12), we promptly obtain

$$\left(1 - \frac{1}{M_0^2} \nabla^2 \right) \nabla^2 h(\mathbf{r}) = -\frac{\kappa}{2} M \delta^3(\mathbf{r}) \quad ,$$

where M_0^2 is supposed to be positive since we are assuming the absence of tachyons in the dynamical field (see the previous section). This equation is solved in detail in Ref. [4]. The solution is

$$h = \frac{\kappa M}{8\pi} \frac{1 - e^{-M_0 r}}{r} \quad .$$

From (9) and (12), we have

$$\nabla^2 \left[\frac{2}{\kappa^2} \gamma_{00} + \alpha \nabla^2 h \right] = \frac{M}{\kappa} \delta^3(\mathbf{r}) \quad ,$$

whose solution is

$$h_{00} = \frac{M\kappa}{16\pi} \left[-\frac{1}{r} - \frac{1}{3} \frac{e^{-M_0 r}}{r} \right] \quad .$$

Note that $h_{00}(r) = \frac{2}{\kappa} V(r)$ (see the preceding section).

From (9) and (12), we finally get

$$\nabla^2 \left[\frac{2}{\kappa^2} \gamma_{ij} + \alpha \left(\partial_i \partial_j - \delta_{ij} \nabla^2 \right) h \right] = 0 \quad ,$$

whose solution can be written as

$$h_{ij} = \frac{\kappa M}{8\pi M_0^2} \left[\delta_{ij}, (r) + \frac{x^i x^j}{r^2} \Lambda(r) \right] \quad ,$$

where

$$, (r) \equiv \frac{1}{3r^3} \left(1 - e^{-M_0 r} \right) - \frac{M_0}{3r^2} e^{-M_0 r} - \frac{M_0^2}{3r} e^{-M_0 r} - \frac{M_0^2}{2r} \left(1 - e^{-M_0 r} \right) \quad ,$$

$$\Lambda(r) \equiv -\frac{1}{r^3} \left(1 - e^{-M_0 r} \right) + \frac{M_0}{r^2} e^{-M_0 r} + \frac{M_0^2}{3r} e^{-M_0 r} \quad .$$

Of course, $h_{i0} = 0$. Therefore, the gravitational field outside the Sun takes the form

$$h_{\mu\nu}(\mathbf{r}) = \frac{\kappa M}{8\pi M_0^2} \begin{pmatrix} -\frac{M_0^2}{2r} \left(1 + \frac{1}{3} e^{-M_0 r} \right) & 0 & 0 & 0 \\ 0 & , + \frac{x^2}{r^2} \Lambda & \frac{xy}{r^2} \Lambda & \frac{xz}{r^2} \Lambda \\ 0 & \frac{yx}{r^2} \Lambda & , + \frac{y^2}{r^2} \Lambda & \frac{yz}{r^2} \Lambda \\ 0 & \frac{zx}{r^2} \Lambda & \frac{zy}{r^2} \Lambda & , + \frac{z^2}{r^2} \Lambda \end{pmatrix}$$

Since in the next section we will need the momentum space gravitational field, i. e. ,

$$h_{\mu\nu}(\mathbf{k}) = \int d^3\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} h_{\mu\nu}(\mathbf{r}) \quad,$$

we use the remainder of this section to evaluate $h_{\mu\nu}(\mathbf{k})$. This calculation is rather involved. Using much algebra and the integrals listed in appendix B, some of which only exist as distributions, we arrive at the following result

$$h_{\mu\nu}(\mathbf{k}) = -\frac{\kappa M}{4}\begin{pmatrix} \frac{1}{k^2} + \frac{Q}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{k^2} - \frac{Q}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{k^2} - \frac{Q}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{k^2} - Q \end{pmatrix} \quad,$$

where $Q \equiv \dfrac{1}{M_0^2 + k^2}$, and $k \equiv |\mathbf{k}|$.

IV. Prediction of $R + R^2$ gravity for the deflection angle of a photon grazing the Sun's surface

Let us now consider the scattering of a photon by the gravitational field of the Sun, treated as an external field. The Lagrangian function of the system is

$$\mathcal{L} = -\frac{F^{\mu\nu}F_{\mu\nu}}{4} - \frac{\kappa}{8}\left[h^{\rho\theta}\eta_{\rho\theta}\eta^{\mu\alpha}\eta^{\nu\beta} - 4h^{\mu\alpha}\eta^{\nu\beta}\right]F_{\mu\nu}F_{\alpha\beta} \quad.$$

The corresponding photon-external-gravitational-field vertex is shown in Fig. 3. Therefore, the Feynman amplitude for the process displayed in Fig. 3 is of the form

$V_{\mu\nu}(p \cdot p') = \frac{\kappa}{2}h^{\lambda\rho}(\mathbf{k})\left[-\eta_{\mu\nu}\eta_{\lambda\rho}p\cdot p' + \eta_{\lambda\rho}p_\nu p'_\mu\right.$
$\left.+2\left(\eta_{\mu\nu}p_\lambda p'_\rho - \eta_{\nu\rho}p_\lambda p'_\mu - \eta_{\mu\lambda}p_\nu p'_\rho + \eta_{\mu\lambda}\eta_{\nu\rho}p\cdot p'\right)\right]i$

Figure 3. The photon-external-gravitational-field vertex.

$$\mathcal{M}_{rr'} = \epsilon_r^\mu(\mathbf{p})\epsilon_{r'}^\nu(\mathbf{p}')\mathcal{M}_{\mu\nu} \quad,$$

whereupon

$$\begin{aligned} \mathcal{M}_{\mu\nu} &= i\frac{\kappa}{2}h^{\lambda\rho}(\mathbf{k})\left[-\eta_{\mu\nu}\eta_{\lambda\rho}p\cdot p' + \eta_{\lambda\rho}p_\nu\cdot p'_\mu\right. \\ &+ 2\left.\left(\eta_{\mu\nu}p_\lambda p'_\rho - \eta_{\nu\rho}p_\lambda p'_\mu - \eta_{\mu\lambda}p_\nu p'_\rho + \eta_{\mu\lambda}\eta_{\nu\rho}p\cdot p'\right)\right] \quad, \end{aligned}$$

where $\epsilon_r^\mu(\mathbf{p})$ and $\epsilon_{r'}^\nu(\mathbf{p}')$ are the polarization vectors for the initial and final photons, respectively.

The unpolarized cross-section for the process is proportional to

$$X = \frac{1}{2} \sum_{r=1}^{2} \sum_{r'=1}^{2} |\mathcal{M}_{rr'}|^2 \quad ,$$

where we have averaged over initial polarizations and summed over final polarizations. Hence

$$\begin{aligned} X &= \frac{\kappa^2}{8} \left[4 \left(p \cdot p' \right)^2 h_{\mu\nu} h^{\mu\nu} - 2h^2 \left(p \cdot p' \right)^2 + 2 \left(4p \cdot p' h h_{\lambda\theta} p^{\lambda} p'^{\theta} \right. \right. \\ &\quad - \left. \left. 8 h_{\theta\rho} h^{\rho}{}_{\lambda} p^{\lambda} p'^{\theta} p \cdot p' + 4 h_{\lambda\theta} h_{\rho\mu} p^{\lambda} p^{\theta} p'^{\rho} p'^{\mu} \right) \right] \quad . \end{aligned}$$

Since $M_0^2 + |\mathbf{k}|^2 \approx M_0^2$, the momentum space gravitational field can be rewritten as

$$\begin{aligned} h_{00}\left(\mathbf{k}\right) &= -\frac{\kappa M}{4} \left[\frac{1}{k^2} + \frac{1}{3} \frac{1}{M_0^2} \right] \quad , \\ h_{11}\left(\mathbf{k}\right) &= h_{22}\left(\mathbf{k}\right) = -\frac{\kappa M}{4} \left[\frac{1}{k^2} - \frac{1}{3} \frac{1}{M_0^2} \right] \quad , \\ h_{33}\left(\mathbf{k}\right) &= -\frac{\kappa M}{4} \left[\frac{1}{k^2} - \frac{1}{M_0^2} \right] \quad . \end{aligned}$$

Therefore,

$$X = \frac{\kappa^4 M^2}{16} \frac{E^4}{k^4} \left(1 + \cos\theta \right)^2 \quad ,$$

where E is the energy of the incident photon and θ is the scattering angle. The unpolarized cross-section for the process in hand can be obtained from the expression [5]

$$\frac{d\sigma}{d\Omega} = \frac{1}{4} \frac{X}{(2\pi)^2} \quad . \tag{13}$$

Thus, the unpolarized cross-section for the scattering of a photon by the Sun's gravitational field, treated as an external field, on the basis of $R+R^2$ gravity, is given by

$$\frac{d\sigma}{d\Omega} = \frac{\kappa^4 M^2}{256 (2\pi)^2} \cot^4 \frac{\theta}{2} \quad ,$$

which, for small angles, reduces to

$$\frac{d\sigma}{d\Omega} = \frac{16 G^2 M^2}{\theta^4} \quad . \tag{14}$$

On the other hand,

$$\frac{d\sigma}{d\Omega} = \left| \frac{r}{\sin\theta} \frac{dr}{d\theta} \right| \quad .$$

For small θ, we get

$$\frac{d\sigma}{d\Omega} = \left| \frac{r}{\theta} \frac{dr}{d\theta} \right| \quad . \tag{15}$$

From (14) and (15) we then have

$$r^2 = \frac{16 G^2 M^2}{\theta^2} \quad .$$

So, we come to the conclusion that the prediction of $R+R^2$ gravity for the deflection angle of a photon passing near to the Sun is exactly the same as that given by Einstein's theory, namely,

$$\theta = 1.75 \ arcsec.$$

Actually, it is a paradox that two quite different gravitational theories could lead to the same value for the solar gravitational deflection. Where such a strange coincidence came from? We discuss this important matter in the next section.

V. A general result

We have argued that the quadratic term of $R + R^2$ gravity theory should give a non negligible contribution to the gravitational solar deflection of starlight $(E \sim 10^5 cm^{-1})$. However, a careful calculation at the tree level showed that the deflection angle of a photon passing near to the Sun's surface is the same as that predicted by Einstein's theory. In other words, the quadratic term does not contribute anything to the gravitational solar deflection. How can we explain this unexpected result? First of all, we note that at the tree level, since

$$h_{\mu\nu}(\mathbf{k}) = h_{\mu\nu}^{(E)}(\mathbf{k}) + \left[\frac{\kappa M}{6M_0^2} \left(-\frac{1}{2}\eta_{\mu\nu} + \delta_\mu^3 \delta_\nu^3 \right) \right] \quad ,$$

whereupon

$$h_{\mu\nu}^{(E)}(\mathbf{k}) = \int d^3\mathbf{r} e^{-i\mathbf{k}\cdot\mathbf{r}} h_{\mu\nu}^{(E)}(\mathbf{r}) \quad ,$$

where $h_{\mu\nu}^{(E)}(\mathbf{r})$ is the solution of the linearized Einstein's equation supplemented by the usual harmonic coordinate condition, we may write the Feynman amplitude as

$$\mathcal{M}_{\mu\nu} = \mathcal{M}_{\mu\nu}^{(E)} + \mathcal{M}_{\mu\nu}^{(R^2)} \quad ,$$

where

$$
\begin{aligned}
\mathcal{M}_{\mu\nu}^{(R^2)} \quad \equiv \quad & i\frac{\kappa}{2} \left[\frac{\kappa M}{6M_0^2} \left(-\frac{1}{2}\eta^{\lambda\rho} + \delta_3^\lambda \delta_3^\rho \right) \right] [-\eta_{\mu\nu}\eta_{\lambda\rho} p \cdot p' \\
+ \quad & \eta_{\lambda\rho} p_\nu p'_\mu + 2 \left(\eta_{\mu\nu} p_\lambda p'_\rho - \eta_{\nu\rho} p_\lambda p'_\mu - \eta_{\mu\lambda} p_\nu p'_\rho + \eta_{\mu\lambda}\eta_{\nu\rho} p \cdot p' \right)] \quad .
\end{aligned}
$$

Of course, $\mathcal{M}_{\mu\nu}^{(R^2)} \equiv 0$. Therefore,

$$\mathcal{M}_{\mu\nu} = \mathcal{M}_{\mu\nu}^{(E)} \quad ,$$

which clearly shows why the two theories lead to the same result for the solar gravitational deflection.

At the classical level, in turn, the field equation for $R + R^2$ gravity, in the weak field approximation and in the gauge $A_\mu \equiv -\gamma_{\mu\alpha}{}^{,\alpha} + \frac{\alpha\kappa^2}{2} R_{,\mu} = 0$, where $\gamma_{\mu\nu} \equiv h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$, is given by

$$\Box h_{\mu\nu} - \frac{1}{3} R\eta_{\mu\nu} = \kappa \left[\frac{T\eta_{\mu\nu}}{6} - \frac{T_{\mu\nu}}{2} \right] \quad .$$

In this equation R must be replaced by its respective first-order expression:

$$R = \frac{1}{2}\Box h - \gamma_{\mu\nu}{}^{,\mu\nu} \quad .$$

In this gauge, the Sun's gravitational takes the form

$$h_{\mu\nu}(\mathbf{r}) = h_{\mu\nu}^{(E)}(\mathbf{r}) + \phi(\mathbf{r})\eta_{\mu\nu} \quad ,$$

where

$$\phi(\mathbf{r}) \equiv -\frac{M\kappa}{48\pi}\frac{e^{-M_0 r}}{r} \quad .$$

Neglecting the product of $h_{\mu\nu}^{(E)}$ by ϕ, the metric can be written as

$$
\begin{aligned}
g_{\mu\nu} \quad &= \quad \left[h_{\mu\nu}^{(E)} + \phi\eta_{\mu\nu} \right]\kappa + \eta_{\mu\nu} \\
&= \quad (1 + \kappa\phi)g_{\mu\nu}^{(E)} \quad .
\end{aligned}
$$

Thus, *in the absence of tachyons $(\alpha > 0)$ and assuming that $1 + \kappa\Phi > 0$ (this condition is necessary for the conformal transformation not to invert the signature of the metric), the linearized $R + R^2$ gravity theory is conformally related to Einstein's linearized theory.*
The previous result clearly shows why we arrive at the same value for the solar gravitational deflection whether we use $R + R^2$ gravity or Einstein's gravity.

To conclude, we call attention to the fact that Whitt[6] and Teyssandier[7] studied the equivalence of the above theories in a quite different context.

Appendix A: The propagator for $R + R^2$ gravity

In order to calculate the propagator for $R + R^2$ gravity, we must first write out the Lagrangian density $\mathcal{L}_g = \left(\dfrac{2R}{\kappa^2} + \dfrac{\alpha}{2} R^2 \right) \sqrt{-g}$ in a purely quadratic form in the gravitational field $h^{\mu\nu}$. Going over to momentum space and using a substantial amount of Lorentz algebra on symmetric rank-two tensors, we arrive at the following expression for $\mathcal{L} = \mathcal{L}_g + \mathcal{L}_{gf}$, with the gauge fixing Lagrangian density given by $\mathcal{L}_{gf} = \left(\partial_\nu h^{\mu\nu} - \dfrac{1}{2} \partial^\mu h \right)^2$,

$$\mathcal{L} = \frac{1}{2} h_{\mu\nu} \mathcal{P}^{\mu\nu,\rho\theta} h_{\rho\theta} \quad,$$

whereupon

$$
\begin{aligned}
\mathcal{P}^{\mu\nu,\rho\theta} = \;& A \eta^{\mu\nu} \eta^{\rho\theta} + B \left(\eta^{\mu\rho} \eta^{\nu\theta} + \eta^{\nu\rho} \eta^{\mu\theta} \right) \\
& + C k^\mu k^\nu k^\rho k^\theta + D \left(\eta^{\rho\theta} k^\mu k^\nu + \eta^{\mu\nu} k^\rho k^\theta \right) \quad,
\end{aligned}
$$

where

$$
\begin{aligned}
A &\equiv -\frac{k^2}{2} + \alpha \kappa^2 k^4 \quad, \\
B &\equiv \frac{k^2}{2} \quad, \\
C &\equiv \kappa^2 \alpha \quad, \\
D &\equiv -\alpha \kappa^2 k^2 \quad.
\end{aligned}
$$

Since $\mathcal{P}^{\mu\nu,\rho\theta}$ is a fourth-rank tensor dependent only on the 4-vector k^α, Lorentz-covariance therefore imposes the following general form for its inverse

$$
\begin{aligned}
\mathcal{P}_{\rho\theta,\lambda\gamma} = \;& \mathcal{A} \eta_{\rho\theta} \eta_{\lambda\gamma} + \mathcal{B} \left(\eta_{\rho\lambda} \eta_{\gamma\theta} + \eta_{\theta\lambda} \eta_{\rho\gamma} \right) \\
& + \mathcal{C} k_\rho k_\theta k_\lambda k_\gamma + \mathcal{D} \left(\eta_{\rho\theta} k_\lambda k_\gamma + \eta_{\lambda\gamma} k_\rho k_\theta \right) \quad.
\end{aligned}
$$

Now, taking into account that

$$\mathcal{P}^{\mu\nu,\rho\theta} \mathcal{P}_{\rho\theta,\lambda\gamma} = \frac{1}{2} \left(\delta^\mu_{\;\lambda} \delta^\nu_{\;\gamma} + \delta^\mu_{\;\gamma} \delta^\nu_{\;\lambda} \right) \quad,$$

we promptly obtain

$$
\begin{aligned}
\mathcal{A} &= -\left[\frac{-2 + 2\alpha\kappa^2 k^2}{-2 + 3\alpha\kappa^2 k^2} \right] \mathcal{B} \quad, \\
\mathcal{B} &= \frac{1}{2k^2} \quad, \\
\mathcal{C} &= \frac{2}{k^2} \mathcal{D} \quad, \\
\mathcal{D} &= \frac{2\kappa^2 \alpha^2}{-2 + 3\kappa^2 k^2 \alpha} \mathcal{B} \quad.
\end{aligned}
$$

The graviton propagator is then given by

$$\Delta_{\rho\theta,\lambda\gamma}(\mathbf{k}) = i \mathcal{P}_{\rho\theta,\lambda\gamma}(\mathbf{k}) \quad.$$

Appendix B: A table of integrals for the calculation of $h_{\mu\nu}(k)$ (Some of the integrals below only exist as distributions)

1. $\displaystyle \int_0^\infty e^{-M_0 r} \sin kr\, dr = \frac{k}{M_0^2 + k^2}$

2. $\displaystyle \int_0^\infty \frac{e^{-M_0 r} \sin kr}{r} dr = tan^{-1} \frac{k}{M_0}$

3. $\displaystyle \int_0^\infty \frac{e^{-M_0 r} \sin kr}{r^2} dr = k - b\, tan^{-1} \frac{k}{M_0} - \frac{k}{2} ln\left(M_0^2 + k^2 \right)$

4. $\displaystyle \int_0^\infty \frac{e^{-M_0 r} \sin kr}{r^3} dr = \frac{1}{2} \left(M_0^2 - k^2 \right) tan^{-1} \frac{k}{M_0} - \frac{3 M_0 k}{2} + \frac{M_0 k}{2} ln\left(M_0^2 + k^2 \right)$

5. $\displaystyle \int_0^\infty \frac{e^{-M_0 r} \sin kr}{r^4} dr = -\frac{11 k^3}{36} + \frac{11 M_0^2 k}{12} + \frac{M_0}{6} \left(3k^2 - M_0^2 \right) tan^{-1} \frac{k}{M_0}$

$$+ \frac{1}{12} \left(k^3 - 3 M_0^2 k \right) ln\left(M_0^2 + k^2 \right)$$

6. $\int_0^\infty e^{-M_0 r} \cos kr\, dr = \dfrac{M_0}{M_0^2 + k^2}$

7. $\int_0^\infty \dfrac{e^{-M_0 r} \cos kr}{r} dr = -\dfrac{1}{2} \ln\left(M_0^2 + k^2\right)$

8. $\int_0^\infty \dfrac{e^{-M_0 r} \cos kr}{r^2} dr = -k\, \tan^{-1}\dfrac{k}{M_0} + \dfrac{M_0}{2} \ln\left(M_0^2 + k^2\right) - M_0$

9. $\int_0^\infty \dfrac{e^{-M_0 r} \cos kr}{r^3} dr = -\dfrac{3k^2}{4} + M_0 k\, \tan^{-1}\dfrac{k}{M_0} + \dfrac{1}{4}\left(k^2 - M_0^2\right) \ln\left(M_0^2 + k^2\right) + \dfrac{3M_0^2}{4}$

10. $\int_0^\infty \sin kr\, dr = \dfrac{1}{k}$

11. $\int_0^\infty \dfrac{\sin kr}{r^2} dr = k\left(1 - \ln k\right)$

12. $\int_0^\infty \dfrac{\sin kr}{r^4} dr = -\dfrac{11k^3}{36} + \dfrac{k^3}{6} \ln k$

13. $\int_0^\infty \dfrac{\cos kr}{r^3} dr = \dfrac{k^2}{4}\left(-3 + 2\ln k\right)$

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