

q -distributions in complex systems: a brief review

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The nonextensive statistical mechanics proposed by Tsallis is today an intense and growing research field. Probability distributions which emerges from the nonextensive formalism (q -distributions) have been applied to an impressive variety of problems. In particular, the role of q -distributions in the interdisciplinary field of complex systems has been expanding. Here, we make a brief review of q -exponential, q -Gaussian and q -Weibull distributions focusing some of their basic properties and recent applications. The richness of systems analyzed may indicate future directions in this field.

Keywords: q -exponential, q -Gaussian, q -Weibull, Nonextensive statistics

1. INTRODUCTION

Common characteristics of complex systems include long-range correlations, multifractality and non-Gaussian distributions with asymptotic power law behavior. Typically, such systems are not well described by approaches based on the usual statistical mechanics. In this scenario, a new formalism capable of providing a better description of complex systems is welcome. This is the case of the generalized (nonextensive) statistical mechanics proposed by Tsallis - nowadays, an intense and growing research field[1–4].

Concepts related with nonextensive statistical mechanics have found applications in a variety of disciplines including physics, chemistry, biology, mathematics, geography, economics, medicine, informatics, linguistics among others[5–7]. Probability distributions which emerge from the nonextensive formalism - also called q -distributions - have been applied to an impressive variety of problems in diverse research areas including the interdisciplinary field of complex systems.

In the present work we focus on q -exponential, q -Gaussian and q -Weibull distributions. We summarized some of their basic properties and provide useful references of recent applications. The richness of systems analyzed may indicate future directions in this research line.

2. q -EXPONENTIAL DISTRIBUTION

The q -exponential distribution is given by the probability density function (pdf)

$$p_{qe}(x) = p_0 \left[1 - (1-q) \frac{x}{x_0} \right]^{1/(1-q)} \quad (1)$$

for $1 - (1-q)x/x_0 \geq 0$. If $p_0 = (2-q)/x_0$, eq. (1) is normalized.

In the limit $q \rightarrow 1$, eq. (1) recovers the usual exponential distribution in the same way in which the q -exponential function, defined as $e_q^{-x} \equiv [1 - (1-q)x]^{1/(1-q)}$, recovers exponential function in the limit $q \rightarrow 1$ ($e_1^{-x} \equiv e^{-x}$). If $q < 1$, eq. (1) has a finite value for any finite real value of x since, by definition, $p_{qe}(x) = 0$ for $1 - (1-q)x/a < 0$. If $q > 1$, eq. (1) exhibits power law asymptotic behavior,

$$p_{qe}(x) \sim x^{-1/(q-1)}. \quad (2)$$

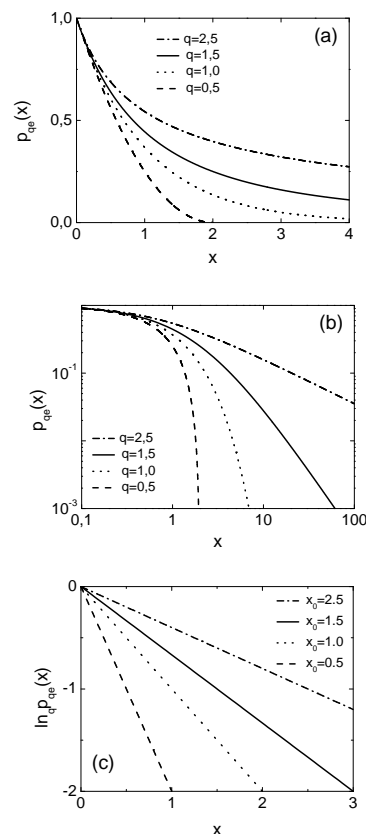


FIG. 1: q -exponential distribution. a) Plot of $p_{qe}(x)$ versus x , with $p_0 = x_0 = 1$ and typical values of q . b) Log-log plot of the curves in a). c) $\ln p_{qe}(x)$ versus x for $p_0 = 1$ and typical values of x_0 .

Note also that $p_{qe}(x) \simeq 1 + x$ for small x , independently of the q value. Figures 1a and 1b show $p_{qe}(x)$ versus x for typical values of q .

The q -exponential distribution, for $q > 1$, corresponds to the Zipf-Mandelbrot law[8] and a Burr-type distribution[9]. In this sense, the q -exponential is a generalization of these distributions for $q < 1$. Thus, by choosing suitable values for q , q -exponentials may be used to represent both short and long tailed distributions. This feature also holds for the other q -distributions.

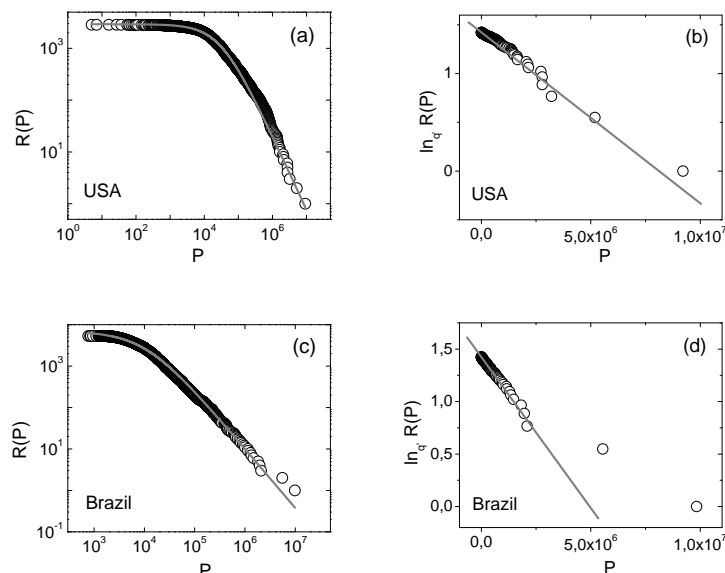


FIG. 2: **Population of cities.** a) Empirical cdf $R(P)$, where P is the population of USA cities. The solid line is a q -exponential, given by eq. (3), with $q' = 1.7$ ($q \simeq 1.4$), $x'_0 = 21,250$ and $c' = 2,919$. b) $\ln_q R(P)$ versus P , with $q' = 1.7$, for the same data shown in (a). The solid line is a linear fit to the data. c) Empirical cdf $R(P)$, where P is the population of Brazilian cities. The solid line is a q -exponential, given by eq. (3), with $q' = 1.7$ ($q \simeq 1.4$), $x'_0 = 7,073$ and $c' = 6,968$. d) $\ln_q R(P)$ versus P , with $q' = 1.7$, for the same data shown in (c). The solid line is a linear fit to the data.

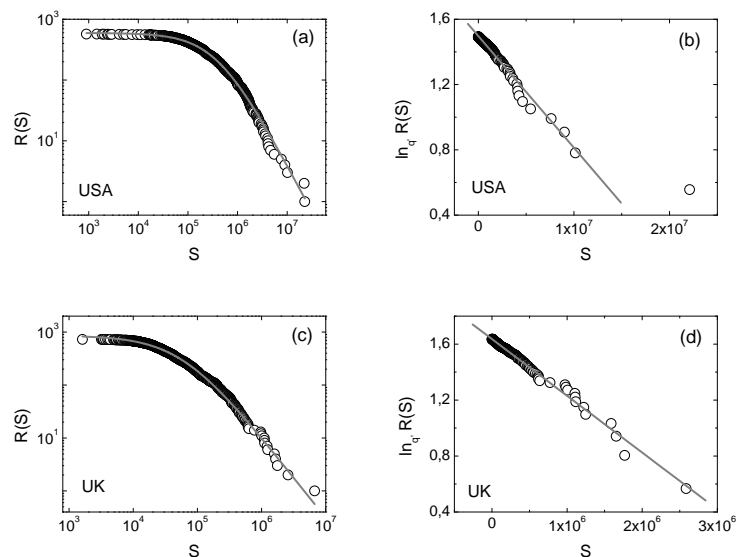


FIG. 3: **Circulation of magazines.** a) Empirical cdf $R(S)$, where S is the circulation of 570 USA magazines in 2004. The solid line is a q -exponential, given by eq. (3), with $q' = 1.65$ ($q \simeq 1.4$), $x'_0 = 255,204$ and $c' = 594$. b) $\ln_q R(S)$ versus S , with $q' = 1.65$, for the same data shown in (a). The solid line is a linear fit to the data. c) Empirical cdf $R(S)$, where S is the circulation of 727 UK magazines in 2005. The solid line is a q -exponential, given by eq. (3), with $q' = 1.65$ ($q \simeq 1.4$), $x'_0 = 37,493$ and $c' = 860$. d) $\ln_q R(S)$ versus S , with $q' = 1.65$, for the same data shown in (c). The solid line is a linear fit to the data.

The cumulative distribution function (cdf) associated to eq. (1) is given by

$$\begin{aligned}
 R_{qe}(x) &= \int_x^\infty p_{qe}(y) dy \\
 &= p'_0 \left[1 - (1 - q') \frac{x}{x'_0} \right]^{1/(1-q')}, \quad (3)
 \end{aligned}$$

defined for $q < 2$, with $q' = 1/(2 - q)$, $x'_0 = x_0/(2 - q)$ and

$p'_0 = p_0 x_0 / (2 - q)$. Observe that $R_{qe}(x)$ and $p_{qe}(x)$ exhibit the same mathematical form.

It is possible to visualize q -exponential distributions as straight lines in graphs with appropriate scales. Applying the q -logarithm function, defined as $\ln_q x \equiv [x^{(1-q)} - 1]/(1 - q)$, with $\ln_1 x \equiv \ln(x)$, in both sides of eq. (1), we have

$$\ln_q p_{qe}(x) = \ln_q p_0 - [1 + (1 - q) \ln_q p_0] \frac{x}{x_0}. \quad (4)$$

A similar result holds for $R_{qe}(x)$. Figure 1c shows $\ln_q p_{qe}(x)$ versus x for typical values of x_0 .

The q -exponential function given by eq. (1) has been employed in a growing number of theoretical and empirical works on a large variety of themes. Examples include scale-free networks[10–14], dynamical systems[15–27], algebraic structures[28–31] among other topics in statistical physics[32–36].

As specific examples of q -exponential distributions in complex systems, let us consider results on population of cities[37] and circulation of magazines[38]. Figure 2 shows the cumulative distribution of the population of cities in the USA and Brazil. Figure 3 shows the cumulative distribution of circulation of magazines in the USA and UK. In both cases - population of cities and circulation of magazines - the empirical data are consistent with a q -exponential distribution, with $q \simeq 1.4$.

q -exponential distributions have also been applied in the empirical study of stock markets[39–42], DNA sequences[43], family names[44], human behavior[45–47], geomagnetic records[48, 49], train delays[50], reaction kinetics[51], air networks[52], hydrological phenomena[53], fossil register[54], basketball[55], earthquakes[56–58], world track records[59], voting processes[60], internet[61], individual success[62], citations of scientific papers[63, 64], football[65], linguistics[66, 67] and solar neutrinos[68, 69].

3. q -GAUSSIAN DISTRIBUTION

The q -Gaussian distribution is specified by the pdf

$$p_{qg}(x) = p_0 \left[1 - (1-q) \left(\frac{x}{x_0} \right)^2 \right]^{1/(1-q)}, \quad (5)$$

for $1 - (1-q)(x/x_0)^2 \geq 0$ and $p_{qg}(x) = 0$ otherwise. It is normalized if $p_0 = (2/x_0) \sqrt{(q-1)/\pi} \Gamma[1/(q-1)] / \Gamma[1/(q-1) - 1/2]$. In addition, eq. (5) presents unit variance if $x_0^2 = 5 - 3q$, with $q < 5/3$.

In the limit $q \rightarrow 1$, eq. (5) recovers the usual Gaussian distribution, so $q \neq 1$ indicates a departure from Gaussian statistics. For $q > 1$, the tails of q -Gaussian decrease as power laws,

$$p_{qg}(|x|) \sim |x|^{-2/(q-1)}. \quad (6)$$

Figures 4a and 4b show $p_{qg}(x)$ for typical values of q .

Applying the q -logarithm function in both sides of eq. (5), we have

$$\ln_q p_{qg}(x) = \ln_q p_0 - [1 + (1-q) \ln_q p_0] \left(\frac{x}{x_0} \right)^2. \quad (7)$$

Figure 4c shows $\ln_q p_{qg}(x)$ versus x^2 for typical values of x_0 .

Recent works have been focused on the study of mathematical properties of q -Gaussian functions[70–78], including methods for generating random numbers which follow q -Gaussian distributions[79, 80]. q -Gaussians have been employed in the study of a wide range of themes including probabilistic models[81, 82], stellar plasmas[83], porous-medium equation[84], Bose-condensed gases[85–87],

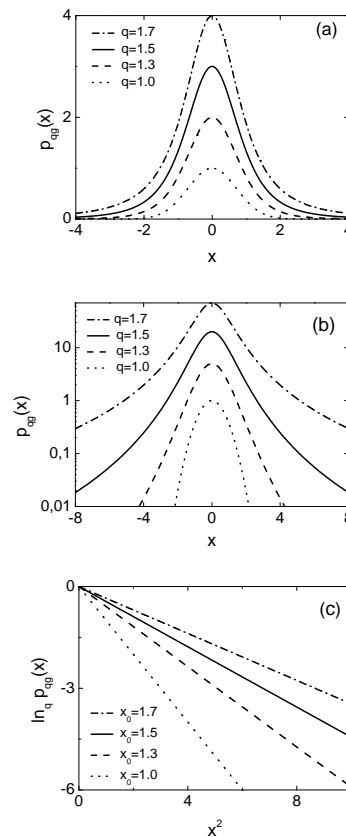


FIG. 4: q -Gaussian distribution. a) Plot of $p_{qg}(x)$ versus x , with $p_0 = x_0 = 1$, for typical values of q . Some curves were vertically shifted for a better visualization. b) The same curves shown in a), but for mono-log scale. Some curves were also shifted. c) $\ln_q p_{qg}(x)$ versus x^2 for $p_0 = 1$ and typical values of x_0 .

dynamical systems[88–90], polymeric networks[91], small-world networks[92], fingering processes[93], processes with stochastic volatility[94, 95] and nonlinear diffusion[96, 97].

In order to illustrate a recent application of q -Gaussian distributions in complex systems, we mention here results on the dynamics of earthquakes[98]. Figure 5 shows the distribution of energy differences between successive earthquakes at the San Andreas Fault. The empirical data is consistent with a q -Gaussian distribution, with $q = 1.75$.

Other recent applications of q -Gaussian distribution include stock markets[99–107], DNA molecules[108], the solar wind[109–111], galaxies[112], optical lattices[113], cellular aggregates[114] and the atmosphere[115].

4. q -WEIBULL DISTRIBUTION

The q -Weibull distribution is given by the pdf

$$p_{qw}(x) = p_0 \frac{r x^{r-1}}{x_0^r} \left[1 - (1-q) \left(\frac{x}{x_0} \right)^r \right]^{1/(1-q)}, \quad (8)$$

for $1 - (1-q)(x/x_0)^r \geq 0$ and $p_{qw}(x) = 0$ otherwise. Eq. (8) is normalized if $p_0 = 2 - q$.

In the limits $q \rightarrow 1$, $r \rightarrow 1$, and $q \rightarrow 1$, $r \rightarrow 1$, eq. (8) recovers Weibull, q -exponential and exponential distributions, re-

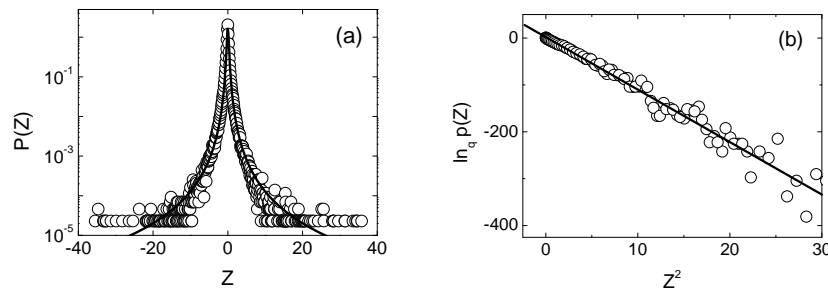


FIG. 5: **Earthquakes.** a) Empirical pdf $P(Z)$, where $Z = E(t+1) - E(t)$ is the energy difference between successive earthquakes at the San Andreas Fault in the period 1966-2006. The solid line is a q -Gaussian, given by eq. (5), with $q = 1.75$, $x_0 = 0.25$ and $p_0 = 1.63$. b) $\ln_q P(Z)$ versus Z^2 , with $q = 1.75$, for small values of Z . The solid line is a linear fit to the data.

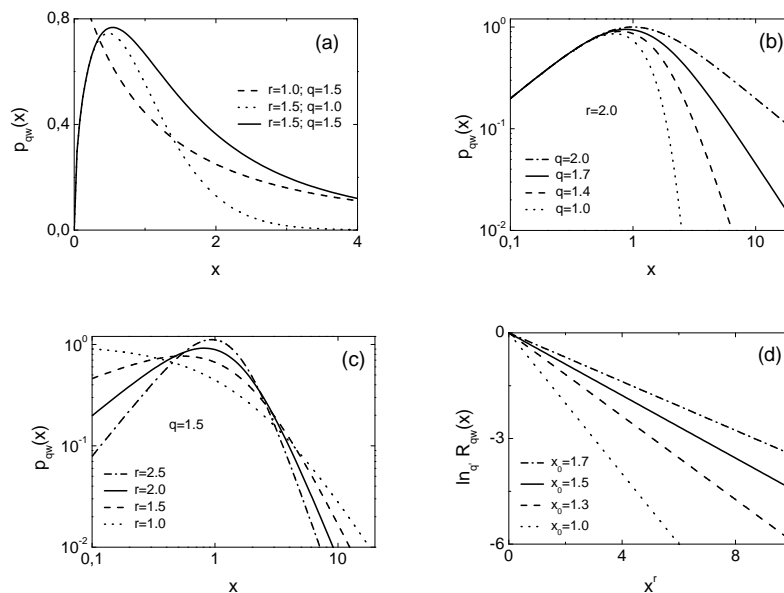


FIG. 6: **q -Weibull distribution.** a) Plot of $p_{qw}(x)$ versus x , with $p_0 = x_0 = 1$, and typical values of q and r . b) Log-log plot of $p_{qw}(x)$ versus x , with $p_0 = x_0 = 1$, $r = 2$ and typical values of q . c) a) Log-log plot of $p_{qw}(x)$ versus x , with $p_0 = x_0 = 1$, $q = 1.5$ and typical values of r . d) $\ln_q R_{qw}(x)$ versus x^r for $p'_0 = 1$ and typical values of x_0 .

spectively. If $q < 1$, $p_{qw}(x)$ has a finite limit since $p_{qw}(x) = 0$ for $1 - (1-q)(x/x_0)^r < 0$. If $q > 1$, $p_{qw}(x)$ exhibits power law behavior both for small and large values of x . More specifically,

$$p_{qw}(x) \sim x^{-\xi}, \quad (9)$$

with $\xi = (1-r)$ for small x and $\xi = r[(2-q)/(q-1)] + 1$ for large x . Figures 6a, 6b and 6c show $p_{qw}(x)$ versus x for typical values of q and r .

The cdf associated to $p_{qw}(x)$ is given by

$$R_{qw}(x) = p'_0 \left[1 - (1-q') \left(\frac{x}{x_0} \right)^r \right]^{1/(1-q')}, \quad (10)$$

with $q' = 1/(2-q)$, $(x'_0)^r = x_0^r/(2-q)$ and $p'_0 = p_0/(2-q)$. Applying the q -logarithm function in both sides of the cdf R_{qw} , we have

$$\ln_q R_{qw}(x) = \ln_q p'_0 - [1 + (1-q) \ln_q p'_0] \left(\frac{x}{x_0} \right)^r. \quad (11)$$

Figure 6c shows $\ln_q R_{qw}(x)$ versus x^r for typical values of x_0 .

If $p_{qw}(x)$ is normalized ($p_0 = 2-q$), Eq. (11) reduces to $\ln_q R_{qw}(x) = -(x/x_0)^r$. In this case,

$$\ln[-\ln_q(R_{qw}(x))] = r \ln x - r \ln x_0. \quad (12)$$

As specific example of q -Weibull distribution in complex systems, we now consider results on citations in scientific journals[116]. Figure 7 shows the distribution of the impact factor of scientific journals in comparison with a q -Weibull curve. The empirical data is consistent with a q -Weibull distribution, with $q = 1.45$ and $r = 1.50$.

Other recent works have been related to q -Weibull distributions. For example, new classes of generalized asymmetric distributions have been introduced which include q -Weibull as a special case[117, 118]. q -Weibull has also been applied in the study of fractal kinetics[119], dielectric breakdown in oxides[120], relaxation in heterogeneous systems[121], clone victims and highway lengths[55] among others.

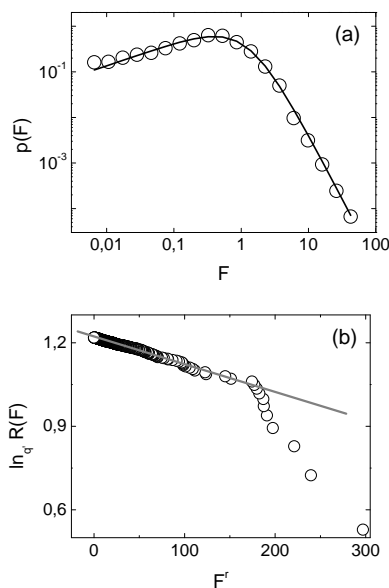


FIG. 7: **Citations in scientific journals.** a) Empirical pdf $p(F)$, where F is the 2004 impact factor for 5912 scientific journals. The solid line is a q -Weibull distribution, given by eq. (8), with $r = 1.5$, $q = 1.45$, $x_0 = 0.74$ and $p_0 = 0.58$. b) $\ln_q R(F)$ versus F^r , with $q' = 1.82$ ($q = 1.45$) and $r = 1.5$. The solid line is a linear fit to the data.

5. BASIS FOR q -DISTRIBUTIONS

From the viewpoint of the principle of the maximum entropy, some q -distributions optimizes generalized entropies - more general entropic measures than the standard Boltzmann-Gibbs entropy. A striking example is the q -entropy proposed by Tsallis[1]

$$S_q = \frac{1 - \sum_{i=1}^W p_i^q}{q-1}, \quad (13)$$

where W is the total number of microstates of the system, p_i are the occupation probabilities and q is a real parameter. The standard Boltzmann-Gibbs entropy is recovered in the limit $q \rightarrow 1$.

The maximization of S_q subject to specific constraints generates occupation probabilities following a q -exponential distribution. The q -exponential optimizes other generalized entropic measures such as the Renyi and normalized Tsallis entropies. However, only Tsallis entropy can provide an appropriate basis for the q -exponential distribution since it presents several properties essential for an entropy[122, 123]. Changing the constraints, the maximization of S_q also generates occupation probabilities following a q -Gaussian distribution.

Formally, q -distributions can arise when the exponential function of the original distribution is replaced by a q -

exponential function. For example, this basic procedure applied in standard exponential, Gaussian and Weibull distributions leads to q -exponential, q -Gaussian and q -Weibull, respectively[55]. This viewpoint suggests the consideration of other q -distributions which could be obtained by simply replacing its exponential function by a q -exponential one.

q -distributions can also emerge from compound distributions[124]

$$p_q(x) = \int_0^\infty p(x, \lambda) f(\lambda) d\lambda, \quad (14)$$

where $f(\lambda)$ is a Gamma function. For example, if $p(x, \lambda)$ is a Weibull distribution, $p_q(x)$ is given by a q -Weibull distribution[120]. Naturally, other forms for $f(\lambda)$ may be considered to obtain alternative distributions. In a physical context, this scenario has been explored with success in superstatistics where nonequilibrium situations with local fluctuations of the environment are taken into account[125–127].

The generalized distributions considered here can also be obtained from the following ordinary differential equation:

$$\frac{dy}{dx} = \rho y^q. \quad (15)$$

In fact, if ρ is constant, the solution of eq. (15) is a q -exponential; if $\rho \propto x$, the solution is a q -Gaussian. If y is the cdf and $\rho \propto x^r$, we have a q -Weibull. By considering further terms in eq. (15), other q -distributions can be obtained[128]. q -distributions can also emerge in other contexts. For instance, q -Gaussian arises from the non-linear diffusion (porous media) equation[84] and from a generalization of the central limit theorem[3]. Another example is the q -lognormal distribution which emerges from generalized cascades[28].

6. CONCLUSION

The present work presents a brief overview of recent applications of some q -distributions largely used in the context of Tsallis statistics. It illustrates how q -exponential, q -Gaussian and q -Weibull distributions have been applied in the study of a wide variety of systems in several fields.

The success of q -distributions in describing diverse systems is in part due to its ability of exhibit heavy-tails and model power law phenomena - a typical characteristic of complex systems. The positive and exciting results obtained with q -distributions also indicate possible applications of Tsallis nonextensive statistical mechanics. Naturally, further work may be necessary to explore possible relations between the analyzed systems and the present theory.

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