

About the Non-Integrability in the Friedmann-Robertson-Walker Cosmological Model

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We study the non integrability of the Friedmann-Robertson-Walker cosmological model, in continuation of the work [5] of Coehlo, Skea and Stuchi. Using Morales-Ramis theorem ([10]) and applying a practical non-integrability criterion deduced from it, we find that the system is not completely integrable for almost all values of the parameters λ and Λ , which was already proved by the authors of [5] applying Kovacic's algorithm. Working on a level surface $H = h$ with $h \neq 0$ and $h \neq -\frac{1}{4\lambda}$ and using the Morales-Ramis-Simo "higher variational" theory ([11]), we prove that the hamiltonian system cannot be integrable for particular values of λ among the exceptional values and that it is completely integrable in two special cases ($\lambda = \Lambda = -m^2$ and $\lambda = \Lambda = -\frac{m^2}{3}$). We conjecture that there is no other case of complete integrability and give detailed arguments towards this.

Keywords: Hamiltonian systems; Integrability; Morales-Ramis-Simo Theorem; Computer algebra

I. THE PROBLEM

The Friedmann-Robertson-Walker model ([7]) is a model which explains some features of the observed universe at the present time. Although it does not describe the real universe in an essential way because of too many symmetries, it remains a fundamental model.

We consider the Friedmann-Robertson-Walker (FRW) universe ([5]) where the metric takes the form

$$ds^2 = a^2(v) \left(-dv^2 + \frac{1}{1-kr^2} dr^2 + r^2 d\Omega^2 \right)$$

where v is the time; $a(v)$ is the scale factor, which means that if a distance is measured as d_0 today, then at any other instant, it is d_0 times a ; $k = 0, 1, -1$ is the curvature of space-time and $d\Omega^2$ is the distance element on a two-sphere.

The dynamics is described by Raychaudhuri equation and Klein-Gordon equation ([7]) which derive from the Hamiltonian ([5],[7])

$$H = \frac{1}{2} \left(-(p_a^2 + k a^2) + (p_\phi^2 + k \phi^2) + m^2 a^2 \phi^2 + \frac{\lambda}{2} \phi^4 + \frac{\Lambda}{2} a^4 \right)$$

After the canonical transformation of variables $p_a = ip_a$ and $a = -ia$ (suggested from [4]), and assuming $k = 1$, we get the Hamiltonian:

$$H = \frac{1}{2} (\dot{a}^2 + \dot{\phi}^2) + \frac{1}{2} a^2 + \frac{1}{2} \phi^2 - \frac{1}{2} m^2 a^2 \phi^2 + \frac{1}{4} \lambda \phi^4 + \frac{1}{4} \Lambda a^4$$

where a is the scale factor of the universe; ϕ is the scalar field with self-coupling constant λ and with mass m (which we will assume to be non zero); Λ is the cosmological constant.

The associated hamiltonian system which describes the FRW cosmological model is $\dot{X} = J\nabla(X)$ where $X = (a, \phi, \dot{a}, \dot{\phi})$ and J is the 4×4 matrix satisfying $J^2 = -Id_4$.

In [5], Coehlo, Skea and Stuchi use the Morales-Ramis theory and the Kovacic algorithm (which solves second order linear differential equations) to show that the FRW model is not completely integrable except eventually when λ and Λ satisfy the conditions

$$\lambda = -\frac{2m^2}{(n+1)(n+2)}, n \in \mathbf{N}$$

and

$$\Lambda = -\frac{2m^2}{(N+1)(N+2)}, N \in \mathbf{N}$$

Furthermore, numerical evidence of integrability is illustrated in [5] when $\lambda = \Lambda = -m = -1$ and when $\lambda = \Lambda = -\frac{m^2}{3}$.

This paper is a continuation of [5]. Our aims are:

- to show that the variational equation, computed along a given particular solution of the hamiltonian system, can be seen as a direct sum of two Lamé equations (when $\lambda \neq 0$);
- to find the result of non-integrability already proved in [5] when $\lambda \neq -\frac{2m^2}{(n+1)(n+2)}, n \in \mathbf{N}$ or $\Lambda \neq -\frac{2m^2}{(N+1)(N+2)}, N \in \mathbf{N}$ using a criterion of non integrability ([2, 3]) deduced from Morales-Ramis theorem ([10]);
- to prove that the system is not completely integrable in the particular case

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$\lambda = -\frac{2m^2}{(n+1)(n+2)}$, with $n \in \{2, \dots, 10\}$ using Morales-Ramis-Simo theorem ([11]);

- to prove that the system is completely integrable in the two cases $\lambda = \Lambda = -m^2$ ($n = N = 0$) and $\lambda = \Lambda = \frac{-m^2}{3}$ ($n = N = 1$).
- to conjecture that the system is completely integrable if, and only if, $\lambda = \Lambda = -m^2$ or $\lambda = \Lambda = \frac{-m^2}{3}$.

The particular case $\lambda = 0$ is a case of non integrability which will be studied in the annex.

These results can be understood as an extension of [5], as we solve questions that were left open there. Since this work was submitted, we have been aware of several other recent works on other configurations of the Friedman-Robertson-Walker models using the Hamiltonian viewpoint and the variational approach that we use below ([6], [9]).

In this paper (as well as in the above references), we use the Morales-Ramis theory from [10]. To show that our Hamiltonian differential system does not admit new first integral, we find a particular solution, linearize the system along this solution; applying differential Galois theory to the linearized equation then allows to conclude to the non-integrability of the model.

This last point may be tedious. In [5, 6, 9], the Kovacic algorithm was used; in [6, 9], additional techniques (systems with homogeneous potentials) were further used. Here, we will use a criterion (from [2, 3]) which is easy to apply and generalizes to equations of higher order: if the linearized

(variational) equation is irreducible and has local solutions with logarithms, then the original system was not integrable (a more general version of our criterion is theorem 3 in [3]).

The paper is structured as follows. In section 2, we study the structure of the variational equation. In section 3, we apply our criterion to rediscover the non-integrability cases of [5]. In section 4, we go deeper in the analysis (higher variational equation) to state our main theorem and conjecture about all integrability cases of this model.

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II. THE (FIRST) VARIATIONAL EQUATION

The Hamiltonian H can be written:

$$H = \frac{1}{2}(\dot{a}^2 + \dot{\phi}^2) + \frac{1}{2}\left(\phi^2 + \frac{\lambda}{2}\phi^4 + a^2(1 - m^2\phi^2) + a^4\frac{\Lambda}{2}\right)$$

We choose a particular solution ([1]) $X_0 = (a_0, \phi_0, \dot{a}_0, \dot{\phi}_0)$ on the level surface

$$a = \dot{a} = 0 \text{ and } h = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}\left(\phi^2 + \frac{\lambda}{2}\phi^4\right)$$

The variational system along the solution X_0 is

$$Y' = J\mathcal{H}(H, X_0)Y = \begin{pmatrix} \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \\ \begin{matrix} f_1 = -1 + m^2\phi_0^2 & 0 \\ 0 & f_2 = -1 - 3\lambda\phi_0^2 \end{matrix} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \end{pmatrix} Y$$

We note that ϕ_0^2 is solution to the equation

$$v^2 = -2\lambda v^3 - 4v^2 + 8hv$$

So, if $\lambda \neq 0$, then

$$\phi_0^2 = -\frac{2}{\lambda}\wp(t) - \frac{2}{3\lambda}$$

where $\wp(t) = \wp(t, g_2, g_3)$ is the Weierstrass function with parameters

$$g_2 = \frac{4}{3} + 4h\lambda \text{ et } g_3 = \frac{8}{27} + \frac{4}{3}h\lambda$$

The variational equation can then be seen as a direct sum of two Lamé equations

$$(E_1) y_1''(t) = f_1(t) y_1(t) \text{ with } f_1(t) = -\frac{2m^2}{\lambda} \wp(t) - 1 - \frac{2m^2}{3\lambda}$$

and

$$(E_2) y_2''(t) = f_2(t) y_2(t) \text{ with } f_2(t) = 6\wp(t) + 1$$

This point will be crucial in our practical application of the "higher variational" theorem of Morales-Ramis-Simo in section 4.

III. NON-INTEGRABILITY FOR ALMOST ALL VALUES OF λ

With the change of variable $x = \wp(t)$ the differential equation

$$y''(t) = (\alpha \wp(t) + \beta) y(t)$$

is equivalent to the differential equation

$$(4x^3 - g_2x - g_3)y''(x) + (6x^2 - \frac{g_2}{2})y'(x) - (\alpha x + \beta)y(x) = 0$$

and the connected components of the identity of the differential Galois groups of these two equations are isomorphic ([1] page 4 for this result, and more generally [13] for differential Galois theory).

We first assume that $h = 0$ (which induces the relation $g_2^3 - 27g_3^2 = 0$).
The equation

$$(E_1) y_1''(t) = f_1(t) y_1(t)$$

becomes

$$(\tilde{E}_1) : \frac{4}{27} (3x-2)(3x+1)^2 y''(x) +$$

$$\frac{2}{3} (3x-1)(3x+1)y'(x) + \left(\frac{2m^2}{\lambda}x + 1 + \frac{2}{3} \frac{m^2}{\lambda} \right) y(x) = 0$$

The exponents at the singular point $\frac{-1}{3}$ are $\frac{-1}{2}$ and $\frac{1}{2}$, the formal solutions at that point are $s_1 = \sqrt{3x+1}(1 + \dots)$ and $-\frac{m^2}{2\lambda} s_1 \ln(3x+1) + s_2$. So if $m \neq 0$ then there are logarithmic terms in the local solutions at the point $\frac{-1}{3}$.
We now will show that the equation is most of the time irreducible, so that our non-integrability criterion applies.

The exponents at the other singular points are:

★ at the point $2/3$: 0 and $\frac{1}{2}$

★ at the point ∞ : the roots of the indicial equation (E_∞) : $2\lambda k^2 - \lambda k + m^2 = 0$.

A necessary condition for the equation (\tilde{E}_1) to be reducible is that it possesses an exponential solution ; from [12] and the form of the local solutions at $x = 1/3$, such a solution should be of the form

$$y(x) = (3x+1)^{1/2} (3x-2)^e p(x)$$

where $p(x)$ is a polynomial of degree d ; $e = 0$ or $\frac{1}{2}$ (exponent at $2/3$) and

$$d + \frac{1}{2} + e + e_\infty = 0$$

with e_∞ being an exponent at infinity (see [12] for the expression of this exponential solution).

A necessary condition for the existence of such a solution is that the indicial equation at infinity (E_∞) has a root $-d - \frac{1}{2}$ or $-d - 1$ with d a natural integer. This implies

$$\lambda = -\frac{2m^2}{(2d+1)(2d+2)} \quad (\text{for } e = 0)$$

or

$$\lambda = -\frac{2m^2}{(2d+2)(2d+3)} \quad (\text{for } e = \frac{1}{2})$$

which is equivalent to

$$\lambda = -\frac{2m^2}{(n+2)(n+1)} \text{ with } n \in \mathbf{N}$$

So, if

$$\lambda \neq -\frac{2m^2}{(n+2)(n+1)}$$

with $n \in \mathbf{N}$ then the equation (\tilde{E}_1) is irreducible. Furthermore, it has formal solutions with logarithmic terms; so according to theorem 3, section 3.2 of [3] (or theorem 8 page 106 of [2] for a shorter version), the connected component of the identity of the differential Galois group of (\tilde{E}_1) (and so the one of (E_1)) is not abelian. Then, according to Morales-Ramis theorem ([10]) the Hamiltonian system is not completely integrable (this was already proven using Kovacic's algorithm in [5]).

Now, if $\lambda = -\frac{2m^2}{(n+2)(n+1)}$ with $n \in \mathbf{N}$, then there are still logarithmic terms but there maybe exponential solutions,

which one checks experimentally. Then the differential Galois group of (\tilde{E}_1) is abelian and one cannot conclude using Morales-Ramis (first) theorem. So we will use higher order variational equations and the extension of the theorem of Morales-Ramis-Simo ([11]).

First of all, we change the level of the surface and instead of taking $h = 0$, we choose h such that $g_2^3 - 27g_3^2 \neq 0$. The (first) variational equation has then no formal solution with logarithmic terms at the singularity 0 of $\wp(t)$ but it is a direct sum of two Lamé-Hermite type equations. Practically, it enables to use lemma 9 page 27 of [11] which only requires a local study at the point 0 of the equation (the presence of logs in local solutions at zero of one of the higher order variational equations will be enough to conclude).

Remark 1 *As it was noticed in [5], because of the symmetry of the Hamiltonian function H in the variables, we can also conclude that the system is not completely integrable when $\Lambda \neq -\frac{2m^2}{(N+2)(N+1)}$ with $N \in \mathbf{N}$.*

IV. NON-INTEGRABILITY FOR SOME EXCEPTIONAL VALUES OF λ , INTEGRABILITY IN TWO SPECIAL CASES AND CONJECTURE.

In the following of the section, we assume

$$\lambda = -\frac{2m^2}{(n+1)(n+2)}, n \in \mathbf{N}.$$

We are going to use the theorem of Morales-Ramis-Simo and higher variational equations ([11]).

We choose the value h of the Hamiltonian such that $h \neq 0$ and $h \neq -\frac{1}{4\lambda}$ so that $g_2^3 - 27g_3^2 \neq 0$ (where g_2, g_3 are the parameters of the Weirstrass \wp -function of section 2). As a consequence, the first variational equation

$$(VE_1) : Y' = \begin{pmatrix} \begin{matrix} \boxed{0 & 0} \\ \boxed{0 & 0} \end{matrix} & \begin{matrix} \boxed{1 & 0} \\ \boxed{0 & 1} \end{matrix} \\ \begin{matrix} \boxed{(n+1)(n+2)\wp(t) + \frac{(n+1)(n+2)}{3} - 1} & \boxed{0} \\ \boxed{0} & \boxed{6\wp(t) + 1} \end{matrix} & \begin{matrix} \boxed{0 & 0} \\ \boxed{0 & 0} \end{matrix} \end{pmatrix} Y$$

is a direct sum of two Lamé-Hermite type equations. So, according to lemma 9 page 27 of [11], one can check whether the Galois groups of the higher order variational equations are virtually abelian or not thanks to the computation of the monodromy along the singularity 0 of $\wp(t)$.

The successive variational equations are obtained using this special well known trick [8] : let us denote

$$\dot{X} = F(X) = J\nabla H(X)$$

the Hamiltonian system and let $X_0 = (a_0, \phi_0, \dot{a}_0, \dot{\phi}_0)$ be a particular solution of it. Let

$$X = X_0 + \varepsilon X_1 + \varepsilon^2 \frac{X_2}{2} + \dots + \varepsilon^k \frac{X_k}{k!} + \dots$$

then

$$\dot{X} = \dot{X}_0 + \varepsilon \dot{X}_1 + \varepsilon^2 \frac{\dot{X}_2}{2} + \dots + \varepsilon^k \frac{\dot{X}_k}{k!} + \dots$$

and using a Taylor expansion, we get

$$\dot{X} = F(X_0) + F'(X_0)\delta + F''(X_0)\frac{\delta^2}{2} + \dots + F^{(i)}(X_0)\frac{\delta^i}{i!} + \dots$$

where $\delta = X - X_0 = \varepsilon X_1 + \varepsilon^2 \frac{X_2}{2} + \dots + \varepsilon^k \frac{X_k}{k!} + \dots$.

So for each $k \geq 1$, equalling the coefficients of the ε^k in both expressions of \dot{X} , we get the k th variational equation (VE_k) :

$$\begin{cases} (VE_1) \dot{X}_1 = F'(X_0)X_1 \\ (VE_2) \dot{X}_2 = F'(X_0)X_2 + F''(X_0)\overline{X_1}^2 \\ (VE_3) \dot{X}_3 = F'(X_0)X_3 + 3F''(X_0)\overline{X_1}\overline{X_2} + F^{(3)}(X_0)\overline{X_1}^3 \\ (VE_4) \dot{X}_4 = F'(X_0)X_4 + F''(X_0)(4\overline{X_1}\overline{X_3} + 3\overline{X_2}^2) + 6F^{(3)}(X_0)\overline{X_2}\overline{X_1}^2 + F^{(4)}(X_0)\overline{X_1}^4 \\ \vdots \end{cases}$$

where \bar{X}_i is a particular series solution (at 0) of (VE_i) .

Furthermore, here, for $j \geq 4$, the matrix $F^{(j)}(x_0)$ is zero, so we get the following variational equations

$$\begin{cases} (VE_1) \dot{X}_1 = F'(X_0)X_1 \\ (VE_2) \dot{X}_2 = F'(X_0)X_2 + F''(X_0)\bar{X}_1^2 \\ (VE_3) \dot{X}_3 = F'(X_0)X_3 + 3F''(X_0)\bar{X}_1\bar{X}_2 + F^{(3)}(X_0)\bar{X}_1^3 \\ (VE_4) \dot{X}_4 = F'(X_0)X_4 + F''(X_0)(4\bar{X}_1\bar{X}_3 + 3\bar{X}_2^2) + 6F^{(3)}(X_0)\bar{X}_2\bar{X}_1^2 \\ (VE_5) \dot{X}_5 = F'(X_0)X_5 + F''(X_0)(10\bar{X}_2\bar{X}_3 + 5\bar{X}_1\bar{X}_4) + \\ \quad F^{(3)}(X_0)(10\bar{X}_3\bar{X}_1^2 + 15\bar{X}_2^2\bar{X}_1) \\ \vdots \end{cases}$$

When $\lambda = \frac{-2m^2}{(n+1)(n+2)}$ (with $n \in \mathbf{N}$), the first variational equation has formal solutions at 0 of valuations $n + 2 - 1, -n - 1 - 1, -2 - 1$ and $3 - 1$. We choose for \bar{X}_1 the solution of valuation $-n - 2$. Then we apply the method of variation of constants. Let Y be a solution of (VE_2) then $Y = \mu V$ where V is a fundamental matrix of series solutions of (VE_1) (at the point 0). We get $\mu' = V^{-1}(F''(X_0)\bar{X}_1^2)$. The coefficient in t of degree -1 of μ' is the null vector. So we may integrate and obtain μ as a (vector of) power series, from which we deduce $\bar{X}_2 = \mu V$ and we go on. We now want to study the third variational equation. We consider $Y = vV$ and we compute v' .

For $n \in \{0, \dots, 10\}$, we always find a *generical* non zero residue at this stage (which means that the local solution will have a logarithm and that the system will not be integrable). For all values of Λ and m , we can always choose a level surface h such that this residue will be non zero except in two situations : when $n = 0$ and $\Lambda = \lambda = -m^2$ and when $n = 1$ and $\Lambda = \lambda = -m^2/3$ (see annex 5.2).

In both cases the residues cancel for the third variational equation. We made the computations choosing \bar{X}_1 as a linear combination of the series solutions $Y_{i,j}$ and found no non zero residue performing the computations until the 8th variational equation. So we tried to look for (polynomial) first integrals and found an additional one, in involution with H in both cases. As the degree of freedom of the system is 2 we have proved the complete integrability of the system in the two cases $\Lambda = \lambda = -m^2$ and $\Lambda = \lambda = -m^2/3$.

We sum up our results in the following theorem:

Theorem 1 *Let*

$$H = \frac{1}{2}(\dot{a}^2 + \dot{\phi}^2) + \frac{1}{2}a^2 + \frac{1}{2}\phi^2 - \frac{1}{2}m^2a^2\phi^2 + \frac{1}{4}\lambda\phi^4 + \frac{1}{4}\Lambda a^4$$

be the hamiltonian function of the Friedmann-Robertson-Walker system and let n be such that

$$\lambda = \frac{-2m^2}{(n+1)(n+2)} \quad (\text{resp. } \Lambda = \frac{-2m^2}{(n+1)(n+2)}).$$

1. *If $n \notin \mathbf{N}$, then the differential Galois group of the first variational equation is not virtually abelian and the system is not completely integrable.*

2. *For $n \in \{2, \dots, 10\}$, then the differential Galois group of the third variational equation is not virtually abelian and the system is not completely integrable.*
3. *If $n \in \{0, 1\}$ and $\lambda \neq \Lambda$, then the differential Galois group of the third variational equation is not virtually abelian and the system is not completely integrable.*
4. *If $n \in \{0, 1\}$ and $\lambda = \Lambda$, then there is an additional first integral I_n in involution with H and functionally independent of H :*

- *if $n=0$ and $\lambda = \Lambda = -m^2$,*

$$I_0 = \phi \dot{a} - a \dot{\phi}$$

- *if $n=1$ and $\lambda = \Lambda = -m^2/3$,*

$$I_1 = \dot{a}\dot{\phi} + a\phi - \frac{m^2}{3}(a\phi^3 + a^3\phi)$$

so the system is completely integrable in these two cases.

Proof

For the first point, the proof is in section 3 (using the level surface $h = 0$ and the first variational equation). For the second and the third points, the proof is in section 4 and in annex 5.2 (using the level surface $h \notin \{0, -1/(4\lambda)\}$ and the third variational equation).

Lastly, for the fourth point, it suffices to check that $dI_n(X)$ cancels when X is solution to the hamiltonian system; that $\langle \nabla H, J\nabla I_n \rangle$ cancels and that ∇H and ∇I_n are linearly independent. That proves that the hamiltonian system has *two* first integrals I_n and H which are in involution and functionally independent. As the degree of freedom of the system is *two*, we conclude that the system is completely integrable. \square

Now here is the announced conjecture:

Conjecture 1 *The Friedmann-Robertson-Walker system is completely integrable if and only if*

$$n \in \{0, 1\} \text{ and } \Lambda = \lambda = \frac{-2m^2}{(n+1)(n+2)}.$$

According to proposition 1, to prove completely this conjecture, it suffices to prove that the third variational equation has a non virtually abelian Galois group when n is an integer ≥ 11 and such that $\lambda = \frac{-2m^2}{(n+1)(n+2)}$. It seems that for $n \geq 2$, the residue (see appendix 5.2) always depends on h and hence is not zero.

V. ANNEX

A. Variational equations and logarithmic terms

We give the intermediate computations for the k th variational equations where $k = 1, \dots, 3$ without giving the complete developments in series of the solutions. The computations were made using Maple.

- The matrices $F^{(k)}(X_0)$.

$$F''(X_0) = \begin{pmatrix} 0_{2,2} & 0_{2,2} & 0_{2,2} & 0_{2,10} \\ A_1 & 0_{2,2} & A_2 & 0_{2,10} \end{pmatrix}$$

$$F^{(3)}(X_0) = \begin{pmatrix} 0_{2,2} & 0_{2,2} & 0_{2,2} & 0_{2,10} & 0_{2,2} & 0_{2,2} & 0_{2,2} & 0_{2,42} \\ A_3 & 0_{2,2} & A_4 & 0_{2,10} & A_4 & 0_{2,2} & A_5 & 0_{2,42} \end{pmatrix}$$

$$F^{(4)}(X_0) = 0$$

where

$$A_1 = \begin{pmatrix} 0 & 2m^2\phi_0 \\ 2m^2\phi_0 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 2m^2\phi_0 & 0 \\ 0 & -6\lambda\phi_0 \end{pmatrix}$$

$$A_3 = \begin{pmatrix} -6\lambda & 0 \\ 0 & 2m^2 \end{pmatrix}, A_4 = \begin{pmatrix} 0 & 2m^2 \\ 2m^2 & 0 \end{pmatrix}, A_5 = \begin{pmatrix} 2m^2 & 0 \\ 0 & -6\lambda \end{pmatrix}$$

- First variational equation

$$(VE_1); Y' = F'(X_0) Y$$

A fundamental system of solutions is

$$V = (Y_{1,1} \ Y_{1,2} \ Y_{2,1} \ Y_{2,2})$$

For $i = 1, 2$, $Y_{1,i} = {}^t (y_{1,i}, 0, y_{1,i}, 0)$ and $Y_{2,i} = {}^t (0, y_{2,i}, 0, y_{2,i})$, where $y_{1,1}$ has valuation $-n-1$, $y_{1,2}$ has valuation $n+2$; $y_{2,1}$ has valuation -2 and $y_{2,2}$ has valuation 3 . Let

$$\bar{X}_1 = Y_{1,1}$$

- Second variational equation

$$(VE_2); Y' = F'(X_0) Y + B_2$$

with

$$B_2 = F''(X_0)\bar{X}_1^2 = {}^t (0, 0, 0, 2m^2\phi_0(t)y_{1,1}^2)$$

A solution of (VE_2) satisfies

$$\bar{X}_2(t) = \mu(t) V(t)$$

with

$$\mu'(t) = V^{-1} B_2 = {}^t \left(0, 0, \frac{-2m^2 y_{2,2} y_{1,1}^2 \phi_0(t)}{d_2}, \frac{2m^2 y_{2,1} y_{1,1}^2 \phi_0(t)}{d_2} \right)$$

and $d_2 = y_{2,1} y_{2,2} - y_{2,2} y_{2,1}$.

There is no term of degree -1 in t in the development in 0 of $\mu'(t)$.

- Third variational equation

$$(VE_3); Y' = F'(X_0) Y + B_3$$

with

$$B_3 = 3F''(X_0)\bar{X}_1\bar{X}_2 + F^{(3)}(X_0)\bar{X}_1^3$$

$$B_3 = {}^t (0, 0, 6y_{1,1}(m^2\phi_0(t)(\mu[3]y_{2,1} + \mu[4]y_{2,2}) - \Lambda y_{1,1}^2), 0)$$

A solution of (VE_3) satisfies

$$\bar{X}_3(t) = v(t) V(t)$$

with

$$v'(t) = V^{-1} B_3 = {}^t \left(\frac{-6y_{1,2}y_{1,1}(m^2\phi_0(t)(\mu[3]y_{2,1} + \mu[4]y_{2,2}) - \Lambda y_{1,1}^2)}{d_1}, \frac{6y_{1,1}y_{1,1}(m^2\phi_0(t)(\mu[3]y_{2,1} + \mu[4]y_{2,2}) - \Lambda y_{1,1}^2)}{d_1}, 0, 0 \right)$$

and $d_1 = y_{1,1} y_{1,2} - y_{1,2} y_{1,1}$.

There is a generical non zero residue in the expression of $v'[1]$ for $n \in \{0, \dots, 9\}$ as is shown in next tabular.

B. Numerical results

n	residue of $v'[1](t)$ (third variational equation)
0	$m^2 + \Lambda$ (cancels iff $\Lambda = -m^2 = \lambda$)
1	$-3\Lambda - m^2$ (cancels iff $\Lambda = -m^2/3 = \lambda$)
2	$200hm^4 - 56m^2 + 75\Lambda hm^2 + 144\Lambda$
3	$-23079\Lambda hm^2 - 16080\Lambda - 102116hm^4 + 25880m^2$
4	$2901720\Lambda hm^2 + 1070400\Lambda - 2294240m^2 +$ $11336508hm^4 + 1806675h^2m^6 + 245310\Lambda h^2m^4$
5	$-65602125\Lambda hm^2 - 15132600\Lambda + 38370360m^2 -$ $233835475hm^4 - 16308985\Lambda h^2m^4 - 115917879h^2m^6$
6	$-4636672395264m^2 + 34065966173760hm^4 + 10266629808960\Lambda hm^2 +$ $1629059475456\Lambda + 5029205015016\Lambda h^2m^4 + 34946518111596h^2m^6 +$ $176211274785h^3m^6\Lambda + 1776954866960h^3m^8$
7	$-884443483627200\Lambda hm^2 - 102737197946880\Lambda + 318902786173440m^2 -$ $2769109330856400hm^4 - 713790270416328\Lambda h^2m^4 - 4888091932092711h^2m^6 -$ $74435577371925h^3m^6\Lambda - 749842264480100h^3m^8$
8	$18188769231360000\Lambda hm^2 + 1616329562112000\Lambda - 5374385551872000m^2 +$ $54256909133952000hm^4 + 148014408493728720h^2m^6 + 45655069391570927h^3m^8 +$ $1156297133832750h^4m^{10} + 87598957238520h^4m^8\Lambda + 21816271124470080\Lambda h^2m^4 +$ $4521413998439388h^3m^6\Lambda$
9	$-200227669786811520000\Lambda hm^2 - 14066834095480320000\Lambda + 49499389514082816000m^2 -$ $573191906897586816000hm^4 - 6638577966765689670h^4m^8\Lambda - 333689292899038270800\Lambda h^2m^4 -$ $2251082650346871561360h^2m^6 - 88506922209720557950h^4m^{10} - 114667992412647561855h^3m^6\Lambda -$ $1163101502461355812811h^3m^8$
10	$56049210857730877610302800h^3m^8 + 8543161503524246389650552h^4m^{10} +$ $71977144524368931702817920h^2m^6 + 7837671931869489710000h^5m^{10}\Lambda +$ $130350944562836789792000h^5m^{12} + 4847238854979929210880000\Lambda hm^2 +$ $276184874161891639296000\Lambda - 1019619628144943549644800m^2 +$ $13391281016157722259456000hm^4 + 633972179303882072411835h^4m^8\Lambda +$ $10703074858075603962921600\Lambda h^2m^4 + 5493288230826857567469000h^3m^6\Lambda$

C. The particular case $\lambda = 0$

If $\lambda = 0$, we choose a particular solution on the level surface $h = 0$, $X_0 = (0, \phi_0, 0, \dot{\phi}_0)$ where $\phi_0(t) = e^{it}$. The variational equation becomes

$$Y' = \begin{pmatrix} \begin{array}{|cc|} \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array} & \begin{array}{|cc|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} \\ \hline \begin{array}{|cc|} \hline -1 + m^2 e^{2it} & 0 \\ \hline 0 & -1 \\ \hline \end{array} & \begin{array}{|cc|} \hline 0 & 0 \\ \hline 0 & 0 \\ \hline \end{array} \end{pmatrix} Y$$

The scalar equation $y''(t) = (m^2 e^{2it} - 1)y(t)$ is equivalent (with $x = e^{2it}$) to:

$$4x^2 y''(x) + 4xy'(x) + (m^2 x - 1)y(x) = 0$$

The exponents at 0 are $-1/2$ and $1/2$, they differ from an integer and one finds formal solutions at 0 with log when $m \neq 0$. Furthermore, the exponents at infinity are $1/4 + m/2i\sqrt{x}$ and $1/4 - m/2i\sqrt{x}$ so the sum of the exponents at 0 and at infinity cannot be an integer : there is no exponential solution and the equation is irreducible. One concludes to the non integrability using our criterion from [2, 3].

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