Actions for the Bosonic String with the Curved Worldsheet

Davoud Kamani
Faculty of Physics, Amirkabir University of Technology (Tehran Polytechnic)
P.O.Box: 15875-4413, Tehran, Iran
e-mail: kamani@cic.aut.ac.ir

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At first we introduce an action for the string, which leads to a worldsheet that always is curved. For this action we study the Poincaré symmetry and the associated conserved currents. Then, a generalization of the above action, which contains an arbitrary function of the two-dimensional scalar curvature, will be introduced. An extra scalar field enables us to modify these actions to Weyl invariant models.

Keywords: Curved worldsheet; 2d scalar curvature; Poincaré symmetry

I. INTRODUCTION

The two-dimensional models have widely been used in the context of the two-dimensional gravity (*e.g.* see [1–4] and references therein) and string theory. From the 2d-gravity point of view, higher-dimensional gravity models, by dimensional reduction reduce to the 2d-gravity [1–3]. From the string theory point of view, the (1+1)-dimensional actions are fundamental tools of the theory. However, 2d-gravity and 2d- string theory are closely related to each other.

The known sigma models for string, in the presence of the dilaton field $\Phi(X)$, contain the two-dimensional scalar curvature $R(h_{ab})$,

$$S_\Phi = \frac{1}{4\pi}\int d^2\sigma\sqrt{h}R\Phi(X). \tag{1}$$

In two dimensions the combination $\sqrt{h}R$ is total derivative. Thus, in the absence of the dilaton field, this action is a topological invariant that gives no dynamics to the worldsheet metric h_{ab}.

In fact, in the action (1), the dilaton is not the only choice. For example, replacing the dilaton field with the scalar curvature R, leads to the R^2-gravity [1, 4, 5]. In particular the Polyakov action is replaced by a special combination of the worldsheet fields, which include an overall factor R^{-1}. Removing the dilaton and replacing it with another quantities motivated us to study a class of two-dimensional actions. They are useful in the context of the non-critical strings with curved worldsheet, and the 2-dimensional gravity.

Instead of the dilaton field, we introduce some combinations of h_{ab}, R and the induced metric on the worldsheet, *i.e.* γ_{ab}, which give dynamics to h_{ab}. These non-linear combinations can contain an arbitrary function $f(R)$ of the scalar curvature R. We observe that these dynamics lead to the constraint equation for h_{ab}, extracted from the Polyakov action.

For the flat spacetime, these models have the Poincaré symmetry. In addition, they are reparametrization invariant. However, for any function $f(R)$, they do not have the Weyl symmetry. Therefore, the string worldsheet at most is conformally flat. By introducing an extra scalar field in these actions, they also find the Weyl symmetry. Note that a Weyl non-invariant string theory has noncritical dimension, *e.g.* see [6].

This paper is organized as follows. In section 2, we introduce a new action for the string in which the corresponding worldsheet always is curved. In section 3, the Poincaré symmetry of this string model will be studied. In section 4, the generalized form of the above action will be introduced and it will be analyzed.

II. CURVED WORLDSHEET IN THE CURVED SPACETIME

We consider the following action for the string, which propagates in the curved spacetime

$$S = -T\int d^2\sigma\sqrt{h}R\left(R - \frac{1}{2\pi\alpha'}h^{ab}\gamma_{ab}\right), \tag{2}$$

where $h = -\det h_{ab}$, and T is a dimensionless constant. In addition, R denotes the two-dimensional scalar curvature which is made from h_{ab}. The string coordinates are $\{X^\mu(\sigma,\tau)\}$. The induced metric on the worldsheet, *i.e.* γ_{ab}, is also given by

$$\gamma_{ab} = g_{\mu\nu}(X)\partial_a X^\mu(\sigma,\tau)\partial_b X^\nu(\sigma,\tau), \tag{3}$$

where $g_{\mu\nu}(X)$ is the spacetime metric.

In two dimensions, the symmetries of the curvature tensor imply the identity

$$R_{ab} - \frac{1}{2}h_{ab}R = 0. \tag{4}$$

Therefore, the variation of the action (2) leads to the following equation of motion for h^{ab},

$$R_{ab} - \frac{1}{2\pi\alpha'}\gamma_{ab} = 0. \tag{5}$$

This implies that the energy-momentum tensor, extracted from the action (2), vanishes.

Contraction of this equation by h^{ab} gives $R = \frac{1}{2\pi\alpha'}h^{ab}\gamma_{ab}$. Introducing this equation and the equation (5) into (4) leads to

$$T^{(\text{Polyakov})}_{ab} \equiv \gamma_{ab} - \frac{1}{2}h_{ab}(h^{a'b'}\gamma_{a'b'}) = 0. \tag{6}$$

This is the constraint equation, extracted from the Polyakov action. Note that the energy-momentum tensor, due to the action (2), is proportional to the left-hand-side of the equation (5). Thus, it is different from (6).

The equation of motion of the string coordinate $X^\mu(\sigma,\tau)$ also is

$$\partial_a(\sqrt{h}Rh^{ab}\partial_b X^\mu) + \sqrt{h}Rh^{ab}\Gamma^\mu_{\nu\lambda}\partial_a X^\nu \partial_b X^\lambda = 0. \qquad (7)$$

Presence of the scalar curvature R distinguishes this equation from its analog, extracted from the Polyakov action.

Now consider those solutions of the equations of motion (5) and (7), which admit constant scalar curvature R. For these solutions, the equation (7) reduces to the equation of motion of the string coordinates, extracted from the Polyakov action with the curved background. However, for general solutions the scalar curvature R depends on the worldsheet coordinates σ and τ, and hence this coincidence does not occur.

A. The model in the conformal gauge

Under reparametrization of σ and τ, the action (2) is invariant. That is, in two dimensions the general coordinate transformations $\sigma \to \sigma'(\sigma,\tau)$ and $\tau \to \tau'(\sigma,\tau)$, depend on two free functions, namely the new coordinates σ' and τ'. By means of such transformations any two of the three independent components of h_{ab} can be eliminated. A standard choice is a parametrization of the worldsheet such that

$$h_{ab} = e^{\phi(\sigma,\tau)}\eta_{ab}, \qquad (8)$$

where $\eta_{ab} = diag(-1,1)$, and $e^{\phi(\sigma,\tau)}$ is an unknown conformal factor. The choice (8) is called the conformal gauge. Since the action (2) does not have the Weyl symmetry (a local rescaling of the worldsheet metric h_{ab}) we cannot choose the gauge $h_{ab} = \eta_{ab}$.

The scalar curvature corresponding to the metric (8) is

$$R = -e^{-\phi}\partial^2\phi, \qquad (9)$$

where $\partial^2 = \eta^{ab}\partial_a\partial_b$. Thus, the action (2) reduces to

$$S' = -T\int d^2\sigma e^{-\phi}\partial^2\phi\left(\partial^2\phi + \frac{1}{2\pi\alpha'}\eta^{ab}\gamma_{ab}\right). \qquad (10)$$

According to the gauge (8), this action describes a conformally flat worldsheet.

III. POINCARÉ SYMMETRY OF THE MODEL

In this section we consider flat Minkowski space, *i.e.* $g_{\mu\nu}(X) = \eta_{\mu\nu}$. Therefore, the equations of motion are simplified to

$$R_{ab} - \frac{1}{2\pi\alpha'}\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu = 0, \qquad (11)$$

$$\partial_a(\sqrt{h}Rh^{ab}\partial_b X^\mu) = 0. \qquad (12)$$

The Poincaré symmetry reflects the symmetry of the background in which the string is propagating. It is described by the transformations

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu,$$
$$\delta h^{ab} = 0, \qquad (13)$$

where $a^\mu{}_\nu$ and b^μ are independent of the worldsheet coordinates σ and τ, and $a_{\mu\nu} = \eta_{\mu\lambda}a^\lambda{}_\nu$ is antisymmetric. Thus, from the worldsheet point of view, these transformations are global symmetries. Under these transformations the action (2) is invariant.

A. The conserved currents

The Poincaré invariance of the action (2) is associated to the following Noether currents

$$\mathcal{J}^{\mu\nu a} = \frac{T}{2\pi\alpha'}\sqrt{h}Rh^{ab}(X^\mu\partial_b X^\nu - X^\nu\partial_b X^\mu),$$
$$\mathcal{P}^{\mu a} = \frac{T}{2\pi\alpha'}\sqrt{h}Rh^{ab}\partial_b X^\mu, \qquad (14)$$

where the current $\mathcal{P}^{\mu a}$ is corresponding to the translation invariance and $\mathcal{J}^{\mu\nu a}$ is the current associated to the Lorentz symmetry. According to the equation of motion (12) these are conserved currents

$$\partial_a\mathcal{J}^{\mu\nu a} = 0,$$
$$\partial_a\mathcal{P}^{\mu a} = 0. \qquad (15)$$

B. The covariantly conserved currents

It is possible to construct two other currents from (14), in which they be covariantly conserved. For this, there is the useful formula

$$\nabla_a K^a = \frac{1}{\sqrt{h}}\partial_a(\sqrt{h}K^a), \qquad (16)$$

where K^a is a worldsheet vector. Therefore, we define the currents $J^{\mu\nu a}$ and $P^{\mu a}$ as in the following

$$J^{\mu\nu a} = \frac{1}{\sqrt{h}}\mathcal{J}^{\mu\nu a},$$
$$P^{\mu a} = \frac{1}{\sqrt{h}}\mathcal{P}^{\mu a}. \qquad (17)$$

According to the equations (15) and (16), these are covariantly conserved currents, *i.e.*,

$$\nabla_a J^{\mu\nu a} = \nabla_a P^{\mu a} = 0. \qquad (18)$$

The currents (17) can also be written as

$$J^{\mu\nu}_a = \frac{T}{2\pi\alpha'}R(X^\mu\partial_a X^\nu - X^\nu\partial_a X^\mu),$$
$$P^\mu_a = \frac{T}{2\pi\alpha'}R\partial_a X^\mu. \qquad (19)$$

Since there is $\nabla_a h_{bc} = 0$, the conservation laws (18) also imply the covariantly conservation of the currents (19).

IV. GENERALIZATION OF THE MODEL

The generalized form of the action (2) is

$$I=-T\int d^2\sigma\sqrt{h}R\left(f(R)-\frac{1}{2\pi\alpha'}h^{ab}\gamma_{ab}\right), \tag{20}$$

where $f(R)$ is an arbitrary differentiable function of the scalar curvature R. The set $\{X^\mu(\sigma,\tau)\}$ describes a string worldsheet in the spacetime. These string coordinates appeared in the induced metric γ_{ab} through the equation (3). Thus, (20) is a model for the string action.

The equation of motion of X^μ is as previous, *i.e.* (7). Vanishing the variation of this action with respect to the worldsheet metric h^{ab}, gives the equation of motion of h^{ab},

$$R_{ab}\frac{df(R)}{dR}-\frac{1}{2\pi\alpha'}\gamma_{ab}=0. \tag{21}$$

The trace of this equation is

$$R\frac{df(R)}{dR}-\frac{1}{2\pi\alpha'}h^{ab}\gamma_{ab}=0. \tag{22}$$

Combining the equations (4), (21) and (22) again leads to the equation (6).

As an example, consider the function $f(R)=\alpha\ln R+\beta$. Thus, the field equation (21) implies that the intrinsic metric h_{ab} becomes proportional to the induced metric γ_{ab}, that is $h_{ab}=\frac{1}{\pi\alpha\alpha'}\gamma_{ab}$.

Since the Poincaré transformations contain $\delta h^{ab}=0$, the generalized action (20) for the flat background metric $g_{\mu\nu}=\eta_{\mu\nu}$, also has the Poincaré invariance. This leads to the previous conserved currents, *i.e.* (14) and (19).

A. Weyl invariance in the presence of a new scalar field

The action (20) under the reparametrization transformations is symmetric. The Weyl transformation is also defined by

$$h_{ab}\longrightarrow h'_{ab}=e^{\rho(\sigma,\tau)}h_{ab}. \tag{23}$$

Thus, the scalar curvature transforms as

$$R\longrightarrow R'=e^{-\rho}(R-\nabla^2\rho), \tag{24}$$

where $\nabla^2\rho=\frac{1}{\sqrt{h}}\partial_a(\sqrt{h}h^{ab}\partial_b\rho)$. The equations (23) and (24) imply that the action (20), for any function $f(R)$, is Weyl non-invariant.

Introducing (23) and (24) into the action (20) gives a new action which contains the field $\rho(\sigma,\tau)$,

$$I'=-T\int d^2\sigma\sqrt{h}(R-\nabla^2\rho)\left(f[e^{-\rho}(R-\nabla^2\rho)]-\frac{1}{2\pi\alpha'}e^{-\rho}h^{ab}\gamma_{ab}\right). \tag{25}$$

We can ignore the origin of this action. In other words, it is another model for string. However, under the Weyl transformations

$$\begin{aligned}h_{ab}&\longrightarrow e^{u(\sigma,\tau)}h_{ab},\\ \rho&\longrightarrow \rho-u,\end{aligned} \tag{26}$$

the action I', for any function f, is symmetric. Note that according to the definition of ∇^2 there is the transformation $\nabla^2\to e^{-u}\nabla^2$.

V. CONCLUSIONS

We considered some string actions which give dynamics to the worldsheet metric h_{ab}. Due to the absence of the Weyl invariance, these models admit at most conformally flat (but not flat) worldsheet. We observed that the constraint equation on the metric, extracted from the Polyakov action, is a special result of the field equations of our string models. Obtaining this constraint equation admits us to introduce an arbitrary function of the scalar curvature to the action. For the case $f(R)=\alpha\ln R+\beta$, the metric h_{ab} becomes proportional to the induced metric of the worldsheet.

By introducing a new degree of freedom we obtained a string action, in which for any function f is Weyl invariant.

Our string models with arbitrary $f(R)$, in the flat background have the Poincaré symmetry. The associated conserved currents are proportional to the scalar curvature R. We also constructed the covariantly conserved currents from the Poincaré currents.

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