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The generalized anti-reflexive solutions for a class of matrix equations $(BX = C, XD = E)$

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Abstract. In this paper, the generalized anti-reflexive solution for matrix equations $(BX = C, XD = E)$, which arise in left and right inverse eigenpairs problem, is considered. With the special properties of generalized anti-reflexive matrices, the necessary and sufficient conditions for the solvability and a general expression of the solution are obtained. Furthermore, the related optimal approximation problem to a given matrix over the solution set is solved. In addition, the algorithm and the example to obtain the unique optimal approximation solution are given.

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Key words: matrix equations, generalized anti-reflexive matrices, optimal approximation.

1 Introduction

Left and right inverse eigenpairs problem is a special inverse eigenvalue problem. That is, giving partial left and right eigenpairs (eigenvalue and corresponding eigenvector) of a matrix A, $(\gamma_j, y_j), j = 1, \ldots, l$; $(\lambda_i, x_i), i = 1, \ldots, h$, a special matrix set $S \subseteq R^{n\times n}$ (In this paper, denote the set of all $n \times n$ real matrices by $R^{n\times n}$), and $h \leq n, l \leq n$, find $A \in S$ such that

$$\begin{cases} Ax_i = \lambda_i x_i, & i = 1, \ldots, h, \\ y_j^T A = \gamma_j y_j^T, & j = 1, \ldots, l. \end{cases}$$

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The prototype of this problem initially arose in perturbation analysis of matrix eigenvalue and in recursive matters. It has profound applications background [1-4].

Let $X = (x_1, \ldots, x_h)$, $Y = (y_1, \ldots, y_l)$, $\Lambda = \text{diag}(\lambda_1, \ldots, \lambda_h)$, $\Gamma = (\gamma_1, \ldots, \gamma_h)$, $Z = X\Lambda$, $W = Y\Gamma$, then the above problem can be described as follows. Giving matrices X, Y, Z, W and a special matrix set S, find $A \in S$ such that

$$\begin{cases} AX = Z, \\ Y^T A = W^T. \end{cases}$$

Actually, this is the problem of seeking the solutions of linear matrix equations. In this paper, we will extend it and obtain the following problem. Giving matrices B, C, D, E and a special matrix set S, find $X \in S$ such that

$$\begin{cases} BX = C, \\ XD = E. \end{cases}$$

This equations is an important class of matrix equations, and has profound applications in engineering and matrix inverse problem $AX = B$ [5, 6]. In recent years, many authors have studied it, and a series of meaningful results have been achieved [7-10]. However, its generalized anti-reflexive solutions have not been concerned with. In this paper, we will discuss this problem.

We now introduce the following notation. $C^{n\times m}$ denote the set of $n \times m$ complex matrices. $OC^{n\times n}$ denote the set of $n \times n$ unitary matrices. A^H, $r(A)$, $\text{tr}(A)$ and A^+ be the conjugate transpose, rank, trace and the Moore-Penrose generalized inverse of a matrix A, respectively. I_n be the identity matrix of size n. For $A, B \in C^{n\times m}$, $\langle A, B\rangle = \text{tr}(B^H A)$ denote the inner product of matrices A and B. The induced matrix norm is called Frobenius norm, i.e. $\|A\| = \langle A, A\rangle^{\frac{1}{2}} = \left(\text{tr}(A^H A)\right)^{\frac{1}{2}}$, then $C^{n\times m}$ is a Hilbert inner product space.

To extend reflexive (anti-reflexive) matrices and centrosymmetric matrices, Chen [11] has introduced two new special classes of matrices, which are generalized reflexive matrices and generalized anti-reflexive matrices. He presented three examples obtained from the altitude estimation of a level network, an electric network and structural analysis of trusses. His investigation indicated that generalized reflexive matrices arise naturally from problem with reflexive symmetry, which account for a great number of real world scientific and engineering applications.

Definition 1. An $n \times n$ complex matrix P is called the generalized reflection matrix if $P = P^H$, $P^2 = I_n$.

Definition 2. Let $A \in C^{n\times m}$, A is called the generalized reflexive matrix (generalized anti-reflexive matrix) with respect to matrix pairs (P, Q) if $A = PAQ$ (or $A = -PAQ$), where P, Q are the $n \times n$ and $m \times m$ generalized reflection matrix, respectively. Denote this class of matrices by $C_r^{n\times m}(P, Q)$ (or $C_a^{n\times m}(P, Q)$), then the following results can be easily deduced.

$$\begin{aligned} C_r^{n\times m}(P, Q) &= \{A | A = PAQ, A \in C^{n\times m}\}, \\ C_a^{n\times m}(P, Q) &= \{A | A = -PAQ, A \in C^{n\times m}\}. \end{aligned}$$

Definition 3. $C_{rP}^{n\times m} = \{X | PX = X, X \in C^{n\times m}\}$, $C_{aP}^{n\times m} = \{X | PX = -X, X \in C^{n\times m}\}$.

From Definition 2, 3, it is easy to see that if P, Q are two given $n \times n$ and $m \times m$ generalized reflection matrices, respectively, then $C_r^{n\times m}(P, Q)$ (or $C_a^{n\times m}(P, Q)$) is a closed linear subspace of $C^{n\times m}$, and $C_{rP}^{n\times m}$ (or $C_{aP}^{n\times m}$) is also a closed linear subspace of $C^{n\times m}$. Throughout, we always assume that P, Q are two given $n \times n$ and $m \times m$ generalized reflection matrices, respectively. From Definition 2, 3 and this assumption, it is easy to prove the following results.

1) $X \in C_r^{n\times m}(P, Q) \Leftrightarrow X^H \in C_r^{m\times n}(Q, P)$,
   $X \in C_a^{n\times m}(P, Q) \Leftrightarrow X^H \in C_a^{m\times n}(Q, P)$.

2) $C^{n\times m} = C_r^{n\times m}(P, Q) \oplus C_a^{n\times m}(P, Q)$,
   $C^{n\times m} = C_{rP}^{n\times m} \oplus C_{aP}^{n\times m}$.

The notation $V_1 \oplus V_2$ stands for the orthogonal direct sum of linear subspace V_1 and V_2. From this, for any

$$B \in C^{h\times n}, \quad C \in C^{h\times m}, \quad D \in C^{m\times l}, \quad E \in C^{n\times l},$$

we have the following results.

$$\begin{aligned} &B = B_1 + B_2,\, PB_1^H = B_1^H,\, PB_2^H = -B_2^H,\, B_1B_2^H = 0. \\ &C = C_1 + C_2,\, QC_1^H = C_1^H,\, QC_2^H = -C_2^H,\, C_1C_2^H = 0. \\ &D = D_1 + D_2,\, QD_1 = D_1,\, QD_2 = -D_2,\, D_2^H D_1 = 0. \\ &E = E_1 + E_2,\, PE_1 = E_1,\, PE_2 = -E_2,\, E_2^H E_1 = 0. \end{aligned} \tag{1.1}$$

In Definition 2, if $P = Q$, then A is a reflexive matrix (or an anti-reflexive matrix) with respect to P [12]. We denote the set of all reflexive matrices (anti-reflexive matrices) by $C_r^{n\times n}(P)$ (or $C_a^{n\times n}(P)$). So $C_r^{n\times n}(P)$ (or $C_a^{n\times n}(P)$) is a special case of $C_r^{n\times n}(P, Q)$ (or $C_a^{n\times n}(P, Q)$).

In this paper, we consider the following problems.

Problem 1. Given $B \in C^{h\times n}$, $C \in C^{h\times m}$, $D \in C^{m\times l}$, $E \in C^{n\times l}$, find $X \in C_a^{n\times m}(P, Q)$ such that

$$\begin{cases} BX = C, \\ XD = E. \end{cases}$$

Problem 2. Given $X^* \in C^{n\times m}$, find $\hat{X} \in S_E$ such that

$$\|X^* - \hat{X}\| = \min_{\forall X \in S_E} \|X^* - X\|,$$

where S_E is the solution set of Problem 1.

Problem 2 is the optimal approximation problem of Problem 1. It occurs frequently in experimental design [13]. Here the matrix X^* may be a matrix obtained from experiments, but it may not satisfy the structural requirement (generalized anti-reflexive matrices with respect to matrix pairs (P, Q)) and/or matrix equations ($BX = C, XD = E$). The optimal estimate $\hat{X}$ is the matrix that satisfies both restrictions and is the optimal approximation of X^*. See for instance [14, 15].

This paper is organized as follows. In section 2, we first study the special properties of matrices in $C_a^{n\times m}(P, Q)$. Then using these properties and the results of [7], we obtain the solvability conditions and the general solutions of Problem 1. Section 3 is devoted to derive the unique approximation solution of Problem 2 by applying the methods of space decomposition. Finally, the algorithm and the example to obtain the unique approximation solution are given.

2 Solvability conditions of Problem 1

First, we discuss the properties of matrices in $C_a^{n\times m}(P,Q)$.

Lemma 1.

1) *If* $X \in C_a^{n\times m}(P,Q)$, $A^H \in C_{rP}^{n\times h}\big(C_{aP}^{n\times h}\big)$, *then* $(AX)^H \in C_{aQ}^{m\times h}\big(C_{rQ}^{m\times h}\big)$.

2) *If* $X \in C_a^{n\times m}(P,Q)$, $A \in C_{rQ}^{m\times l}\big(C_{aQ}^{m\times l}\big)$, *then* $XA \in C_{aP}^{n\times l}\big(C_{rP}^{n\times l}\big)$.

Proof.

1) If $A^H \in C_{rP}^{n\times h}$, then

$$Q(AX)^H = QX^H PPA^H = -(AX)^H.$$

Hence, $(AX)^H \in C_{aQ}^{m\times h}$. If $A^H \in C_{aP}^{n\times h}$, then

$$Q(AX)^H = QX^H PPA^H = (AX)^H.$$

Hence, $(AX)^H \in C_{rQ}^{m\times h}$. We can prove 2) by the same methods. □

Lemma 2 [16]. *Let* $E \in C^{n\times h}$, $F \in C^{n\times l}$ *and* $F^H E = 0$. *Then we have*

$$(EF)^+ = \begin{pmatrix} E^+ \\ F^+ \end{pmatrix}.$$

Lemma 3.

1) *If* $B^H \in C_{rP}^{n\times h}\big(C_{aP}^{n\times h}\big)$, $C^H \in C_{aQ}^{m\times h}\big(C_{rQ}^{m\times h}\big)$, *then* $B^+B \in C_r^{n\times n}(P)$, $B^+C \in C_a^{n\times m}(P,Q)$.

2) *If* $D \in C_{rQ}^{m\times l}\big(C_{aQ}^{m\times l}\big)$, $E \in C_{aP}^{n\times l}\big(C_{rP}^{n\times l}\big)$, *then* $DD^+ \in C_r^{m\times m}(Q)$, $ED^+ \in C_a^{n\times m}(P,Q)$.

Proof. We only prove 1), and 2) can be proved by the same methods.

1) Since $B^H \in C_{rP}^{n\times h}$, it is easy to prove the following equations

$$PB^H = B^H, \qquad B^+ = PB^+. \tag{2.1}$$

From this, we have $PB^+BP = B^+B$, i.e. $B^+B \in C_r^{n\times n}(P)$. The equations (2.1) implies the following equations

$$\left(B^H\right)^+ = \left(PB^H\right)^+ = \left(B^H\right)^+ P, \qquad PB^+ = B^+. \tag{2.2}$$

Since $C^H \in C_{aQ}^{m\times h}$, it is also easy to obtain the following equations

$$QC^H = -C^H, \qquad C = -CQ. \tag{2.3}$$

Combining (2.2) and (2.3), we obtain

$$PB^+CQ = -B^+C \quad \text{i.e.} \quad B^+C \in C_a^{n\times m}(P, Q).$$

If $B^H \in C_{aP}^{n\times h}$, $C^H \in C_{rQ}^{m\times h}$, we can also prove the conclusion by the same methods. □

Lemma 4. *Let $K \in C_r^{n\times n}(P)$, $G \in C_r^{m\times m}(Q)$, $F \in C^{n\times m}$, denote $M = KFG$. Then the following statements are true.*

1) *If $F \in C_r^{n\times m}(P, Q)$, then $M \in C_r^{n\times m}(P, Q)$.*
2) *If $F \in C_a^{n\times m}(P, Q)$, then $M \in C_a^{n\times m}(P, Q)$.*
3) *If $F = F_1 + F_2$, where $F_1 \in C_r^{n\times m}(P, Q)$, $F_2 \in C_a^{n\times m}(P, Q)$, then $M \in C_r^{n\times m}(P, Q)$ if and only if $KF_2G = 0$. In addition, we have $M = KF_1G$.*

Proof.

1) $PMQ = PKFGQ = PKPPFQQGQ = KFG = M$. Hence, $M \in C_r^{n\times m}(P, Q)$.
2) $PMQ = PKFGQ = PKPPFQQGQ = K(-F)G = -M$. Hence, $M \in C_a^{n\times m}(P, Q)$.
3) $M = KFG = K(F_1 + F_2)G = KF_1G + KF_2G$, from 1) and 2), $KF_1G \in C_r^{n\times m}(P, Q)$, $KF_2G \in C_a^{n\times m}(P, Q)$. If $M \in C_r^{n\times m}(P, Q)$, then $M - KF_1G \in C_r^{n\times m}(P, Q)$, but $M - KF_1G = KF_2G \in C_a^{n\times m}(P, Q)$. According to conclusion 2) of Definition 2, we have $KF_2G = 0$, i.e. $M = KF_1G$. If $KF_2G = 0$, it is clear that $M = KF_1G + KF_2G = KF_1G \in C_r^{n\times m}(P, Q)$. □

Lemma 5.

1) *If* $A \in C_r^{n\times n}(P)$*,* $B \in C_a^{n\times m}(P, Q)$*, then* $AB \in C_a^{n\times m}(P, Q)$*.*

2) *If* $A \in C_{rP}^{n\times m}$*,* $B \in C_{aP}^{n\times h}$*, then* $A^H B = 0$*,* $A^+ B = 0$*.*

Proof.

1) $PABQ = PAPPBQ = -AB$. So, $AB \in C_a^{n\times m}(P, Q)$.

2) From Definition 3, we have

$$A^H B = A^H P^H P B = (PA)^H P B = A^H(-B) = -A^H B.$$

So, $A^H B = 0$. From Definition 3, we also have

$$A^+ B = A^+ P P B = (PA)^+ P B = A^+(-B) = -A^+ B.$$

So, $A^+ B = 0$. □

Denote

$$\tilde{B}^H = \left(B_1^H \, B_2^H\right), \quad \tilde{C}^H = \left(C_2^H \, C_1^H\right), \quad \tilde{D} = \left(D_1 \, D_2\right), \quad \tilde{E} = \left(E_2 \, E_1\right), \tag{2.4}$$

where $B_1, B_2, C_1, C_2, D_1, D_2, E_1, E_2$ are given by (1.1).

Lemma 6. *If* B, C, D, E *are given by* (1.1)*,* $\tilde{B}, \tilde{C}, \tilde{D}, \tilde{E}$ *are denoted by* (2.4)*,* $X \in C_a^{n\times m}(P, Q)$*, then matrix equations* $(BX = C, XD = E)$ *are equivalent to* $(\tilde{B}X = \tilde{C}, X\tilde{D} = \tilde{E})$*.*

Proof. According to (1.1), $BX = C$ is equivalent to

$$B_1 X + B_2 X = C_1 + C_2.$$

From Lemma 1, $(B_1 X)^H \in C_{aQ}^{m\times h}$, $(B_2 X)^H \in C_{rQ}^{m\times h}$. According to conclusion 2) of Definition 2, we have

$$B_1 X = C_2, \qquad B_2 X = C_1\,,$$

i.e.

$$\begin{pmatrix} B_1 \\ B_2 \end{pmatrix} X = \begin{pmatrix} C_2 \\ C_1 \end{pmatrix}.$$

So $BX = C$ is equivalent to $\tilde{B}X = \tilde{C}$. By applying the similar methods, we can prove that $XD = E$ is equivalent to $X\tilde{D} = \tilde{E}$. □

Lemma 7 [7]. *If B, C, D, E are given by* (1.1)*, then* $(BX = C, XD = E)$ *has a solution in $C^{n\times m}$ if and only if*

$$CD = BE, \qquad C = BB^+C, \qquad E = ED^+D. \tag{2.5}$$

Moreover, its general solution can be expressed as

$$\begin{aligned} X = \; & B^+C + (I_n - B^+B)ED^+ \\ & + (I_n - B^+B)F(I_m - DD^+), \quad \forall F \in C^{n\times m}. \end{aligned} \tag{2.6}$$

Theorem 1. *If B, C, D, E are given by* (1.1)*, then Problem* 1 *has a solution in $C_a^{n\times m}(P, Q)$ if and only if*

$$C_2D_2 = B_1E_1, \qquad C_1 = B_2B_2^+C_1, \qquad E_1 = E_1D_2^+D_2. \tag{2.7}$$

$$C_1D_1 = B_2E_2, \qquad C_2 = B_1B_1^+C_2, \qquad E_2 = E_2D_1^+D_1. \tag{2.8}$$

Moreover, the general solution can be expressed as

$$X = X_0 + KFG, \quad \forall F \in C_a^{n\times m}(P, Q), \tag{2.9}$$

where

$$\begin{aligned} X_0 &= B_1^+C_2 + \big(I_n - B_2^+B_2\big)E_2D_1^+ + B_2^+C_1 + \big(I_n - B_1^+B_1\big)E_1D_2^+, \\ K &= I_n - B_1^+B_1 - B_2^+B_2, \qquad G = I_m - D_1D_1^+ - D_2D_2^+. \end{aligned} \tag{2.10}$$

Proof. Necessity: Since Problem 1 has a solution in $C_a^{n\times m}(P, Q)$, from Lemma 6, matrix equations $(\tilde{B}X = \tilde{C}, X\tilde{D} = \tilde{E})$ has a solution in $C_a^{n\times m}(P, Q) \subseteq C^{n\times m}$. From Lemma 7, we have

$$\tilde{C}\tilde{D} = \tilde{B}\tilde{E}, \qquad \tilde{C} = \tilde{B}\tilde{B}^+\tilde{C}, \qquad \tilde{E} = \tilde{E}\tilde{D}^+\tilde{D}. \tag{2.11}$$

Combining (1.1), (2.4), Lemma 5 and according to the first equality of (2.11), we have

$$\begin{pmatrix} C_2 \\ C_1 \end{pmatrix} (D_1 \ D_2) \ = \ \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} (E_2 \ E_1),$$

$$\begin{pmatrix} 0 & C_2 D_2 \\ C_1 D_1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & B_1 E_1 \\ B_2 E_2 & 0 \end{pmatrix}.$$

i.e.

$$C_2 D_2 = B_1 E_1, \qquad C_1 D_1 = B_2 E_2. \tag{2.12}$$

Combining Lemma 2, Lemma 5, (1.1), (2.4), and according to the second equality of (2.11), we have

$$\begin{pmatrix} C_2 \\ C_1 \end{pmatrix} \ = \ \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \left(B_1^+ \ B_2^+\right) \begin{pmatrix} C_2 \\ C_1 \end{pmatrix},$$

$$\begin{pmatrix} C_2 \\ C_1 \end{pmatrix} \ = \ \begin{pmatrix} B_1 B_1^+ & 0 \\ 0 & B_2 B_2^+ \end{pmatrix} \begin{pmatrix} C_2 \\ C_1 \end{pmatrix},$$

i.e.

$$C_1 = B_2 B_2^+ C_1, \qquad C_2 = B_1 B_1^+ C_2. \tag{2.13}$$

Using the similar methods, from the third equality of (2.11), we also have

$$E_1 = E_1 D_2^+ D_2, \qquad E_2 = E_2 D_1^+ D_1. \tag{2.14}$$

Combining (2.12)–(2.14) yields (2.7) and (2.8).

Sufficiency: From Lemma 2, Lemma 5, (2.7) and (2.8) are equivalent to (2.11) if (1.1), (2.4) hold. From Lemma 7, it is easy to see that matrix equations $(\tilde{B}X = \tilde{C}, X\tilde{D} = \tilde{E})$ has a solution in $C^{n\times m}$. Moreover, the general solution can be expressed as

$$X = X_0 + (I_n - \tilde{B}^+\tilde{B})F(I_m - \tilde{D}\tilde{D}^+), \quad \forall F \in C^{n\times m}, \tag{2.15}$$

where

$$X_0 = \tilde{B}^+\tilde{C} + (I_n - \tilde{B}^+\tilde{B})\tilde{E}\tilde{D}^+. \tag{2.16}$$

According to (1.1), (2.4), Lemma 2 and Lemma 5, we have

$$\begin{aligned} X_0 &= \begin{pmatrix} B_1^+ & B_2^+ \end{pmatrix} \begin{pmatrix} C_2 \\ C_1 \end{pmatrix} + \left(I_n - \begin{pmatrix} B_1^+ & B_2^+ \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \right) \begin{pmatrix} E_2 & E_1 \end{pmatrix} \begin{pmatrix} D_1^+ \\ D_2^+ \end{pmatrix} \\ &= B_1^+ C_2 + \left(I_n - B_2^+ B_2 \right) E_2 D_1^+ + B_2^+ C_1 + \left(I_n - B_1^+ B_1 \right) E_1 D_2^+. \end{aligned}$$

So X_0 in (2.16) is equivalent to X_0 in (2.10). From Lemma 3, Lemma 5, (2.11), it is easy to prove

$$X_0 \in C_a^{n\times m}(P, Q), \qquad \tilde{B} X_0 = \tilde{C}, \qquad X_0 \tilde{D} = \tilde{E}.$$

Hence X_0 is a solution of matrix equations $(\tilde{B}X = \tilde{C}, X\tilde{D} = \tilde{E})$ in $C_a^{n\times m}(P, Q)$. From Lemma 6, X_0 is also a solution of matrix equations $(BX = C, XD = E)$ in $C_a^{n\times m}(P, Q)$.

In the following, we show that the general solution of Problem 1 can be expressed as (2.9) if (1.1), (2.4), (2.7) and (2.8) hold. Denote by S_E the solution set of Problem 1 and S the set consisting of X expressed by (2.15) (S is the solution set of matrix equations $(\tilde{B}X = \tilde{C}, X\tilde{D} = \tilde{E})$ in $C^{n\times m}$). Denote

$$K = I_n - \tilde{B}^+ \tilde{B}, \qquad G = I_m - \tilde{D}\tilde{D}^+. \tag{2.17}$$

From (1.1), (2.4), Lemma 2, we have

$$K = I_n - B_1^+ B_1 - B_2^+ B_2, \qquad G = I_m - D_1 D_1^+ - D_2 D_2^+.$$

So K, G in (2.17) are equivalent to K, G in (2.10), respectively. Since $C_a^{n\times m}(P, Q) \subseteq C^{n\times m}$, it is clear that $S_E \subseteq S$. According to Lemma 3, we have

$$\begin{aligned} K &= I_n - \tilde{B}^+ \tilde{B} = I_n - B_1^+ B_1 - B_2^+ B_2 \in C_r^{n\times n}(P), \\ G &= I_m - \tilde{C}\tilde{C}^+ = I_m - C_1 C_1^+ - C_2 C_2^+ \in C_r^{m\times m}(Q). \end{aligned}$$

According to Lemma 4, $X = X_0 + EFG \in C_a^{n\times m}(P, Q)$ if and only if $F \in C_a^{n\times m}(P, Q)$, i.e. (2.15) is equivalent to (2.9) or $S = S_E$ if and only if $F \in C_a^{n\times m}(P, Q)$ in (2.15). From Lemma 6, the general solution of Problem 1 can be expressed as (2.9). □

3 The solution of Problem 2

According to (2.9), it is easy to prove that if Problem 1 has a solution in $C_a^{n\times m}(P, Q)$, then the solution set S_E is a nonempty closed convex set. We can claim that for any given $X^* \in C^{n\times m}$, there exists the unique optimal approximation for Problem 2.

Theorem 2. *Given $X^* \in C^{n\times m}$, if B, C, D, E are denoted by* (1.1) *and they satisfy the conditions of Theorem* 1*, then Problem 2 has the unique solution $\hat{X} \in S_E$. Moreover $\hat{X}$ can be expressed as*

$$\hat{X} = X_0 + KX_2^*G, \tag{3.1}$$

where X_0, K, G are given by (2.10)*, and X_2^* is given by the following equation.*

$$X_2^* = \frac{1}{2}\left(X^* - PX^*Q\right) \tag{3.2}$$

Proof. Denote $K_1 = I_n - K$, it is easy to prove that matrices K and K_1 are orthogonal projection matrices satisfying $KK_1 = 0$. Denote $G_1 = I_m - G$, it is also easy to prove that matrices G and G_1 are orthogonal projection matrices satisfying $GG_1 = 0$.

According to conclusion 2) of Definition 2, for any $X^* \in C^{n\times m}$, there exist only $X_1^* \in C_r^{n\times m}(P, Q)$ and only $X_2^* \in C_a^{n\times m}(P, Q)$, which satisfy that

$$X^* = X_1^* + X_2^*, \qquad \langle X_1^*, X_2^*\rangle = 0,$$

where

$$X_1^* = \frac{1}{2}(X^* + PX^*Q), \qquad X_2^* = \frac{1}{2}(X^* - PX^*Q).$$

From this, combining the invariance of Frobenius norm under orthogonal transformations and the methods of space decomposition, and according to (2.9), for any $X \in S_E$, we have

$$\begin{aligned}
\|X^* - X\|^2 &= \|X_1^* + X_2^* - X\|^2 = \|X_2^* - X\|^2 + \|X_1^*\|^2 \\
&= \|X_2^* - X_0 - KFG\|^2 + \|X_1^*\|^2 \\
&= \|(K + K_1)(X_2^* - X_0) - KFG\|^2 + \|X_1^*\|^2 \\
&= \|K(X_2^* - X_0) - KFG\|^2 + \|K_1(X_1^* - X_0)\|^2 + \|X_2^*\|^2
\end{aligned}$$

$$\begin{aligned} &= \|K(X_2^* - X_0)(G + G_1) - KFG\|^2 + \|K_1(X_2^* - X_0)\|^2 + \|X_1^*\|^2 \\ &= \|K(X_2^* - X_0)G - KFG\|^2 + \|K(X_2^* - X_0)G_1\|^2 \\ &\quad + \|K_1(X_2^* - X_0)\|^2 + \|X_1^*\|^2. \end{aligned}$$

Since $\|K(X_2^* - X_0)G_1\|^2$, $\|K_1(X_2^* - X_0)\|^2$, $\|X_1^*\|^2$ are constants, it is obvious that $\min_{\forall X \in S_E} \|X^* - X\|$ is equivalent to

$$\min_{\forall F \in C_a^{n\times m}(P,Q)} \|K(X_2^* - X_0)G - KFG\|. \tag{3.3}$$

According to the definitions of K, X_0, and G, it is easy to prove $KX_0G = 0$.
So, (3.3) is equivalent to

$$\min_{\forall F \in C_a^{n\times m}(P,Q)} \|KX_2^*G - KFG\|. \tag{3.4}$$

It is clear that $F = X_2^* + K_1F^*G_1$, $\forall F^* \in C_a^{n\times m}(P, Q)$ is a solution of (3.4). Substituting this result to (2.11) yields (3.1). □

Algorithm

1. Input B, C, D, E, P, Q, X^*.
2. According to (1.1) compute $B_1, B_2, C_1, C_2, D_1, D_2, E_1, E_2$.
3. Compute $C_1D_1, C_2D_2, B_1E_1, B_2E_2, B_1B_1^+C_2, B_2B_2^+C_1, E_1D_2^+D_2, E_2D_1^+D_1$, if (2.7), (2.8) hold, then go to 4; otherwise stop.
4. According to (2.10) compute X_0, K, G.
5. According to (3.2) compute X_2^*.
6. Calculate $\hat{X}$ from (3.1).

Numerical analysis

Theorem 2 leads naturally to this numerical algorithm for the solution of Problem 2. The process will then be numerically stable, the reason is that the singular value decomposition is numerically stable. We can also test that as X^* approximates a solution of Problem 1, X^* becomes closer to the unique solution $\hat{X}$ of Problem 2.

Example ($n = 10,\ m = 8,\ h = 5,\ l = 4$)

$$B = \begin{pmatrix} 1.5 & 3.6 & -7.8 & 1.9 & 1+3i & 1-i & 7.3 & 2.7 & 8.2 & -6.7 \\ 2i & 5.3 & 4.1 & 7.5 & 3.1 & -2.9 & 1.6 & 3.4 & -3i & 8.5 \\ 3.5 & 7.6 & 1-i & 5.8 & 9.6 & 7.5 & 4.2 & -5.1 & 3.5 & 7-2i \\ 3-5i & 6.7 & 1.3 & 4.6 & 6.5 & 5.9 & 10.1 & -7.6 & 1.8 & 4i \\ 7-2i & 5.6 & 7.5 & 6.9 & 12.6 & -10.3 & 3.5 & 5.7 & 7.1 & 10.9 \end{pmatrix},$$

$C = 1.0e + 002\ \times$

From first column to fourth column

$$\left(\begin{matrix} 0.6659 - 0.0020i & -0.3750 - 0.0180i & 0.3401 - 0.1735i & -0.0656 - 0.0335i \\ 0.0118 - 0.0990i & 0.1222 - 0.0510i & -1.1070 + 0.0140i & -0.3035 - 0.0907i \\ -0.0523 + 0.0300i & -0.2872 - 0.0545i & -0.4139 + 0.1610i & -0.6031 - 0.0203i \\ -0.1025 - 0.0450i & -0.5622 + 0.1950i & -0.4017 + 0.1120i & -0.7739 + 0.0728i \\ 0.3143 - 0.0180i & -0.2712 + 0.0780i & -1.8750 + 0.1495i & -0.3922 + 0.0607i \end{matrix} \right.$$

From fifth column to eighth column

$$\left. \begin{matrix} 0.3543 - 0.0105i & 0.2491 - 0.2340i & -0.2982 + 0.0130i & 0.6948 - 0.0250i \\ 0.3231 - 0.0868i & 0.5957 - 0.1135i & -0.1347 - 0.0640i & 0.0260 - 0.0860i \\ 0.3317 - 0.0050i & 0.2884 + 0.0090i & -0.3730 + 0.0040i & 0.4630 + 0.0475i \\ 0.3166 + 0.1212i & 0.3854 + 0.1315i & -0.5566 + 0.1300i & 0.6913 + 0.0200i \\ 0.2585 + 0.0735i & 0.2333 - 0.0785i & -0.1809 + 0.0520i & -0.0302 + 0.0080i \end{matrix} \right),$$

$$D = \begin{pmatrix} 1.3 & 5.7 & 2.9 & 4.5 \\ -3.5 & 4.6 & -0.9 & -5.1 \\ 2.7 & -1.6 & 1.1 & 6.2 \\ 2.1 & 5-3i & 5.3 & 2.3 \\ -5.1 & 7.5 & -3.1 & 1.7 \\ 1.5 & 0.7 & 4.2 & -1.2 \\ 2.9 & -3.2 & 2.4 & 1.8 \\ -4.6 & 1.8 & 6.4 & 3.6 \end{pmatrix},$$

$$E = \begin{pmatrix} 10.635 & -26.995 + 5.25i & -8.95 & -0.825 \\ -14.72 & -30.43 + 7.95i & 17.82 & -49.12 \\ 41.145 + 4.05i & -68.815 + 1.65i & -11.53 + 1.65i & -5.18 + 9.3i \\ -29.695 & 32.875 + 7.2i & 5.44 & 17.435 \\ -0.755 - 1.275i & -17.49 + 4.875i & -36.27 - 0.775i & 8.72 + 0.425i \\ 20.585 - 0.525i & -5.385 - 1.55i & 2.915 - 1.325i & 45.69 - 0.575i \\ -25.55 & 18.705 + 7.35i & 3.615 & 18.995 \\ 22.14 - 2.25i & 34.51 - 8.25i & -1.195 - 6.3i & 2.11 + 1.8i \\ -3.375 & 45.85 - 6.3i & 43.58 & 39.975 \\ -43.6 & 60.05 - 2.25i & -14.67 & -73.64 \end{pmatrix},$$

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},$$

$$X^* = \begin{pmatrix} 4.7 & -3.5 & 3.9 & -2.3 & 2.6 & -1.9 & -3.6 & 5.1 \\ -2.9 & 2.8 & 0.8 & 2.8 & 1.7 & 4.5 & 7.3 & 6.9 \\ 1.3 & 1.6 & 1.2 & 1.6 & 2.5 & 2.1 & 9.5 & -2.9 \\ 9.6 & -2.1 & -1.5 & -2.5 & 2.8 & 0.7 & 2.1 & 7.5 \\ -1.5 & 1.4 & 7.5 & -0.1 & 0.5i & -3.5 & -1.5 & 1.6 \\ 2.7 & -1.8 & 9.6 & 0.2 & 1.9 & 6.9 & 3.8 & -3.1 \\ 1.7 & 3.2 & -1.7 & -2.1 & 2.3 & 1.4 & 1.9 & 7.6 \\ 8.1 & 5.5 & 2.2 & 7.3 & 4.3 & 2-3i & 2.6 & 1.5 \\ 4.3 & 2.9 & -1.2 & 6.8 & 1.2 & 7.6 & 5.9 & 1.6 \\ 1.9 & 5.8 & -9.7 & 3.2 & 8.1 & 7.6 & 2.5 & 5.3 \end{pmatrix}.$$

It is easy to see that B, C, D, E, P, Q, X^* satisfy the required properties. Using the software "MATLAB", we obtain the unique solution $\hat{X}$ of Problem 2.

$$\hat{X} = \begin{pmatrix} 0.9 & -3.9 & -1.85 & -1.75 & -2.1 & -0.35 & -2.6 & -0.4 \\ -4.35 & 0.45 & -3.4 & -2.65 & -0.75 & 7.1 & 1 & 2.2 \\ -2.1 & -3.25 & -0.4+1.5i & -1.35 & -2.4 & -0.05 & 4 & -2.75 \\ 3.2 & -1.9 & -1.45 & -2.4 & 2.45 & 1.2 & -2.75 & 2.8 \\ 0.15 & -0.65 & 0.3 & -1 & -0.1+0.25i & -6.55 & 0.8 & -1.1 \\ 0.65 & -0.15 & 6.55 & 0.1-0.25i & 1 & -0.3 & 1.1 & -0.8 \\ 1.9 & -3.2 & -1.2 & -2.45 & 2.4 & 1.45 & -2.8 & 2.75 \\ 3.25 & 2.1 & 0.05 & 2.4 & 1.35 & 0.4-1.5i & 2.75 & -4 \\ 3.9 & -0.9 & 0.35 & 2.1 & 1.75 & 1.85 & 0.4 & 2.6 \\ -0.45 & 4.35 & -7.1 & 0.75 & 2.65 & 3.4 & -2.2 & -1 \end{pmatrix}.$$

4 Conclusions

In this paper, we considered the generalized anti-reflexive solutions of matrix equations ($BX = C$, $XD = E$), i.e. Problem 1. We also considered the nearest solution to a given matrix in Frobenius norm, i.e. Problem 2. The solvability conditions and the explicit formula for the solution are given. According to Theorem 1 and 2, the algorithm is presented to compute the nearest solution. The numerical example is given to illustrate the results obtained in this paper correction.

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REFERENCES

[1] J.H. Wilkinson, *The Algebraic Problem*. Oxford University Press, Oxford, 1965.

[2] M. Arav, D. Hershkowitz, V. Mehrmann and H. Schneider, *The recursive inverse eigenvalue problem*. SIAM J. Matrix Anal. Appl., **22** (2000), 392–412.

[3] R. Loewy and V. Mehrmann, *A note on the symmetric recursive inverse eigenvalue problem*. SIAM J. Matrix Anal. Appl., **25** (2003), 180–187.

[4] F.L. Li, X.Y. Hu and L. Zhang, *Left and right eigenpairs problem of skew-centrosymmetric matrices*. Appl. Math. Comput., **177** (2006), 105–110.

[5] S.K. Mitra and M.L. Puri, *Shorted operaters and generalized inverses of matrices*. Linear Algebra Appl., **25** (1979), 45–56.

[6] Z.Y. Peng, *The inverse eigenvalue problem for Hermitian anti-reflexive matrices and its approximation*. Appl. Math. Comput., **162** (2005), 1377–1389.

[7] S.K. Mitra, *The matrix equations* $AX = C, XB = D$. Linear Algebra Appl., **59** (1984), 171–181.

[8] K-W.E. Chu, *Singular value and generalized singular value decomposition and the solution of linear matrix equations*. Linear Algebra Appl., **88/89** (1987), 83–98.

[9] S.K. Mitra, *A pair of simultaneous linear matrix equations* $A_1XB_1 = C_1$, $A_2XB_2 = C_2$ *and a matrix programming problem*. Linear Algebra Appl., **131** (1990), 107–123.

[10] Y.L. Cheng, *The iterative methods of solving matrix equations* $AX = C$, $XB = D$. J. Nanjing Norm. Univ. (Natural Science Edition), **22**(1) (1999): 1–3.

[11] H.C. Chen, *Generalized reflexive matrices: Special properties and applications*. SIAM J. Matix Anal. Appl., **19** (1998), 140–153.

[12] J.L. Chen and X.H. Chen, *Special Matrices*. Qinghua University Press, Beijing, China, 2001 (in Chinese).

[13] T. Meng, *Experimental design and decision support*. In: C. Leondes (Ed), Expert System, The Technology of Knowledge Management and Decision Making for the 21st Century, Vol. 1, Academic Press, 2001.

[14] N.J. Higham, *Computing a nearest symmetric positive semidefinite matrix*. Linear Algebra Appl., **103** (1988), 103–118.

[15] L. Zhang, *The approximation on the closed convex cone and its numerical application*. Hunan Ann. Math., **6** (1986), 43–48.

[16] G.H. Golub and Ch.F. Van Loan, *Matrix Computations*. Johns Hopkins University Press, Baltimore Maryland, 1989.