Chebyshev polynomials for solving two dimensional linear and nonlinear integral equations of the second kind

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Abstract. In this paper, an efficient method is presented for solving two dimensional Fredholm and Volterra integral equations of the second kind. Chebyshev polynomials are applied to approximate a solution for these integral equations. This method transforms the integral equation to algebraic equations with unknown Chebyshev coefficients. The high accuracy of this method is verified through some numerical examples.

Mathematical subject classification: 65R20, 41A50, 41A55, 65M70.

Key words: Chebyshev polynomials, two dimensional integral equations, collocation method.

1 Introduction

Two dimensional integral equations provide an important tool for modeling a numerous problems in engineering and science [2, 12]. These equations appear in electromagnetic and electrodynamic, elasticity and dynamic contact, heat and mass transfer, fluid mechanic, acoustic, chemical and electrochemical process, molecular physics, population, medicine and in many other fields [6, 7, 8, 14, 18, 20].

The Nystrom method [10] and collocation method [3, 21] are the most important approaches of the numerical solution of these integral equations. Recently,

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some new methods such as differential transform method are applied for solving two dimensional linear and nonlinear Volterra integral equations [1, 5, 19].

In this work, we will apply the Chebyshev polynomials for solving two dimensional integral equations of second kind. The use of the Chebyshev series for the numerical solution of linear integral equations has previously been discussed in [17] and references therein. The privilege of the method is simplicity and spectral accuracy [4, 11]. The examples confirm that the method is considerably fast and highly accurate as sometimes lead to exact solution. Also, this method lead to continuous solution covering all the domain.

The paper is organized in the following way. In Section 2 the famous Chebyshev polynomials and its application are introduced [16]. In Section 3 the method of solution of the linear two dimensional integral equation of second kind is described. In Section 4 we will try to generalize this method for nonlinear cases. In Section 5 some examples are chosen to show the ability and high accuracy of the method.

2   Chebyshev polynomials

Definition 1.   *If $t = \cos\theta$ $(0 \leqslant \theta \leqslant \pi)$, the function*

$$T_n(t) = \cos(n\theta) = \cos(n \arccos t), \tag{1}$$

is a polynomial of t of degree $n(n = 0, 1, 2, \ldots)$. T_n is called the Chebyshev polynomial of degree n [16]. When θ increase from 0 to π , t decrease from 1 to -1. Then the interval $[-1, 1]$ is domain of definition of $T_n(t)$. It satisfies the orthogonality condition

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} = \begin{cases} 0, & n \neq m, \\ \dfrac{\pi}{2}, & n = m \neq 0, \\ \pi, & n = m = 0 \end{cases} \tag{2}$$

Remark 1 (Chebyshev series expansion).   Let be $g(x)$ a function on $[a, b]$. For a given arbitrary natural number M, Chebyshev series expansion of $g(x)$ have the form

$$g(x) \simeq \sum_{k=0}^{M} a_k T_k \left(\frac{2}{b-a}x - \frac{b+a}{b-a} \right), \quad x \in [a, b], \tag{3}$$

where

$$a_k = \frac{2}{\pi c_k} \int_{-1}^{1} \frac{g\left(\frac{b-a}{2}x + \frac{b+a}{2}\right) T_k(x)}{\sqrt{1-x^2}} dx, \quad k = 1, 2, \cdots, M. \quad (4)$$

and

$$c_k = \begin{cases} 2, & k = 0, \\ 1, & k \geq 1. \end{cases} \quad (5)$$

Definition 2. *Suppose $f(x, t)$ be a continuous function on $[-1, 1] \times [-1, 1]$. For a given natural number N, we set*

$$f(x, t) \approx \sum_{i=0}^{N} \sum_{j=0}^{N} a_{ij} T_i(x) T_j(t), \qquad (x, t) \in [-1, 1] \times [-1, 1], \quad (6)$$

where

$$a_{ij} = \frac{\langle T_i(x), \langle f(x, t), T_j(t) \rangle \rangle}{\langle T_i(x), T_i(x) \rangle \cdot \langle T_j(t), T_j(t) \rangle}, \quad (7)$$

and $\langle \cdot, \cdot \rangle$ denotes the inner product in function space $L^2([-1, 1] \times [-1, 1])$.

Remark 2.   This paper discusses using Chebyshev polynomials of the first kind to approximate functions on finite interval, that is, on the interval $[-1, 1]$. Practically, other polynomials, which are orthogonal on finite interval, can also be applied for approximating functions. But the partial sums of a first-kind Chebyshev expansion of a continuous function in $[-1, 1]$ cover faster than the partial sums of an expansion in any other orthogonal polynomials [16].

3   Solution of linear two dimensional integral equation

Consider the two dimensional linear Fredholm and Volterra integral equations as follows

$$u(x, t) - \int_{-1}^{1} \int_{-1}^{1} k(x, t, y, z) u(y, z) dy dz = f(x, t),$$

$$(x, t) \in [-1, 1] \times [-1, 1], \quad (8)$$

and

$$u(x,t) - \int_{-1}^{t} \int_{-1}^{x} k(x,t,y,z)u(y,z)dydz = f(x,t),$$

$$(x,t) \in [-1,1] \times [-1,1],$$

(9)

where $u(x,t)$ is an unknown scalar-valued function, $f(x,t)$ and $k(x,t,y,z)$ are continuous functions on $[-1,1]^2$ and $[-1,1]^4$ respectively. For the case which integration domain is $[a,b] \times [c,d]$, we can use suitable change of variable to obtain these intervals.

At first, we consider two dimensional linear Fredholm integral equations are defined in (4). Function $u(x,t)$ defined over $[-1,1] \times [-1,1]$ may be represented by Chebyshev series as

$$u(x,t) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} a_{ij} T_i(x) T_j(t), \qquad (x,t) \in [-1,1] \times [-1,1].$$

(10)

If the infinite series in (6) is truncated, then (6) can be written as

$$u(x,t) \approx u_N(x,t) = \sum_{i=0}^{N} \sum_{j=0}^{N} a_{ij} T_i(x) T_j(t),$$

(11)

where N is any natural number. The method of collocation solves the (4) using the approximation (7) through the equations

$$R_N(x_r, t_s) = u_N(x_r, t_s) - \int_{-1}^{1} \int_{-1}^{1} k(x_r, t_s, y, z) u_N(y, z) dydz$$

$$- f(x_r, t_s) = 0,$$

(12)

for Gauss-Chebyshev-Lobatto as collocation points [16]

$$\begin{cases} x_r = \cos\left(\dfrac{r\pi}{N}\right), & r = 0, 1, \ldots, N, \\ t_s = \cos\left(\dfrac{s\pi}{N}\right), & s = 0, 1, \ldots, N. \end{cases}$$

(13)

The interested reader can see more detail of collocation method in [3, 4, 11].

Similarly, function $k(x_r, t_s, y, z)$ can be expressed as truncated Chebyshev series in the following form

$$k(x_r, t_s, y, z) \approx k_M(x_r, t_s, y, z) = \sum_{p=0}^{M} \sum_{q=0}^{M} k_{pq}^{(r,s)} T_p(y) T_q(z),$$

(14)

for any natural number M. From (3) we have

$$k_{pq}^{(r,s)} = \frac{4}{\pi^2 c_p c_q} \int_{-1}^{1} \int_{-1}^{1} \frac{k(x_r, t_s, y, z)}{\sqrt{1-y^2}\sqrt{1-z^2}} T_p(y) T_q(z) dy dz, \qquad (15)$$

where

$$c_p = \begin{cases} 2 & p = 0, \\ 1 & p \geq 1. \end{cases}$$

By using Gauss-Chebyshev-Lobatto integration rule [13], for a given natural number n we have

$$k_{pq}^{(r,s)} = \frac{4}{n^2 c_p c_q} \sum_{\xi=0}^{n}{}'' \sum_{\eta=0}^{n}{}'' k\left(x_r, t_s, \cos\frac{\xi\pi}{n}, \cos\frac{\eta\pi}{n}\right) \cos\left(\frac{p\xi\pi}{n}\right) \cos\left(\frac{q\eta\pi}{n}\right), \quad (16)$$

where double prime denotes that the first and the last terms are halved. Now, by substituting (7) and (10) into (8) we obtain

$$\sum_{i=0}^{N} \sum_{j=0}^{N} a_{ij} \psi_{ij}^{rs} - f(x_r, t_s) = 0. \qquad (17)$$

where

$$\psi_{ij}^{rs} = T_i(x_r) T_j(t_s) - \sum_{p=0}^{M} \sum_{q=0}^{M} k_{pq}^{(r,s)} \left(\int_{-1}^{1} \int_{-1}^{1} T_i(y) T_j(z) T_p(y) T_q(z) dy dz\right).$$

We define

$$b_{ij}^{(r,s)} = T_i(x_r) T_j(t_s), \qquad (18)$$

$$f_{rs} = f(x_r, t_s), \qquad (19)$$

and

$$w_{pq}^{(i,j)} = \left(\int_{-1}^{1} T_i(y) T_p(y) dy\right) \left(\int_{-1}^{1} T_j(z) T_q(z) dz\right). \qquad (20)$$

But the Chebyshev polynomials are even for even degree and odd for odd degree. Hence,

$$\int_{-1}^{1} T_i(x) T_p(x) dx = \begin{cases} \dfrac{1}{1-(i+p)^2} + \dfrac{1}{1-(i-p)^2} & i+p \text{ is even,} \\ 0 & i+p \text{ is odd.} \end{cases} \qquad (21)$$

So, from (13), (14), (15) and (16) we can obtain the system of linear equations,

$$\sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij}\tilde{w}_{ij}^{(r,s)} = f_{rs}, \qquad r,s = 0,1,2,\ldots,N, \qquad (22)$$

where $\tilde{w}_{ij}^{(r,s)}$ is computed by the following relation

$$\tilde{w}_{ij}^{(r,s)} = b_{ij}^{(r,s)} - \sum_{p=0}^{M}\sum_{q=0}^{M} k_{pq}^{(r,s)} w_{pq}^{(i,j)}, \qquad i,j = 0,1,\ldots,N. \qquad (23)$$

Clearly, the obtained system contains $(N+1)^2$ equations in the same number as unknowns. It can be solved by Newton's iteration method to obtain the value of a_{ij} such that $i,j = 0\ldots N$.

For the Volterra case, this method is valid. We just change (16) with the following equation

$$w_{pq}^{(i,j,r,s)} = \left(\int_{-1}^{x_r} T_i(y)T_p(y)dy\right)\left(\int_{-1}^{t_s} T_j(z)T_q(z)dz\right), \qquad (24)$$

so that, if let

$$\tau_{ip}(x) = \int_{-1}^{x} T_i(y)T_p(y)dy,$$

we will have

$$\tau_{ip}(x) = \frac{1}{4}\begin{cases} 2x^2 - 2 \\[2mm] \dfrac{T_{i+p+1}(x)}{i+p+1} - \dfrac{T_{i+p-1}(x)}{i+p-1} - \dfrac{1}{i+p+1} + \dfrac{1}{i+p-1} + x^2 - 1 \\[4mm] \begin{aligned}&\dfrac{T_{i+p+1}(x)}{i+p+1} + \dfrac{T_{1-i-p}(x)}{1-i-p} + \dfrac{T_{1+i-p}(x)}{1+i-p} + \dfrac{T_{1-i+p}(x)}{1-i+p}\\ &\quad + \dfrac{2}{1-(i+p)^2} + \dfrac{2}{1-(i-p)^2}\end{aligned} \\[6mm] \begin{aligned}&\dfrac{T_{i+p+1}(x)}{i+p+1} + \dfrac{T_{1-i-p}(x)}{1-i-p} + \dfrac{T_{1+i-p}(x)}{1+i-p} + \dfrac{T_{1-i+p}(x)}{1-i+p}\\ &\quad - \dfrac{2}{1-(i+p)^2} - \dfrac{2}{1-(i-p)^2}\end{aligned} \end{cases}$$

when respectively $i+p = 1$, $|i-p| = 1$, $i+p$ is even and $i+p$ is odd.

4   Solution of nonlinear two dimensional integral equation

Consider the two dimensional nonlinear Fredholm and Volterra integral equations as follows

$$u(x, t) - \int_{-1}^{1} \int_{-1}^{1} k(x, t, y, z) F(u(y, z)) dy dz = f(x, t),$$

$$(x, t) \in [-1, 1]^2,$$ (25)

and

$$u(x, t) - \int_{-1}^{t} \int_{-1}^{x} k(x, t, y, z) F(u(y, z)) dy dz = f(x, t),$$

$$(x, t) \in [-1, 1]^2,$$ (26)

where $k(x, t, y, z)$ is continuous on $[-1, 1]^4$, and $f(x, t)$ and $F(u(y, z))$ are continuous on $[-1, 1]^2$. Again, for the case which integration domain is $[a, b] \times [c, d]$, we can use suitable change of variable to obtain this intervals.

Before solving the above equations we exchange them with following equations. The other cases can be approximated in this form using Taylor extension. It reduce the related computation effectively.

$$u(x, t) - \int_{-1}^{1} \int_{-1}^{1} k(x, t, y, z) [u(y, z)]^p \, dy dz = f(x, t),$$

$$(x, t) \in [-1, 1]^2,$$ (27)

and

$$(x, t) - \int_{-1}^{t} \int_{-1}^{x} k(x, t, y, z) [u(y, z)]^p \, dy dz = f(x, t),$$

$$(x, t) \in [-1, 1]^2,$$ (28)

where p is a positive integer number and $p \geq 2$. Correspondingly the linear case, by using (7) and (24) and considering collocation points we have

$$u_N(x_r, t_s) - \int_{-1}^{1} \int_{-1}^{1} k(x_r, t_s, y, z) [u_N(y, z)]^p dy dz - f(x_r, t_s) = 0. \quad (29)$$

Now, we replace (10) into above equation and if we let

$$v_{pq} = \int_{-1}^{1} \int_{-1}^{1} T_p(y) T_q(z) [u_N(y, z)]^p dy dz,$$ (30)

we have

$$\sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij} T_i(x_r) T_j(t_s) - \sum_{p=0}^{M}\sum_{q=0}^{M} k_{pq}^{(r,s)} v_{pq} = f(x_r, t_s), \tag{31}$$

$$r, s = 0, 1, \ldots, N.$$

This is a system of algebraic equations with $(N + 1)$ unknowns and $(N + 1)$ equations which can be solved by Newton's iteration method to obtain the value of a_{ij} such that $i, j = 0 \ldots N$.

In Volterra case, we let $\tilde{v}_{pq}^{(r,s)}$ instead of v_{pq} in (29)

$$\sum_{i=0}^{N}\sum_{j=0}^{N} a_{ij} T_i(x_r) T_j(t_s) - \sum_{p=0}^{M}\sum_{q=0}^{M} k_{pq}^{(r,s)} \tilde{v}_{pq}^{(r,s)} = f(x_r, x_s), \tag{32}$$

$$r, s = 0, 1, \ldots, N,$$

where

$$\tilde{v}_{pq}^{(r,s)} = \int_{-1}^{x_r} \int_{-1}^{t_s} T_p(y) T_q(z) [u_N(y, z)]^p \, dy \, dz. \tag{33}$$

We remind

$$[u_N(x, t)]^p = \sum_{i=0}^{p_N}\sum_{j=0}^{p_N} d_{ij} T_i(x) T_j(t), \tag{34}$$

where d_{ij} is linear or nonlinear combination of a_{ij}. Hence, we can calculate easily v_{pq} and $\tilde{v}_{pq}^{(r,s)}$ by (17) and (21).

Remark 3.   In case $F(u(x, t))$ is strongly nonlinear, the Taylor series can be used to approximate $F(u(x, t))$ as a polynomial in $u(x, t)$. Then the above method can be applied easily for general cases (25) and (26).

5   Numerical results

In this section, the illustrate examples are given to show efficiency the method proposed in Sections 3 and 4. All of the computations have been done using the Maple 12 with just 10 digits precision. In this study, our criterion of accuracy is the maximum absolute error in relevant intervals. In the other word, we investigate the value of infinity norm of error functions.

5.1   Linear examples

For the following cases, we let $M = 6$ and $n = 15$.

Example 1.   Consider the following Fredholm integral equation

$$u(x, t) - \int_{-1}^{1} \int_{-1}^{1} (z \sin x + ty)u(y, z)dydz$$

$$= x \cos t + \frac{4}{3} \sin x - \left(1 + \frac{4}{3} \sin(1)\right)t, \qquad x, t \in [-1, 1],$$

with exact solution

$$u(x, t) = x \cos t - t.$$

By using (18) we obtain approximate solution

$$\begin{aligned} u_N(x, t) &= 1.000000001x - 1.000000002t - 0.4999999824xt^2 \\ &\quad - 1.702672594 \times 10^{-8}x^2t + \cdots - 3.416035551 \times 10^{-7}x^8t^7 \\ &\quad + 0.000000000x^8t^8. \end{aligned}$$

The maximum absolute errors are shown in Table 1 for $N = 3, 5$ and 8. Also you can see Figure 1(a).

Example	$N = 3$	$N = 5$	$N = 8$
1.	1.1×10^{-2}	8.7×10^{-5}	2.4×10^{-9}
2.	4.2×10^{-4}	8.8×10^{-7}	9×10^{-9}
3.	1.2×10^{-2}	8.9×10^{-5}	2.1×10^{-8}
4.	3.2×10^{-4}	6.4×10^{-7}	8×10^{-10}

Table 1 – Maximum absolute errors are presented for Example 1, 2, 3 and 4.

Example 2.   Consider the following Fredholm integral equation

$$u(x, t) - \int_{0}^{1} \int_{0}^{1} (xy + te^z)u(y, z)dydz$$

$$= xe^{-t} + \left(\frac{1}{3}e^{-1} - \frac{7}{12}\right)x - \frac{1}{2}t \qquad x, t \in [0, 1],$$

with exact solution $u(x, t) = xe^{-t} + t$. By using (18) we have

$$
\begin{aligned}
u_N(x, t) = {} & 1.000000001x + 0.9999999972t - 0.999995432xt \\
& + 0.4999883358xt^2 + \cdots + 0.02307911926x^8 t^8.
\end{aligned}
$$

(a)
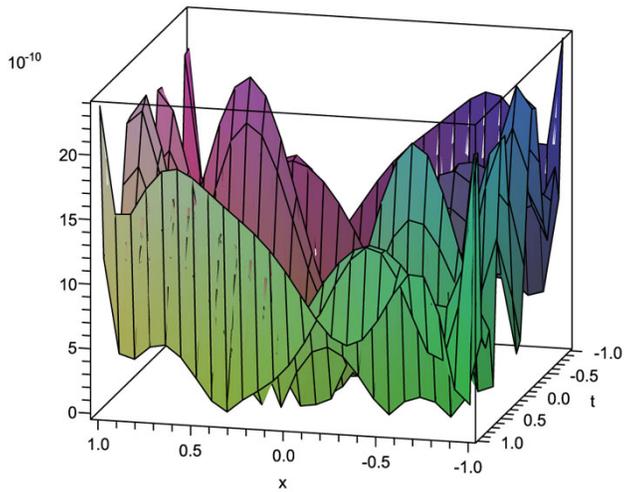
(b)
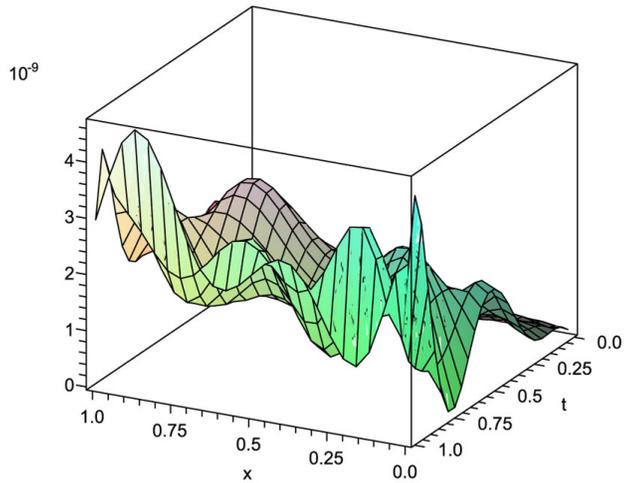
Figure 1 – The error functions of Example 1 and 2 from top to bottom are presented respectively. These examples show the efficiency of the method for Fredholm integral equations.

The maximum absolute errors are presented in Table 1 for $N = 3, 5$ and 8. Also you can see Figure 2(a).

Example 3. Consider the following Volterra integral equation

$$u(x, t) - \int_{-1}^{t} \int_{-1}^{x} (x^2 y^2 + z e^t) u(y, z) dy dz = f(x, t) \qquad x, t \in [0, 1],$$

where

$$f(x, t) = e^{-t} \left(x - \frac{1}{4}x^2 + \frac{1}{4}x^6 \right) - \frac{1}{4}e x^6 + \left(\frac{1}{4}e + \frac{1}{2} \right) x^2 + \frac{1}{2}x^2 t - \frac{1}{2}t - \frac{1}{2},$$

and exact solution is $u(x, t) = x e^{-t}$. By applying (18) and considering (20) we obtain approximate solution as follows

$$u_N(x, t) = 1.000000000x - 0.9999998304xt + 0.4999999899xt^2$$
$$+ \cdots - 5.678168292 \times 10^{-8} x^8 t^8.$$

The maximum absolute errors are shown in Table 1 for $N = 3, 5$ and 8. Also you can see Figure 1(a).

Example 4. Consider the following Volterra integral equation [19]

$$u(x, t) - \int_{0}^{t} \int_{0}^{x} (x y^2 + \cos z) u(y, z) dy dz$$

$$= x \sin t - \frac{1}{4}x^5 + \frac{1}{4}x^5 \cos t - \frac{1}{4}x^2 (\sin t)^2 \qquad x, t \in [0, 1],$$

with exact solution $u(x, t) = x \sin t$. By applying (18) and considering (20) we obtain approximate solution as follows

$$u_N(x, t) = .9999999216xt + 0.0000016358xt^2 + 0.0000013773x^2 t$$
$$+ \cdots + 0.0040792670x^8 t^8.$$

The numerical results are shown in Table 1 are computed errors. Also you can see Figure 2(b).

In Table 1 we investigate the above examples and shows the maximum absolute error. The numerical results shows high accuracy even for small N.

(a)

(b)
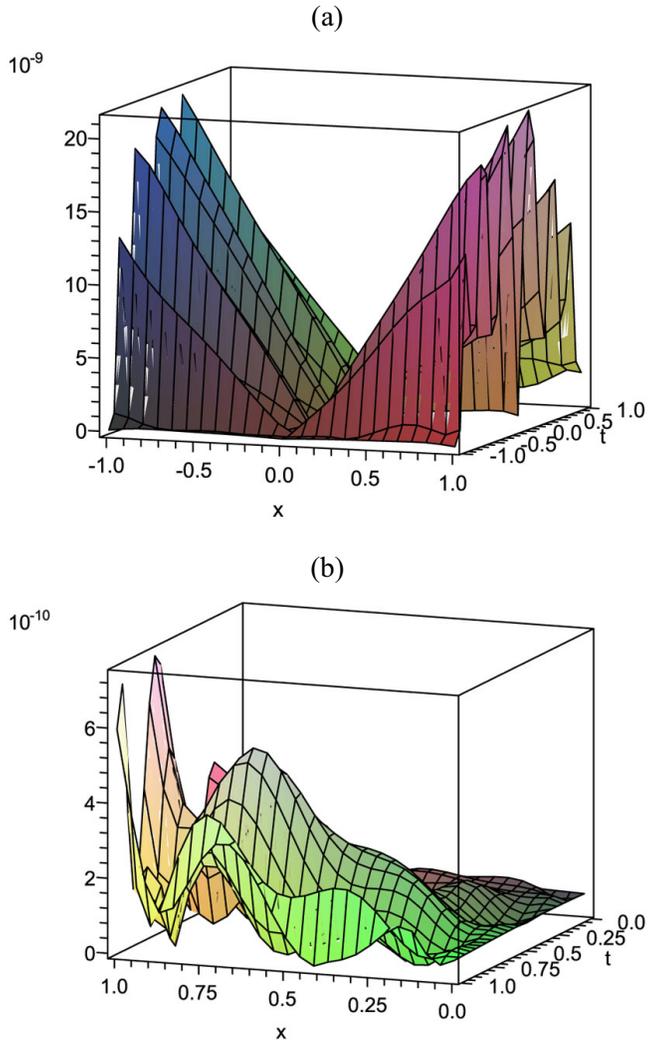
Figure 2 – The error functions of Example 3 and 4 from top to bottom are presented respectively. These examples confirm the efficiency of the method for Volterra integral equations.

Also, comparison between estimated absolute errors of Example 4 for $N = 8$ using presented method and differential transform method [19] are illustrated in Table 2. The results show more accuracy and smoother error function by described method.

(x, t)	$N = 8$ (differential transform method)	$N = 8$ (presented method)
(.2, .2)	2.820844×10^{-13}	2.727401×10^{-11}
(.2, .8)	7.354498×10^{-8}	5.479929×10^{-10}
(.4, .6)	1.107230×10^{-8}	2.279254×10^{-12}
(.4, 1)	1.092336×10^{-6}	3.049593×10^{-10}
(.6, .2)	8.462531×10^{-13}	4.405731×10^{-10}
(.6, .8)	2.206350×10^{-7}	3.243791×10^{-10}
(.8, .4)	5.770791×10^{-10}	7.240839×10^{-10}
(.8, .8)	2.941799×10^{-7}	2.585837×10^{-10}
(1, 6)	2.768074×10^{-8}	2.300421×10^{-10}
(1, 1)	2.730839×10^{-6}	2.773500×10^{-10}

Table 2 – Comparison between estimated absolute errors of Example 4 for $N = 8$ using presented method and differential transform method [19]. The results show more accuracy for described method.

5.2 Nonlinear examples

Example 5. Consider the following Fredholm integral equation

$$u(x, t) - \int_0^1 \int_0^1 (y + z)[u(y, z)]^2 dy dz$$

$$= x \cos t - \frac{1}{8} - \frac{7}{24} \cos(1) \sin(1) - \frac{1}{12}[\cos(1)]^2,$$

where $x, t \in [0, 1]$ and exact solution is $u(x, t) = x \cos t$. If we let $M = 1$, $N = 6$ and $n = 15$, by considering (29) we obtain approximate solution as follows

$$u_N(x, t) = 1.000000000x - 4.895686000 \times 10^{-7}xt - .4999882352xt^2$$
$$- \cdots + 0.000000000x^6 t^6.$$

Also, the value of infinity norm of error function is 2.3×10^{-8}.

Example 6. Consider the following Volterra integral equation [9]

$$u(x, t) - \int_0^t \int_0^x [u(y, z)]^2 dy dz$$

$$= x^2 + t^2 - \frac{1}{45}xt(9x^4 + 10x^2 t^2 + 9t^4) \qquad x, t \in [0, 1],$$
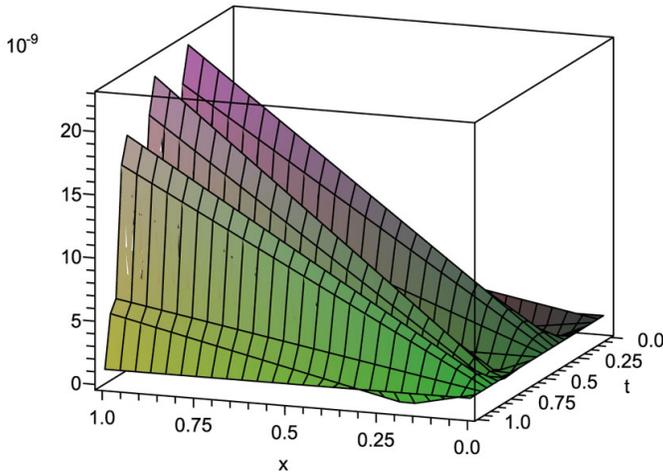
Figure 3 – The error function of Example 5 is shown. This example confirms the efficiency of the method for nonlinear integral equations.

In this case, we let $M = 1$, $N = 2$ and $n = 15$. By considering (30) and solving the obtained system we have

$$a_{00} = \frac{3}{4}, \quad a_{01} = \frac{1}{2}, \quad a_{02} = \frac{1}{8},$$

$$a_{10} = \frac{1}{2}, \quad a_{11} = 0, \quad a_{12} = 0,$$

$$a_{20} = \frac{1}{8}, \quad a_{21} = 0, \quad a_{22} = 0.$$

The values of a_{ij} lead to $u(x, t) = x^2 + t^2$ which is the exact solution.

6   Conclusion

Analytical solution of the two dimensional integral equations are usually difficult. In many cases, it is required to approximate solutions. In this work, the two dimensional linear and nonlinear integral equations of the second kind are solved by using Chebyshev polynomials through collocation scheme. However this method only works when F is a power function as (27) and (28), we know other cases can be rewritten easily in this form using Taylor extension. The privilege of the method is simplicity and spectrally accuracy [4, 11]. The illustrative

examples confirm the validity and efficiency of the method. This method can be extended for the system including such the equations. Also, development of the method can solve the two dimensional integro-differential equations.

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