

Bounds for the subsistence of a problem of heat conduction

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Abstract. In this paper, we consider a slab represented by the interval $0 < x < 1$, at the initial temperature $u_0(x) \geq 0$ and having a heat flux $q(t)$ on the left face and a nonlinear condition on the right face $x = 1$. We consider the corresponding heat conduction problem and we assume that the phase-change temperature is 0^0C . We prove that certain conditions on the data are necessary and sufficient in order to obtain estimations of the occurrence of a phase-change in the material.

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1 Introduction

We consider a slab represented by the interval $0 < x < 1$ and assume that the phase-change temperature is 0^0C . We study the following heat conduction problem:

Problem

$$u_{xx} = u_t, \quad \text{in } D = \{(x, t) : 0 < x < 1, t > 0\}, \quad (1)$$

$$u(x, 0) = u_0(x), \quad 0 \leq x \leq 1, \quad (2)$$

$$u_x(0, t) = q(t), \quad t > 0, \quad (3)$$

$$u_x(1, t) = f(u(1, t)), \quad t > 0, \quad (4)$$

where $u_0(x) \geq 0$ and $q(t) > 0$. We assume that the data satisfy the hypotheses that ensure the existence and uniqueness property of the solution of problem [1, pag. 67]. In this heat conduction problem the material is initially in the liquid phase. We explicit the relation among the heat flux q and the function $f(s)$, in order to have a change of phase in the material by considering that the flux at $x = 1$ depends on the temperature $u(1, t)$, through (4). The solution of the problem is given by [1]:

$$\begin{aligned}
 u(x, t) = & \int_0^1 \{\theta(x - \xi, t) + \theta(x + \xi, t)\} u_0(\xi) d\xi \\
 & - 2 \int_0^t \theta(x, t - \tau) q(\tau) d\tau \\
 & + 2 \int_0^t \theta(x - 1, t - \tau) f(\phi(\tau)) d\tau,
 \end{aligned} \tag{5}$$

where the function $\phi(\tau)$ is the solution of a Volterra integral equation given by:

$$\phi(t) = w(1, t) - 2 \int_0^t \theta(1, t - \tau) q(\tau) d\tau + 2 \int_0^t \theta(0, t - \tau) f(\phi(\tau)) d\tau,$$

with

$$w(x, t) = \int_0^1 \{\theta(x - \xi, t) + \theta(x + \xi, t)\} u_0(\xi) d\xi,$$

and $\theta(x, t)$ is defined by:

$$\theta(x, t) = 1 + 2 \sum_{k=1}^{\infty} e^{-k^2 \pi^2 t} \cos(k \pi x).$$

The reduction of the problem with boundary condition $u_x = f(t, u)$ to an integral equation was studied by [14], [15], [16].

First, from the uniform absolute convergence of the series for θ and its partial derivatives, it is clear that, for $t > 0$, $\theta(x, t) > 0$, the θ function is continuous and its partial derivatives are continuous [see 1]. If the data verify the following conditions:

$$F \geq f(s) \geq f_0 > 0, \quad s > 0, \tag{6}$$

$$u_0(x) \geq 0, u'_0(x) > 0, \quad 0 \leq x \leq 1, \tag{7}$$

$$q(t) > 0, \quad t > 0, \tag{8}$$

then by using the maximum principle we have that $u(0, t) \leq u(x, t)$ in $0 < x < 1$ since we can obtain $u_x(x, t) > 0$ [11].

We will consider the following two possibilities:

1. The heat conduction problem (1)-(4) is defined for all $t \leq t^*$, where $t^* < \infty$.
2. There exists a time $t_{ch} < \infty$ such that $u(0, t_{ch}) < 0$, that means another phase appears (the solid phase) for $t \geq t_{ch}$.

These possibilities depend on the data u_0, q, f for the **Problem**. We try to clarify this dependence by finding necessary and sufficient conditions on u_0, q, f in order to have the two possibilities. In [4] the one-phase Stefan problem with prescribed flux or convective boundary condition at $x = 0$ is studied (see also [5, 6]). This paper was motivated by [2] and [3] (see also [7, 9, 10]). In [8] the author find an exact solution for a particular type of heat flux q . In the paper by Tarzia-Turner (1992) the authors have presented a similar problem with temperature and convective boundary conditions and they proved that certain conditions on the data are necessary and sufficient in order to obtain a change-phase in the material. In the work by Berrone (1994) several similar problems are analyzed too. He considered several methods for the study of the subsistence of the model that involve the reaction-diffusion equations. In [12] the authors have studied a similar problem in spherical coordinates. A large bibliography on the Stefan and related problems was given in [13].

This paper extends the problem introduced in [2] to the case of nonlinear boundary condition.

When the radiation of heat flux from a solid is considered, the heat flux is often taken to proportional to the fourth power of the difference of the boundary temperature of the surroundings. Here, f represents a general radiation law.

In this paper we will study case 1 in Section 2, where we obtain bounds for the time t^* in order to have a heat conduction problem for $t < t^*$, with q a constant function or a bounded function. In Section 3, we consider case 2, we find estimations for the time t_{ch} when another phase will appear. Finally we compare the two cases.

2 Subsistence of the model of heat equation

2.1 The case when the heat flux q is constant

In this subsection we will consider the constant flux $q(t) = q$, in the last remark of this section we will generalize the results to the case of $q(t)$ a bounded function of t . The condition $u(0, t) > 0$ for all $t \leq t^*$ is equivalent to:

$$q < \min_{[0, t^*]} \Phi(t). \quad (9)$$

where

$$\Phi(t) = \frac{\int_0^1 \theta(\xi, t) u_0(\xi) d\xi + \int_0^t \theta(-1, t - \tau) f(\phi(\tau)) d\tau}{\int_0^t \theta(0, t - \tau) d\tau}, \quad (10)$$

by using that $\theta(x, t)$ is positive and even in x . We emphasize that $\Phi(t) > 0$ since $\theta(x, t) > 0$, $f(s) > 0$ and $u_0(x) \geq 0$. Moreover this minimum of Φ exists since we will find a lower bound $\Psi(t)$ for $\Phi(t)$ and $\Psi(t)$ will have a positive minimum. Since q is constant we take the minimum value of Φ in order to ensure $u(0, t) > 0$ for all $t \leq t^*$. We are looking for a function $\Psi(t)$ more simple than the expression we have as a bound for q such that $\Psi(t) \leq \Phi(t)$, then a sufficient condition in order to have the subsistence of the heat conduction problem for all $t \leq t^*$ is:

$$q < \min_{[0, t^*]} \Psi(t).$$

The following inequalities for the function $\theta(x, t)$ are needed.

Lemma 1. *The function θ satisfies the following inequalities:*

1.

$$\int_0^t \theta(0, t - \tau) d\tau \leq \int_0^t 1 + \frac{\sqrt{\pi}}{\sqrt{(t - \tau)}} d\tau = t + 2\sqrt{\pi t}, \quad (11)$$

for all $t > 0$.

2.

$$\int_0^1 \theta(\xi, t) u_0(\xi) d\xi \geq u_0(0), \quad \text{for all } t > 0. \quad (12)$$

3. For $t \leq K$:

$$\int_0^t \theta(-1, t - \tau) f(\phi(\tau)) d\tau > \frac{1}{2} \alpha K^2 f_0,$$

and for $t \geq K$:

$$\begin{aligned} \int_0^t \theta(-1, t - \tau) f(\phi(\tau)) d\tau &> \int_0^K \alpha t f_0 d\tau + \int_K^t (1 - \varepsilon) f_0 d\tau \\ &= \frac{1}{2} \alpha K^2 f_0 + (1 - \varepsilon) f_0 (t - K), \end{aligned} \tag{13}$$

for a suitable election of the parameters K , ε and $\alpha(K)$.

Proof.

1. The inequality (10) comes from the following inequality:

$$\theta(0, t - \tau) \leq \int_0^\infty e^{-k^2 \pi^2 t} dk.$$

This inequality is obtained estimating the function $\theta(0, t - \tau)$ by looking θ as a lower Riemann sum.

2. In this case the inequality follows from:

$$\int_0^1 \theta(\xi, t) d\xi = 1,$$

and that the function $u_0(x)$ is nondecreasing.

3. For the last inequality we use the change of variable:

$$z = t - \tau, \quad dz = -d\tau,$$

now, we obtain the following relation:

$$\int_0^t \theta(-1, t - \tau) f(\phi(\tau)) d\tau = \int_0^t \theta(-1, z) f(\phi(t - z)) dz.$$

Therefore, we can bound the function theta in this manner:

$$\theta(-1, t) \geq g(t) = \begin{cases} \alpha t & t \leq K \\ 1 - \varepsilon & t \geq K, \end{cases} \tag{14}$$

for a suitable election of the parameters K , ε and $\alpha(K)$. For example we can choose $\varepsilon(K) = 1 - \theta(-1, K)$ and $\alpha(K) = \frac{1 - \varepsilon}{K}$ for a given K . This was motivated by the geometrical form of the function $\theta(-1, t - \tau)$ (i.e. increasing, concave, $\theta(-1, 0^+) = 0$ and $\theta(-1, +\infty) = 1$), then by construction we obtain for $t \leq K$:

$$\begin{aligned} \int_0^t \theta(-1, t - \tau) f(\phi(\tau)) d\tau &> \int_0^K \alpha t f_0 d\tau \\ &= \frac{1}{2} \alpha K^2 f_0, \end{aligned} \quad (15)$$

and for $t \geq K$:

$$\begin{aligned} \int_0^t \theta(-1, t - \tau) f(\phi(\tau)) d\tau &> \int_0^K \alpha t f_0 d\tau + \int_K^t (1 - \varepsilon) f_0 d\tau \\ &= \frac{1}{2} \alpha K^2 f_0 + (1 - \varepsilon) f_0 (t - K). \end{aligned} \quad (16 \square)$$

The desired bound for $\Phi(t)$ is obtained by applying this last lemma.

$$\Omega(K, t) = \begin{cases} \frac{u_0(0) + \frac{1}{2} \alpha K^2 f_0}{t + 2\sqrt{\pi t}} & t \leq K \\ \frac{u_0(0) + \frac{1}{2} \alpha K^2 f_0 + (1 - \varepsilon) f_0 (t - K)}{t + 2\sqrt{\pi t}} & t \geq K. \end{cases} \quad (17)$$

We remark that $\varepsilon = \varepsilon(K)$, $\alpha = \alpha(K)$ and we defined for a given t^* the function $\Psi(t) = \Omega(t^*, t)$ (i.e. we replace K by t^*). Then we obtain the following lemma:

Lemma 2. For $t \leq t^*$ holds:

$$\Psi(t) \leq \Phi(t),$$

where

$$\Psi(t) = \Omega(t^*, t).$$

Now we can conclude the following statement.

Theorem 1. For a given t^* , the problem is a heat conduction problem for $t \leq t^*$ if:

$$q < \Psi(t^*).$$

Proof. Using the fact that $\Psi(t)$ is decreasing (for $t \leq t^*$) and $\Psi(t) \leq \Phi(t)$, we obtain the following inequality:

$$q < \Psi(t^*) \leq \Psi(t) \leq \Phi(t),$$

for $t \leq t^*$ and using (9) we can obtain $u(0, t) > 0$ for $t \leq t^*$. □

For a given t^* , we remark that the maximum K for the construction of $\Psi(t)$ is t^* since the $\lim_{t \rightarrow \infty} \Omega(K, t) = (1 - \varepsilon)f_0 > 0$ and for fixed K the function $\Omega(K, t)$ is increasing for $t \geq K$.

From the theorem 1, we can obtain an explicit expression for the time t^* , for this we use the fact that

$$\alpha K = 1 - \varepsilon,$$

obtained by the definition of the function $g(t)$ and the following equation:

$$q = \Psi(t^*) = \frac{u_0(0) + \frac{1}{2}(1 - \varepsilon)t^* f_0}{t^* + 2\sqrt{\pi t^*}}. \tag{18}$$

We can think the last equation as a quadratic equation. We look for their roots. We obtain the following Corollary:

Corollary 1. The time t^* in theorem 1 is given by:

$$t^* = \left(\frac{-q\sqrt{\pi} + \sqrt{q^2\pi - (\frac{1}{2}(1 - \varepsilon)f_0 - q)u_0(0)}}{q - \frac{1}{2}(1 - \varepsilon)f_0} \right)^2. \tag{19}$$

Where we choose the positive root and we needed that:

$$q \geq \frac{1}{2}(1 - \varepsilon)f_0.$$

Remark 1. We emphasize that the time t^* depends on the election of the function $g(t)$, that is t^* depends on K , $\varepsilon(K)$ and $\alpha(K)$.

Remark 2. By the theorem 1 for any $q > 0$, there exists a time $t_q > 0$ that implies there is not a phase-change process for $t \leq t_q$. Therefore a necessary condition in order to have an instantaneous change of phase for the problem is $q(0^+) = +\infty$. If we consider the Problem for a slab $[0, x_0]$ where $x_0 = \infty$, then we can replace the condition (4) by:

$$u(\infty, t) = u_0(\infty), \quad t > 0.$$

In [2] the authors show that the condition $q(0^+) = \infty$ is not sufficient for the case of a semi-infinite domain. In the example they take:

1. $x_0 = +\infty$,
2. $u_0(x) \geq \beta_0 > 0, \quad x > 0$,
3. $q(t) \leq q_0(t) = \frac{\beta_0}{\sqrt{\pi t}}$.

The solution $u(x, t)$ of this problem satisfies the following inequality:

$$u(x, t) \geq \beta \operatorname{erf}\left(\frac{x}{2\sqrt{t}}\right) \geq 0,$$

for $x \geq 0$ and $t > 0$.

Moreover the particular case $q(t) = \frac{\beta_0}{\sqrt{\pi t}}$, then $q(0^+) = +\infty$ and it is not sufficient in order to have an instantaneous change of phase.

It is an open question to prove that this condition is not sufficient for the case of a finite domain.

2.2 The heat flux q is a bounded function of t

We can find bounds for the case of a bounded function $q = q(t)$, where we consider

$$\min_{[0, +\infty)} q(t)$$

in the equation (9).

Theorem 2. For a given time t^* , the Problem is a heat conduction problem for $t \leq t^*$ if:

$$Q \leq \Psi(t^*), \quad \text{where} \quad Q = \min_{[0, +\infty)} q(t).$$

Proof. In this case the condition $u(0, t) > 0$ for $t \leq t^*$ is equivalent to:

$$\int_0^t \theta(0, t - \tau)q(\tau)d\tau < \int_0^1 \theta(\xi, t)u_0(\xi)d\xi + \int_0^t \theta(-1, t - \tau)f(\phi(\tau))d\tau, \tag{20}$$

since $\theta(0, t) > 0$, and as $q(t)$ is bounded we can use the fact that:

$$Q \int_0^t \theta(0, t - \tau)d\tau < \int_0^t \theta(0, t - \tau)q(\tau)d\tau, \tag{21}$$

where

$$Q = \min_{[0, +\infty)} q(t) \leq \min_{[0, t^*]} q(t),$$

in order to obtain the following inequality:

$$Q \leq \min_{[0, t^*]} \frac{\int_0^1 \theta(\xi, t)u_0(\xi)d\xi + \int_0^t \theta(-1, t - \tau)f(\phi(\tau))d\tau}{\int_0^t \theta(0, t - \tau)d\tau} = \min_{[0, t^*]} \Phi(t). \tag{22}$$

Then a sufficient condition to have $u(0, t) > 0$ for $t \leq t^*$ is given by

$$\min_{[0, t^*]} \Phi(t) \geq \min_{[0, t^*]} \Psi(t) = \Psi(t^*) \geq Q. \quad \square$$

3 Existence of a phase-change process

In this section we will find an estimation for the time t_{ch} in order to have a change-phase process in the material. The condition for $u(0, t_{ch}) < 0$ for $t = t_{ch}$ is equivalent to:

$$q > \Phi(t_{ch}), \tag{23}$$

where

$$\Phi(t_{ch}) = \frac{\int_0^1 \theta(\xi, t_{ch}) u_0(\xi) d\xi + \int_0^{t_{ch}} \theta(-1, t_{ch} - \tau) f(\phi(\tau)) d\tau}{\int_0^{t_{ch}} \theta(0, t_{ch} - \tau) d\tau}. \quad (24)$$

We emphasize that this is different from the case in section 2 where we consider for $u(0, t) > 0$ for $t \leq t^*$. We need an upper bound function $\delta(t)$ for the function $\Phi(t)$ such that $\delta(t) \geq \Phi(t)$ (where Φ was defined in section 2). We suppose now that $q > \delta(t_{ch})$; this implies the inequality (23).

The following inequalities for the function θ are needed.

Lemma 3. *The function θ satisfies the following inequalities:*

1.

$$\int_0^t \theta(0, t - \tau) d\tau \geq \frac{2(\pi - 2)}{\sqrt{\pi}} \sqrt{t} + t \quad \text{for all } t > 0. \quad (25)$$

2.

$$\int_0^1 \theta(\xi, t) u_0(\xi) d\xi \leq u_0(1) \quad \text{for all } t > 0. \quad (26)$$

3.

$$\int_0^t \theta(-1, t - \tau) f(\phi(\tau)) d\tau \leq Ft. \quad (27)$$

Proof.

1. The inequality (26) comes from the following inequality:

$$\theta(0, t - \tau) \geq \int_1^\infty e^{-k^2 \pi^2 t} dk.$$

This inequality was obtained estimating the function $\theta(0, t - \tau)$ by looking at θ as an upper Riemann sum.

2. In this case the inequality follows from:

$$\int_0^1 \theta(\xi, t) d\xi = 1,$$

and that the function $u_0(1)$ is the minimum of the $u_0(x)$.

3. For this last inequality we use:

$$\theta(-1, t - \tau) \geq 0,$$

and the hypothesis for the function f . □

The function δ is obtained by using the last lemma:

$$\delta(t) = \frac{u_0(1) + Ft}{C\sqrt{t} + t}, \tag{28}$$

where the constant C is given by:

$$C = \frac{2(\pi - 2)}{\sqrt{\pi}}.$$

The function $\delta(t)$ is decreasing in $[0, t_{\min}]$, where:

$$t_{\min} = \left(\frac{u_0(1) + \sqrt{u_0(1)(u_0(1) + C^2F)}}{CF} \right)^2.$$

Then we obtain the following result:

Theorem 3. *If*

$$q > \delta(t_{ch}),$$

with $t_{ch} \leq t_{\min}$, then another phase appears in the Problem for some $t \leq t_{ch}$.

Remark 3. In this case the heat conduction model is not more valid. We have to express the problem as free boundary problem for a two phase Stefan problem with appropriate conditions from some $t \leq t^*$.

Remark 4. We can take $t^* = t_{ch} \leq t_{\min}$ in Theorem 1. From the fact that:

$$\delta(t_{ch}) \geq \Psi(t_{ch}),$$

we conclude that:

1. If

$$q \leq \Psi(t_{ch}),$$

then $u(0, t) \geq 0$ for all $t \leq t_{ch}$.

2. If

$$q \geq \delta(t_{ch}),$$

then the material changes of phase before that the time t_{ch} .

3. It is an open question what happens if:

$$\delta(t_{ch}) < q < \Psi(t_{ch}).$$

4 Conclusions

In section 2 we have found an expression for t^* for a given q in order to have a heat conduction problem for $t \leq t^*$ (i.e. the material does not change of phase). In section 3 we have found an condition for $t_{ch} < t_{\min}$ for a given q in order to have a change of phase for $t \leq t_{ch}$ (i.e. the material changes of phase before that the time t_{ch}). We have studied this problem through the exact solution (in a complex form with integral equations) and we have considered bounds of their terms in order to make easy the explicit expression of the time for the occurrence or not of the change of phase in the material.

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