An Analytical Solution for the Critical Number of Particles for Stable Bose-Einstein Condensation under the Influence of an Anisotropic Potential

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We have considered a Bose gas in an anisotropic potential. Applying the the Gross-Pitaevskii Equation (GPE) for a confined dilute atomic gas, we have used the methods of optimized perturbation theory and self-similar root approximants, to obtain an analytical formula for the critical number of particles as a function of the anisotropy parameter for the potential. The spectrum of the GPE is also discussed.

Keywords: Bose gas; Anisotropic potential; Critical number of particles; Optimized perturbation theory and critical coupling parameter

I. INTRODUCTION

The experimental realization of BEC with alkali atoms [1-5] opened a new and exciting field involving quantum atomic fluids. This new field has been launched with many interesting theoretical predictions and ideas. When the system is spatially confined with atomic interaction effectively attractive, BEC is stable only below a certain critical value. In this case, the zero-point motion of the atoms serves as a kinetic obstacle against collapse, allowing a metastable BEC to be formed [6-10]. Above the critical number of particles the colapse is unavoidable.

When the ground-state energy becomes complex, the system becomes unstable. That is a general way to identify the critical value for the coupling parameter in the system as well as identifying the critical number of particles.

It was shown by Gammal et al [11-12] within the Gross-Pitaevskii formalism, that the critical number of particles for BEC in cylindrical traps can be obtained numerically for the case attractive interactions. It was also calculated in reference [13] the spectrum of the Gross-Pitaevskii Equation (GPE) for a system composed of attractive bosons confined in a harmonic trap through the Controlled Perturbation Theory. In reference [13], the critical number of particles that ensures real values for the energy spectrum was obtained as a function of the potential anisotropic parameter.

The aim of this work is to consider the analytical spectrum of the Gross-Pitaevskii equation through approximants crossover roots [16] with negative effective interaction strength in a cylindrical symmetry. An analytical formula was derived to analyze the stable critical number of particles, N_c as a function of the anisotropy of the confining potential. Here we used controlled perturbation theory [14] to derive numerically the spectrum of the Gross-Pitaevskii equation, and approximants crossover roots [15] to obtain the analytical expression for the spectrum of the Gross-Pitaevskii equation.

In fact, we used controlled perturbation theory instead of standard (ordinary) perturbation theory since the first one uses control functions that optimize the convergence [13].

II. MODEL

Bose system is confined by a trapping potential given by

$$U = \frac{1}{2}\left(\omega_x^2 r_x^2 + \omega_y^2 r_y^2 + \omega_z^2 r_z^2\right) \tag{1}$$

The interaction potential is considered as a Fermi contact potential as

$$\Phi(\vec{r}) = A\delta(\vec{r}) \quad , \quad \text{where} \quad A = 4\pi h^2 \frac{a_s}{m_0} \tag{2}$$

with a_s being the s-wave scattering length. With the confining potential and the interaction potential, we get the nonlinear Hamiltonian

$$\mathrm{H}(\varphi) = -\frac{\hbar^2\nabla^2}{2m_0} + U(\vec{r}) + NA|\varphi|^2 . \tag{3}$$

Due to the presence of this confining potential, the spectrum of the stationary Gross-Pitaevskii Equation (GPE) is discrete, being defined by the eigenvalue problem

$$\mathrm{H}[\varphi_n]\varphi_n(\vec{r}) = \mathrm{E}_n\varphi_n(\vec{r}) . \tag{4}$$

When the confining presents cylindrically symmetric, so that

$$\omega_x = \omega_y = \omega_r \quad , \tag{5}$$

hence anisotropy parameter is defined as

$$\lambda = \frac{\omega_z}{\omega_r} \quad . \tag{6}$$

The radial oscillator length

$$l_r \equiv \sqrt{\frac{\hbar}{m_0 w_r}} \tag{7}$$

serves to define the dimensionless cylindrical variables, and

$$\vec{r} \equiv \frac{\sqrt{r_x^2 + r_y^2}}{l_r}, \quad z = \frac{r_z}{l_r}. \tag{8}$$

Then one may define the dimensionless coherent wave function

$$\psi(r, \varphi, z) \equiv l_r^{3/2} \varphi(\vec{r})$$

depending on the cylindrical variables $r \in (0, \infty)$, $\varphi \in (0, 2\pi)$ and $z \in (\infty, -\infty)$ and the Hamiltonian, H, is defined as

$$H \equiv \frac{H(\varphi(\vec{r}))}{\hbar\omega_r}$$

The atom-atom coupling parameter can be expressed as

$$g = 4\pi \frac{a_s}{l_r} N. \tag{9}$$

Thus we can write the Hamiltonian in the form

$$\hat{\mathrm{H}} = -\frac{1}{2}\vec{\nabla}^2 + \frac{1}{2}\left(r^2 + \lambda z^2\right) - g|\psi|^2$$

where

$$\vec{\nabla}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}$$

and the coupling parameter, g , is the perturbed Hamiltonian term.

The eigenproblem (4) takes the form $\mathrm{H}\Psi_{nml} = \mathrm{E}_{nml}\Psi_{nml}$, in which $n = 0, 1, 2, ...$ is radial quantum number, $m = 0, \pm 1, \pm 2, ...$ is the azimuthal quantum number, and $l = 0, 1, 2, ...$ is the axial quantum number.

The equilibrium Bose-Einstein condensate corresponds to the ground-state solution of the stationary GPE, when $n = m = l = 0$.

The nonlinear eigenproblem cannot be solved exactly. We can employ the optimized perturbation theory to find the accurate expressions for the spectrum of GPE and for arbitrary values of the coupling parameter. This theory has been applied in several physical problems [13-14]. This approach has been used for calculating the critical temperature of Bose-Einsten condensation in a Dilute Bose Gas [14]. The introduction of optimized perturbation theory provides us the control functions that are optimal choice and provoke the optimal convergence of a calculational procedure [14], this theory was introduced by Yukalov [16].

In applying the optimized perturbation theory to the GPE, the spectrum, in first approximation, can be written as:

$$e(g) \equiv \mathrm{E}(g, u(g), v(g))$$

with the expression

$$\mathrm{E}(g, u, v) = \frac{p}{2}\left(u + \frac{1}{u}\right) + \frac{q}{4}\left(v + \frac{\lambda^2}{v}\right) - \frac{1}{2}\frac{su\sqrt{\lambda}}{vp\sqrt{q}} \tag{10}$$

where p and q are defined as

$$p = 2n + |m| + 1\,,\ q \equiv 2l + 1. \tag{11}$$

The related eigenfunction are [17]

$$\Psi_{nml}^{(0)}(r, \varphi, z) = \left[\frac{2n!u^{|m|+1}}{(n+|m|)!}\right] r^{|m|} \exp\left(-\frac{1}{2}ur^2\right) L_n^{|m|}.\left(ur^2\right) \frac{e^{im\varphi}}{\sqrt{2\pi}} \frac{(v/\pi)^{1/4}}{\sqrt{2^l l!}} \exp\left(-\frac{1}{2}vz^2\right) \mathrm{H}_l\sqrt{v}z \tag{12}$$

where $L_n^m(.)$ are the Laguerre polynomials and $H_k(.)$ are Hermite polynomials.

The fixed-point conditions are, therefore, obtained from conditions

$$\frac{\partial}{\partial u}\mathrm{E}(g, u, v) = 0\,,\ \frac{\partial}{\partial v}\mathrm{E}(g, u, v) = 0$$

which yield the control function equations

$$p\left(1 - \frac{1}{u^2}\right) - \frac{s}{p\lambda} = 0, \quad q\left(1 - \frac{\lambda^2}{v^2}\right) - \frac{s}{p\lambda\sqrt{\lambda q}} = 0 \tag{13}$$

and the effective interaction strength is represented by

$$s \equiv 2p\sqrt{q} I_{nml} \lambda g \quad . \tag{14}$$

Equations (5)-(7) define the spectrum (4) for all quantum numbers $n, m\, and\, l$ and for arbitrary values of the coupling

parameter g. The term I_{nml} can be related with the coefficients a_n through a recursion formula showed in reference [13]. We derive the expressions for the spectrum of GPE and find weak-coupling
limit for the spectrum as

$$e(g) \approx \sum_{n=0}^{k} a_n s_n \quad (s \to 0) \tag{15}$$

with the coefficients

$$a_0 = p + \frac{q\lambda}{2}, \quad a_1 = -\frac{1}{2p\,(q\lambda)^{1/2}}$$

$$a_2 = -\frac{p+2q\lambda}{16p^3\,(q\lambda)^2}, \quad a_3 = -\frac{(p+2q\lambda)^2}{64p^5\,(q\lambda)^{7/2}}.$$

For the strong-coupling limit

$$e(g) = \sum_{n=0}^{k} b_n s^{\beta_n} \quad (s \to \infty) \tag{16}$$

where the coefficients are given by the equations

$$b_0 = \frac{5}{4} \quad 4b_1 = 2p^2 - (q\lambda)^2,$$

$$20b_2 = -3p^4 + 2p^2\,(q\lambda)^2 - 2\,(q\lambda)^4.$$

and the exponents are

$$\beta_0 = \tfrac{2}{5}, \quad \beta_1 = -\tfrac{2}{5}$$
$$\beta_2 = -\tfrac{6}{5}.$$

interpolating the asymptotic limits (15) and (16) by means of the self-similar root approximants [17-18], that results in the following approximants crossover roots function

$$f_b^*(g) = a_0 \left(\ldots \left\{ \left[(1 + A_{b1} s)^{n_1} + A_{b2} s^2 \right]^{n_2} + A_{b3} s^3 \right\}^3 + \ldots + A_k s_k \right)^{n_k}. \tag{17}$$

Here, depending on the approximation order $b = 1, 2, \ldots$ we have the values of A and n in the first order

$$A_{11} = \frac{1.746928}{a_0^{2/5}}, \quad n_{11} = \frac{2}{5};$$

in the second order

$$A_{21} = 2.533913 \frac{\left(2p^2 + (q\lambda)^2\right)^{5/6}}{a_0^{25/6}}, \quad A_{22} = \frac{3.051758}{a_0^5}$$

$$n_{21} = \frac{6}{5}, \quad n_{22} = \frac{1}{5};$$

and in the third order

$$A_{31} = 1.405455 \frac{\left(8p^4 + 12p^2\,(q\lambda)^2 + (q\lambda)^4\right)^{5/6}}{a_0^{125/22}\left[2p^2 + (q\lambda)^2\right]^{5/66}}, \quad n_{31} = \frac{6}{5}$$

$$A_{32} = 6.619620 \frac{\left(2p^2 + (q\lambda)^2\right)^{10/11}}{a_0^{75/11}}, \quad n_{32} = \frac{11}{10}$$

$$A_{33} = \frac{5.331202}{a_0^{15/2}}, \quad n_{33} = \frac{2}{15}.$$

This model was described in several references [16-18].

III. RESULTS

Analysis for the critical number of particles in different traps are performed using an analytical expression from the self-similar root approximants to get an expression for N_c. We need to find the negative value of s_c in which the spectrum (10) become complex. We limit the first order of root approximant,$E_1^*(g)$, which is complex for $s < s_c$, with

$$s_c = -0.572433 a_0^{5/2}.$$

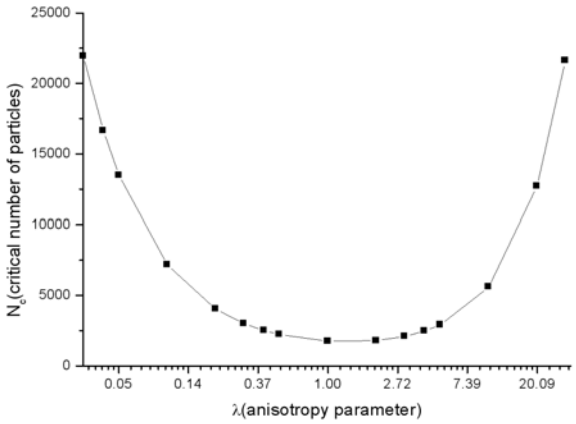

FIG. 1: The behaviour of the critical number of particles in function of several different traps.

Using the eq. (14), we get the equation for the critical coupling parameter

$$g_c = -0.05\frac{(2p+q\lambda)^{5/2}}{p\sqrt{q}I_{nmj}\lambda}.$$

The equation $g = 4\pi\frac{a_s}{l_r}N$, gives us the expression for the critical number of particles

$$N_c = \frac{(2p+q\lambda)^{5/2}}{300p\sqrt{q}I_{nmj}\lambda}\left|\frac{l_r}{a_s}\right|. \tag{18}$$

Using the analytical expression (18) with the values of ^{7}Li Condensate [18-19] and considering the ground state, with $n=m=l=0$, $p=l=1$ and $I_{000}=0.063494$,we can obtain N$_c$ in Figure 1.

From these results, we observe that there are two regimes: cigar-shape ($\lambda << 1$) and the disc-shape ($\lambda >> 1$). Both situations increase N$_c$, while in a spherical-shape N$_c$ reaches its the minimal value. For example, in a cigar-shape trap $\lambda = 0.1$ *and* $N_c = 7157$, in a disc-shape trap $\lambda = 10$ *and* $N_c = 5586$ and in a spherical-shape trap $\lambda = 1$ *with* $N_c = 1350$.These values are in a good agreement with those in references [11,13].

IV. CONCLUSION

The method of the self-similar root approximants resulted in analytical expressions for the spectrum of energy levels, for the critical coupling parameter and for the critical number of particles. We have analytically studied the behaviour of the critical number of particles as a function of the anisotropy parameter, which introduced a better stability in the system.

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