COMPUTATIONAL
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MATHEMATICS

On the eigenvalues of Euclidean distance matrices

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Abstract. In this paper, the notion of equitable partitions (EP) is used to study the eigenvalues of Euclidean distance matrices (EDMs). In particular, EP is used to obtain the characteristic polynomials of regular EDMs and non-spherical centrally symmetric EDMs. The paper also presents methods for constructing cospectral EDMs and EDMs with exactly three distinct eigenvalues.

Mathematical subject classification: 51K05, 15A18, 05C50.

Key words: Euclidean distance matrices, eigenvalues, equitable partitions, characteristic polynomial.

1 Introduction

An $n \times n$ nonzero matrix $D = (d_{ij})$ is called a *Euclidean distance matrix (EDM)* if there exist points $p^1, p^2, \ldots, p^n$ in some Euclidean space $\Re^r$ such that

$$d_{ij} = \| p^i - p^j \|^2 \quad \text{for all} \quad i, j = 1, \ldots, n,$$

where $\| \ \|$ denotes the Euclidean norm.

Let p^i, $i \in N = \{1, 2, \ldots, n\}$, be the set of points that generate an EDM D. An *m-partition* π of D is an ordered sequence $\pi = (N_1, N_2, \ldots, N_m)$ of nonempty disjoint subsets of N whose union is N. The subsets $N_1, \ldots, N_m$ are called the *cells* of the partition. The *n*-partition of D where each cell consists

of a single point is called the *discrete partition*, while the 1-partition of D with only one cell is called the *single-cell partition*.

An m-partition $\pi = (N_1, N_2, \ldots, N_m)$ of an EDM D is said to be *equitable* if for all $i, j = 1, \ldots, m$ (case $i = j$ included), there exist non-negative scalars α_{ij} such that for each $k \in N_i$, the sum of the squared Euclidean distances from p^k to all points p^l, l in N_j, is equal to α_{ij}. i.e.,

$$\forall\, k \in N_i, \quad \sum_{l \in N_j} d_{kl} = \alpha_{ij}, \text{ for all } i, j = 1, \ldots, m. \tag{1}$$

The notion of equitable partitions for graphs, which was introduced by Sachs [13], is related to, among others, automorphism groups of graphs and distance-regular graphs [4]. Schwenk [15] used equitable partitions to find the eigenvalues of the adjacency matrix of a graph. Hayden et al. [7] also used equitable partitions, albeit under the name *block structure*, to investigate EDMs generated by points lying on a collection of concentric spheres. In particular, they devised an algorithm for finding the least number of concentric spheres containing the points that generate a given EDM. Their investigation was based on the block structure of EDMs and the corresponding eigenvectors.

Many of the results on the spectra of graphs obtained using equitable partitions have analogous counterparts in the case of EDMs. In particular, we show (see Theorem 3.1) that the characteristic polynomial of an EDM can be written as the product of the characteristic polynomials of two matrices associated with partitions. Theorem 3.1 is then used to determine the characteristic polynomials of regular EDMs and non-spherical centrally symmetric EDMs. We also present methods for constructing cospectral EDMs and non-regular EDMs with exactly three distinct eigenvalues.

Recently, EDMs have received a great deal of attention for their many important applications. These applications include, among others, molecular conformation problems in chemistry [2], multidimensional scaling in statistics [9], and wireless sensor network localization problems [16].

We denote the identity matrix of order n by I_n and the n-vector of all 1's by e_n. $E_{n,m} = e_n e_m^T$ denotes the $n \times m$ matrix of all 1's, and $E_n = e_n e_n^T$ denotes the square matrix of order n of all 1's. The subscripts of I, e and E will be deleted if the order is clear from the context. For a matrix A, diag A

denotes the vector consisting of the diagonal entries of A. Finally, the spectrum of a matrix A, denoted by $\sigma(A)$, is the multiset of the eigenvalues of A. If A has eigenvalues $\lambda_1, \ldots, \lambda_k$ with multiplicities $m_1, \ldots, m_k$ respectively, then $\sigma(D) = \{\lambda_1^{m_1}, \ldots, \lambda_k^{m_k}\}$.

2   Preliminaries

Let D be an $n \times n$ EDM, the dimension of the affine span of the points $p^1, \ldots, p^n$ that generate D is called the *embedding dimension* of D. Let $J_n := I_n - e_n e_n^T / n$ denote the orthogonal projection on subspace

$$M := e^{\perp} = \left\{ x \in \Re^n : e^T x = 0 \right\}. \tag{2}$$

It is well known [14, 18] that a symmetric matrix D with zero diagonal is an EDM if and only if D is negative semidefinite on M. Hence, EDMs have exactly one positive eigenvalue. Let S_H denote the subset of $n \times n$ symmetric matrices with zero diagonal; and let S_C denote the subset of $n \times n$ symmetric matrices A satisfying $Ae = 0$. Following [3], let $\mathcal{T} : S_H \rightarrow S_C$ and $\mathcal{K} : S_C \rightarrow S_H$ be the two linear maps defined by

$$\mathcal{T}(D) := -\frac{1}{2} JDJ, \tag{3}$$

and

$$\mathcal{K}(B) := \operatorname{diag}(B)e^T + e(\operatorname{diag}(B))^T - 2B. \tag{4}$$

It, then, immediately follows [3] that $\mathcal{T}$ and $\mathcal{K}$ are mutually inverse, and that D in S_H is an EDM of embedding dimension r if and only if $\mathcal{T}(D)$ is positive semidefinite of rank r.

Let D be an $n \times n$ EDM of embedding dimension r generated by the points $p^1, \ldots, p^n$ in $\Re^r$. Then the $n \times r$ matrix

$$P = \begin{bmatrix} p^{1T} \\ \vdots \\ p^{nT} \end{bmatrix}$$

is called a *realization* of D. Given an EDM D, a realization P of D can be obtained by factorizing $\mathcal{T}(D)$ into $\mathcal{T}(D) = PP^T$. Note that if P is a realization

of D, then $P' = PQ$ is also a realization of D for any $r \times r$ orthogonal matrix Q. Obviously, P and P' in this case are obtained from each other by a rigid motion such as a rotation or a translation.

An EDM D is said to be *spherical* if the points that generate D lie on a hypersphere, otherwise, it is said to be *non-spherical*. It is well known [6, 17] that an EDM D of embedding dimension r is spherical if and only if rank $D = r + 1$, and that D is non-spherical if and only if rank $D = r + 2$. A spherical EDM D is said to be *regular* if the points $p^1, \ldots, p^n$ that generate D lie on a hyper-sphere whose center coincides with the centroid of $p^1, \ldots, p^n$. It is not difficult to show that [12, 8] an EDM D is regular if and only if e is an eigenvector of D corresponding to the eigenvalue $\frac{1}{n}e^T De$.

3   Equitable partition for EDMs

It immediately follows from (1) that the discrete partition of D is always equitable with $\alpha_{ij} = d_{ij}$; while the single-cell partition of D is equitable if and only if D is regular. In the latter case, $\alpha_{11} = \frac{1}{n}e^T De$. It also follows from (1) that

$$\forall\, i = 1, \ldots, m \ \text{ and } \ \forall\, k \in N_i \ \text{ we have } \ (De)_k = \sum_{j=1}^{m} \alpha_{ij}. \qquad (5)$$

Let $\pi = (N_1, N_2, \ldots, N_m)$ be an m-partition of an $n \times n$ EDM D where $|N_i| = n_i$ for $i = 1, \ldots, m$. Define the $n \times m$ matrix $P_\pi = (p_{ij})$ such that

$$p_{ij} = \begin{cases} n_j^{-1/2} & \text{if } i \in N_j \\ 0 & \text{otherwise.} \end{cases} \qquad (6)$$

P_π is called the *normalized characteristic matrix* [5] of π since its jth column is equal to $n_j^{-1/2}$ times the characteristic vector of N_j, and since $P_\pi^T P_\pi = I_m$.

Next we present a lemma whose graph adjacency matrix counterpart was proved by Godsil and McKay [5].

Lemma 3.1. *Let π be an m-partition of an EDM D. Then π is equitable if and only if there exists an $m \times m$ symmetric matrix $S = (s_{ij})$ such that*

$$DP_\pi = P_\pi S. \qquad (7)$$

Furthermore, if π is an equitable partition then $s_{ij} = (n_i/n_j)^{1/2}\,\alpha_{ij}$.

Proof.   Assume that π is equitable then for all $k \in N_i$ and for all $j = 1, \dots, m$ we have

$$(DP_\pi)_{kj} = n_j^{-1/2} \sum_{l \in N_j} d_{kl} = n_j^{-1/2} \alpha_{ij} = (n_i)^{-1/2} \left(\frac{n_i}{n_j}\right)^{1/2} \alpha_{ij} = (P_\pi S)_{kj},$$

where $s_{ij} = (n_i/n_j)^{1/2} \alpha_{ij}$.

On the other hand, assume that (7) holds for some partition π. Then for all $k \in N_i$ and all $j = 1, \dots, m$, we have

$$(DP_\pi)_{kj} = n_j^{-1/2} \sum_{l \in N_j} d_{kl} = (P_\pi S)_{kj} = (n_i)^{-1/2} s_{ij}.$$

Therefore, $\sum_{l \in N_j} d_{kl} = (n_j/n_i)^{1/2} s_{ij}$, which is independent of k. Hence π is equitable. $\square$

Given an m-partition π of EDM D with $m \le n - 1$, let $\bar{P}_\pi$ be the $n \times (n - m)$ matrix such that $[P_\pi \ \bar{P}_\pi]$ is an orthogonal matrix. Let $\chi_A(\lambda)$ denote the characteristic polynomial of matrix A. Then we have the following two results:

Theorem 3.1.   *Let π be an equitable m-partition of an $n \times n$ EDM D, where $m \le n - 1$. Then*

$$\chi_D(\lambda) = \chi_S(\lambda) \, \chi_{\bar{S}}(\lambda), \tag{8}$$

where S is defined in (7) and $\bar{S} = \bar{P}_\pi^T D \bar{P}_\pi$.

Proof.   It follows from (7) and the definition of $\bar{P}_\pi$ that $P_\pi^T D \bar{P}_\pi = S P_\pi^T \bar{P}_\pi = 0$. Thus,

$$\det\left(\lambda \, I_n - \begin{bmatrix} P_\pi^T \\ \bar{P}_\pi^T \end{bmatrix} D \begin{bmatrix} P_\pi & \bar{P}_\pi \end{bmatrix}\right) = \det \begin{bmatrix} \lambda I_m - P_\pi^T D P_\pi & 0 \\ 0 & \lambda I_{n-m} - \bar{P}_\pi^T D \bar{P}_\pi \end{bmatrix}.$$

Hence,

$$\det(\lambda I_n - D) = \det(\lambda I_m - S) \det(\lambda I_{n-m} - \bar{S}).   \square$$

Note that in case of discrete partitions, i.e., in case $m = n$, we have $S = D$ thus $\chi_D(\lambda) = \chi_S(\lambda)$ follows trivially. An analogous result for graphs, namely that the characteristic polynomial of S divides the characteristic polynomial of a graph was obtained by Mowshowitz [10], and by Schwenk [15].

Theorem 3.2. *Let π be an equitable m-partition of an $n \times n$ EDM D, where $m \leq n - 1$. Then*

$$\chi_{\bar{S}}(\lambda) \text{ divides } \chi_{-2\mathcal{T}(D)}(\lambda), \tag{9}$$

where $\bar{S}$ is as defined in Theorem 3.1.

Proof. It follows from (5) and the definition of $\bar{P}_\pi$ that $\bar{P}_\pi^T De = 0$ and $\bar{P}_\pi^T e = 0$. Thus,

$$\bar{P}_\pi^T \mathcal{T}(D) P_\pi = 0 \text{ and } \bar{P}_\pi^T \mathcal{T}(D) \bar{P}_\pi = -\frac{1}{2} \bar{P}_\pi^T D \bar{P}_\pi = -\frac{\bar{S}}{2}.$$

Hence,

$$\det\left(\lambda I_n + 2 \begin{bmatrix} P_\pi^T \\ \bar{P}_\pi^T \end{bmatrix} \mathcal{T}(D) \begin{bmatrix} P_\pi & \bar{P}_\pi \end{bmatrix}\right)$$

$$= \det \begin{bmatrix} \lambda I_m + 2 P_\pi^T \mathcal{T}(D) P_\pi & 0 \\ 0 & \lambda I_{n-m} + 2 \bar{P}_\pi^T \mathcal{T}(D) \bar{P}_\pi \end{bmatrix}.$$

Therefore,

$$\det\left(\lambda I_n + 2\mathcal{T}(D)\right) = \det\left(\lambda I_m + 2 P_\pi^T \mathcal{T}(D) P_\pi\right) \det\left(\lambda I_{n-m} - \bar{S}\right). \qquad \square$$

4   Applications of Theorem 3.1

In this section we show that Theorem 3.1 provides a new method for determining the characteristic polynomials of regular EDMs and non-spherical centrally symmetric EDMs. It also provides a method for constructing cospectral EDMs.

Let V_n be the $n \times (n - 1)$ whose columns form an orthonormal basis for subspace M defined in (2), i.e., V_n has full column rank and satisfies

$$V_n^T e_n = 0, \quad V_n^T V_n = I_{n-1}, \quad V_n V_n^T = J_n = \left(I_n - e_n e_n^T / n\right). \tag{10}$$

4.1 $\chi_D(\lambda)$ of regular EDMs

The characteristic polynomial of a regular EDM was obtained in [8, 1]. The method used in [1] is based on a characterization of the nullspace of an EDM in terms of its Gale subspace. Next we determine the characteristic polynomial of regular EDMs as a corollary of Theorem 3.1.

Corollary 4.1 ([8, 1]).   *Let $D \neq 0$ be an $n \times n$ regular EDM of embedding dimension r, then*

$$\chi_D(\lambda) = \lambda^{n-r-1} \left(\lambda - \frac{1}{n} e^T D e \right) \prod_{i=1}^{r} (\lambda + 2\mu_i),$$

where μ_i, for $i = 1, \ldots, r$, are the nonzero eigenvalues of $\mathcal{T}(D)$.

Proof.   Let D be an $n \times n$ regular EDM. Then, as we remarked earlier, the one-cell partition π of D, in this case, is equitable. Since $P_\pi = \frac{1}{\sqrt{n}} e$ and $\bar{P}_\pi = V_n$ we have $S = \frac{1}{n} e^T D e$ and $\bar{S} = V_n^T D V_n = -2 V_n^T \mathcal{T}(D) V_n$. Thus,

$$\chi_S(\lambda) = \lambda - \frac{1}{n} e^T D e \text{ and } \chi_{\bar{S}}(\lambda) = \lambda^{n-r-1} \prod_{i=1}^{r} (\lambda + 2\mu_i)$$

since the nonzero eigenvalues of $\bar{S}$ are equal to the nonzero eigenvalues of the matrix $-2\mathcal{T}(D)$. $\qquad \square$

4.2 $\chi_D(\lambda)$ of non-spherical centrally symmetric EDMs

Let D_1 be the $2n \times 2n$ EDM generated by the points $p^1, \ldots, p^n, \ldots, p^{2n}$, where $p^{n+i} = -p^i$ for all $i = 1, \ldots, n$. Assume that D_1 is non-spherical. Then D_1 is called a *non-spherical centrally symmetric* EDM. It is easy to see that

$$D_1 = \begin{bmatrix} D & A \\ A & D \end{bmatrix},$$

where D is the $n \times n$ EDM generated by the points $p^1, \ldots, p^n$, and

$$A = \text{diag}\,(\mathcal{T}(D))\, e_n^T + e_n \,\text{diag}\,(\mathcal{T}(D))^T + 2\,\mathcal{T}(D). \tag{11}$$

The characteristic polynomial of D_1, which was obtained in [1] using a characterization of the nullspace of an EDM in terms of its Gale subspace, is given by

$$\chi_{D_1}(\lambda) = \lambda^{2n-r-2} \left(\lambda^2 - \lambda \frac{1}{2n} e_{2n}^T D_1 e_{2n} - \frac{2n}{x_1^T x_1} \right) \prod_{i=1}^{r} (\lambda + 2\mu_i), \tag{12}$$

where r is the embedding dimension of D_1, μ_i, $i = 1, \ldots, r$, are the nonzero eigenvalues of $\mathcal{T}(D_1)$, and $x_1 \in \Re^{2n}$ is the unique vector satisfying

$$D_1 x_1 = e_{2n}, \quad e_{2n}^T x_1 = 0, \quad \text{and} \quad x_1 \perp \text{Nullspace of } D_1. \tag{13}$$

Note that such x_1 satisfying (13) exists since D_1 is non-spherical. A simple method to compute x_1 is given in [1]. Next, we establish (12) using Theorem 3.1.

The n-partition π of D_1 corresponding to the normalized characteristic matrix

$$P_\pi = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n \\ I_n \end{bmatrix}$$

is, obviously, equitable. Since

$$\bar{P}_\pi = \frac{1}{\sqrt{2}} \begin{bmatrix} I_n \\ -I_n \end{bmatrix},$$

it immediately follows that $S = P_\pi^T D_1 P_\pi = D + A = 2 (\text{diag} (\mathcal{T}(D)) e_n^T + e_n \text{diag} (\mathcal{T}(D))^T)$ and $\bar{S} = \bar{P}_\pi^T D_1 \bar{P}_\pi = D - A = -4\mathcal{T}(D)$. Hence, the nonzero eigenvalues of $\bar{S}$ = the nonzero eigenvalues of $-4\mathcal{T}(D)$ = the nonzero eigenvalues of $-2\mathcal{T}(D_1)$ since

$$\mathcal{T}(D_1) = \begin{bmatrix} \mathcal{T}(D) & -\mathcal{T}(D) \\ -\mathcal{T}(D) & \mathcal{T}(D) \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \otimes \mathcal{T}(D), \tag{14}$$

where $\otimes$ denotes the Kronecker product. Hence, $\chi_{\bar{S}}(\lambda) = \lambda^{n-r} \prod_{i=1}^r (\lambda + 2\mu_i)$, where μ_i, $i = 1, \ldots, r$, are the nonzero eigenvalues of $\mathcal{T}(D_1)$.

In order to establish (12), we still need to determine $\chi_S(\lambda)$. To this end, note that $\mathcal{T}(D_1) D_1 e_{2n} = 2n \mathcal{T}(D_1) \text{diag} (\mathcal{T}(D_1)) = 0$, where the first equality follows from (4). Then we have the following technical lemma.

Lemma 4.1. *Let D_1 be an $2n \times 2n$ non-spherical EDM satisfying*

$$\mathcal{T}(D_1) D_1 e_{2n} = 0.$$

Then

$$D_1 e_{2n} = \frac{e_{2n}^T D_1 e_{2n}}{2n} e_{2n} + \frac{2n}{x_1^T x_1} x_1, \tag{15}$$

where x_1 satisfies (13).

Proof.   Let $\langle\, y \,\rangle$ denote the subspace generated by $y \in \Re^{2n}$. Then it is shown in [1] that

$$\text{Nullspace of } \mathcal{T}(D_1) = \langle\, e_{2n} \,\rangle \oplus \langle\, x_1 \,\rangle \oplus \text{ Nullspace of } D_1.$$

Thus,

$$D_1 e_{2n} = \beta e_{2n} + \gamma x_1, \tag{16}$$

for some scalars β and γ. The result follows by multiplying equation (16) from the left with e_{2n}^T and x_1^T respectively. $\square$

Therefore, it follows from (15) that $x_1^T = (x^T \;\; x^T)$ for some vector $x \in \Re^n$ and

$$Se_n = (D + A)e_n = \frac{e_{2n}^T D_1 \, e_{2n}}{2n} e_n + \frac{2n}{x_1^T x_1} x. \tag{17}$$

Now, from the definition of V_n in (10) we have $V_n^T S V_n = 0$. Hence,

$$\chi_S(\lambda) = \det\left(\lambda I_n - \begin{bmatrix} V_n^T \\ \dfrac{e_n^T}{\sqrt{n}} \end{bmatrix} S \begin{bmatrix} V_n & \dfrac{e_n}{\sqrt{n}} \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} \lambda I_{n-1} & -\dfrac{2\sqrt{n}}{x_1^T x_1} V_n^T x \\[2mm] -\dfrac{2\sqrt{n}}{x_1^T x_1} x^T V_n & \lambda - \dfrac{1}{2n} e_{2n}^T D_1 e_{2n} \end{bmatrix} \right).$$

Thus,

$$\chi_S(\lambda) = \lambda^{n-2} \left(\lambda \left(\lambda - \frac{1}{2n} e_{2n}^T D_1 e_{2n} \right) - \frac{4n}{(x_1^T x_1)^2} x^T V_n V_n^T x \right)$$

$$= \lambda^{n-2} \left(\lambda \left(\lambda - \frac{1}{2n} e_{2n}^T D_1 \, e_{2n} \right) - \frac{2n}{x_1^T x_1} \right),$$

since $x^T V_n V_n^T x = x^T x = x_1^T x_1 / 2$. Hence, (12) is established.

4.3   Constructing cospectral EDMs

Two EDMs are said to be *cospectral* if they have the same eigenvalues, i.e., if they have the same characteristic polynomial. In this subsection, we present a method for constructing two cospectral EDMs.

Given a scalar $\mu > 1$ and an $n \times n$ regular EDM D generated by the points $p^1, \ldots, p^n$, let D_1 and D_2 be the two $2n \times 2n$ EDMs constructed from D as follows. D_1 is generated by the $2n$ points $p^1, p^2, \ldots, p^n, -\mu p^1, -\mu p^2, \ldots, -\mu p^n$; and D_2 is generated by the $2n$ points $p^1, p^2, \ldots, p^n, \mu p^1, \mu p^2, \ldots, \mu p^n$. Next we show that D_1 and D_2 are cospectral.

Since D is regular, we have $\operatorname{diag}(\mathcal{T}(D)) = \frac{e_n^T D e_n}{2n^2} e_n$. Thus it follows from (3) and (4) that

$$D_1 = \begin{bmatrix} D & A_1 \\ A_1 & \mu^2 D \end{bmatrix}, \quad \text{and} \quad D_2 = \begin{bmatrix} D & A_2 \\ A_2 & \mu^2 D \end{bmatrix},$$

where

$$A_1 = \frac{1}{2n^2} e_n^T D e_n (\mu^2 + 1) e_n e_n^T + 2\mu \mathcal{T}(D), \tag{18}$$

and

$$A_2 = \frac{1}{2n^2} e_n^T D e_n (\mu^2 + 1) e_n e_n^T - 2\mu \mathcal{T}(D). \tag{19}$$

Note that $A_1 e_n = A_2 e_n = \frac{1}{2n} e_n^T D e_n (\mu^2 + 1) e_n$. Therefore, the 2-partition π of D_1 and D_2 corresponding to the normalized characteristic matrix

$$P_\pi = \frac{1}{\sqrt{n}} \begin{bmatrix} e_n & 0 \\ 0 & e_n \end{bmatrix}$$

is equitable. Since

$$\bar{P}_\pi = \begin{bmatrix} V_n & 0 \\ 0 & V_n \end{bmatrix},$$

it immediately follows that

$$S_1 = P_\pi^T D_1 P_\pi = \frac{1}{n} e_n^T D e_n \begin{bmatrix} 1 & \frac{1}{2}(\mu^2 + 1) \\ \frac{1}{2}(\mu^2 + 1) & \mu^2 \end{bmatrix} = P_\pi^T D_2 P_\pi = S_2.$$

Furthermore,

$$\bar{S}_1 = \bar{P}_\pi^T D_1 \bar{P}_\pi = \begin{bmatrix} V_n^T D V_n & -\mu V_n^T D V_n \\ -\mu V_n^T D V_n & \mu^2 V_n^T D V_n \end{bmatrix} = \begin{bmatrix} 1 & -\mu \\ -\mu & \mu^2 \end{bmatrix} \otimes V_n^T D V_n,$$

and

$$\bar{S}_2 = \bar{P}_\pi^T D_2 \bar{P}_\pi = \begin{bmatrix} V_n^T D V_n & \mu V_n^T D V_n \\ \mu V_n^T D V_n & \mu^2 V_n^T D V_n \end{bmatrix} = \begin{bmatrix} 1 & \mu \\ \mu & \mu^2 \end{bmatrix} \otimes V_n^T D V_n.$$

But the eigenvalues of $\bar{S}_1$ are equal to the eigenvalues of $\bar{S}_2$. In particular, the nonzero eigenvalues of $\bar{S}_1 =$ the nonzero eigenvalues of $\bar{S}_2 = (1 + \mu^2)$ times the nonzero eigenvalues of $V_n^T D V_n = (1 + \mu^2)$ times the nonzero eigenvalues of $-2\mathcal{T}(D)$. Therefore, D_1 and D_2 are two cospectral EDMs. Furthermore, D_1 and D_2 are not regular since $\mu > 1$.

5   Constructing EDMs with three distinct eigenvalues

EDMs have two or more distinct eigenvalues since the eigenvalues of an EDM D can not all be equal. Moreover, EDMs with exactly two distinct eigenvalues are precisely those generated by the standard simplex, i.e., EDMs of the form $D = \gamma(E - I)$ for some scalar $\gamma > 0$. The problem of obtaining a complete characterization of EDMs with exactly three distinct eigenvalues, just like its graph counterpart, seems to be difficult.

Neumaier [11] introduced the notion of *strength* for distance matrices as a measure of their regularity. According to Neumaier regular EDMs are of strength 1 while regular EDMs with three distinct eigenvalues, where one of the eigenvalues is a zero are of strength 2. The following theorem follows from a more general result due to Neumaier [11].

Theorem 5.1. *Let D be a regular EDM with exactly three distinct eigenvalues such that its off-diagonal entries have exactly 2 or 3 distinct values. Then any two rows (columns) of D are obtained from each other by a permutation.*

Next we present two classes of non-regular EDMs with exactly 3 distinct eigenvalues. The first class consists of non-regular EDMs of order $n+1$ obtained from $n \times n$ regular EDMs with 3 distinct eigenvalues by adding one row and one column of equal entries.

Theorem 5.2. *Let D be an $n \times n$ regular EDM such that $\sigma(D) = \{e^T De/n,$ $-\lambda_1^{r_1}, -\lambda_2^{n-1-r_1}\}$. Assume that either $(e^T De/n + \lambda_1)\lambda_1 = n\alpha^2$ or $(e^T De/n +$*

$\lambda_2)\lambda_2 = n\alpha^2$ *for some* $\alpha \geq \frac{1}{2n^2}e^T De$, $\alpha \neq \frac{1}{n(n-1)}e^T De$. *Let* γ *be some positive scalar. Then*

$$D_1 = \gamma \begin{bmatrix} D & \alpha e_n \\ \alpha e_n^T & 0 \end{bmatrix}$$

is a non-regular EDM of order $n + 1$ *with exactly 3 distinct eigenvalues.*

Proof.   It is easy to see that D_1 is a non-regular EDM since $\alpha \neq \frac{1}{n(n-1)}e^T De$. Now assume $(e^T De/n + \lambda_1)\lambda_1 = n\alpha^2$. Let V be the $n \times (n-1)$ matrix whose columns form an orthonormal basis for subspace M defined in (2), thus $VV^T = J = I - ee^T/n$. Let

$$Q = \begin{bmatrix} e_n/\sqrt{n} & 0 & V \\ 0 & 1 & 0 \end{bmatrix}.$$

Then

$$Q^T D_1 Q = \gamma \begin{bmatrix} e^T De/n & \alpha n^{1/2} & 0 \\ \alpha n^{1/2} & 0 & 0 \\ 0 & 0 & V^T DV \end{bmatrix}.$$

But it is well known [1, 8] that $\sigma(-2\mathcal{T}(D)) = \sigma(VV^T DVV^T) = \{-\lambda_1^{r_1}, -\lambda_2^{n-1-r_1}, 0\}$. Thus $\sigma(V^T DV) = \{-\lambda_1^{r_1}, -\lambda_2^{n-1-r_1}\}$. Furthermore, the submatrix

$$\begin{bmatrix} e^T De/n & \alpha n^{1/2} \\ \alpha n^{1/2} & 0 \end{bmatrix}$$

has eigenvalues $-\lambda_1$ and $\frac{1}{n}e^T De + \lambda_1$. The result follows since Q is an orthogonal matrix. $\square$

Note that, taking $\alpha = \frac{1}{2n^2}e^T De$ in Theorem 5.2 is equivalent to placing point p^{n+1} at the center of the hyper-sphere containing $p^1, \ldots, p^n$. Also, note that if $\alpha = \frac{1}{n(n-1)}e^T De$, then D_1 becomes a regular EDM.

The second class consists of non-regular EDMs of order $n + 1$ obtained from EDMs of the form $D = \lambda(E_n - I_n)$, $\lambda > 0$, by adding one row and one column of equal entries. The proof of the next theorem is similar to that of Theorem 5.2.

Theorem 5.3.   *Let* γ *be a positive scalar and* $\alpha \geq (n-1)/2n$, $\alpha \neq 1$. *Then*

$$D_1 = \gamma \begin{bmatrix} E_n - I_n & \alpha e_n \\ \alpha e_n^T & 0 \end{bmatrix}$$

is a non-regular EDM with exactly 3 distinct eigenvalues, i.e., $\sigma(D_1) = \{\lambda_1, -\lambda_2, -\gamma^{n-1}\}$ where

$$\lambda_1 = \frac{1}{2}\gamma(n - 1 + \sqrt{(n-1)^2 + 4\alpha^2 n})$$

and

$$\lambda_2 = \frac{1}{2}\gamma(n - 1 - \sqrt{(n-1)^2 + 4\alpha^2 n}).$$

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