GESTÃO & PRODUÇÃO

ISSN 0104-530X (Print)
ISSN 1806-9649 (Online)

ORIGINAL ARTICLE

Monitoring bivariate processes with synthetic control charts based on sample ranges

Gráficos de controle baseado em amplitudes amostrais e regras especiais de decisão para o monitoramento de processos bivariados

Marcela Machado[1], Antonio Costa[2], Felipe Domingues Simões[1]

[1] Universidade Estadual Paulista – UNESP, Faculdade de Engenharia e Ciências – FEG, Departamento de Produção, Guaratinguetá, SP, Brasil. E-mail: marcela.freitas@unesp.br; fdsimoes94@yahoo.com.br

[2] Universidade Federal de Itajubá – UNIFEI, Instituto de Engenharia de Produção e Gestão – IEPG, Itajubá, MG, Brasil. E-mail: fbrancosta@gmail.com

How to cite: Machado, M., Costa, A., & Simões, F. D. (2023). Monitoring bivariate processes with synthetic control charts based on sample ranges. *Gestão & Produção*, 30, e6822. https://doi.org/10.1590/1806-9649-2022v29e6822

Abstract: The RMAX chart was proposed to control the covariance matrix of two quality characteristics. The monitoring statistic of the RMAX chart is the maximum of two standardized sample ranges from bivariate observations of two quality characteristics. In this article, we investigate the performance of two synthetic RMAX charts. The first synthetic chart signals when a second point, not far from the first one, falls beyond the warning limit. The second synthetic chart additionally signals when a sample point falls beyond the control limit. The performance of the synthetic RMAX charts are compared with the performance of the standard RMAX chart and the generalized variance $|\mathbf{S}|$ chart. The proposed charts are the best option to detect moderate or even small changes in the covariance matrix. To detect large changes in the covariance matrix, additional run rules are not necessary.

Keywords: RMAX chart; Bivariate processes; Synthetic run rules.

Resumo: O gráfico RMAX foi proposto para o monitoramento da matriz de covariâncias de duas características de qualidade. A estatística de monitoramento RMAX é o maior valor entre duas amplitudes amostrais padronizadas oriundas de observações bivariadas de duas características de qualidade. Neste artigo, investigou-se o desempenho de dois gráficos RMAX com regras especiais de decisão. O primeiro gráfico sinaliza quando um segundo ponto, não muito distante do primeiro, cai acima do limite de advertência. O segundo gráfico também sinaliza quando um ponto cai acima do limite de controle. O desempenho dos gráficos RMAX com regras especiais de decisão foram comparados com o gráfico RMAX tradicional e o gráfico da variância amostral generalizada $|\mathbf{S}|$. Os gráficos propostos são melhores para detectar moderadas ou até mesmo pequenas perturbações na matriz de covariâncias. Para detectar grandes perturbações na matriz de covariâncias, não é necessário adicionar regras especiais de decisão.

Palavras-chave: Gráfico RMAX; Processos bivariados; Regras especiais de decisão.

Received May 11, 2022 - Accepted Mar. 9, 2023

Financial support: Este trabalho contou com o apoio financeiro do CNPq, processo número 305571/2015-0.

1 Introduction

The formal beginning of Statistical Process Control (SPC) occurred around 1924, when Walter A. Shewhart developed and applied the control charts on Bell Telephone Laboratories. Control charts are designed to detect assignable causes that may occur in production processes. It is standard practice to use an $\overline{X}$ chart for detecting assignable causes that shift the process mean and an R chart for detecting assignable causes that increase the process variability. It is well known that a control chart based on sample ranges is inferior than a control chart based on sample variances in terms of efficiency in detecting process shifts. However, the user´s familiarity with sample ranges is a point in favor of the control charts based on sample ranges, see Woodall (2016).

In general, the Shewhart charts are very simple operationally; however, this operational simplicity, that is, taking samples of fixed size n at regular time intervals and searching for an assignable cause when a point falls outside the control limits, makes the control charts slow in detecting small to moderate shifts in the process parameter being controlled. Since this handicap of Shewhart control charts was recognized, many innovations have been proposed to improve the charts' performance, such as the synthetic version of the Shewhart chart.

The synthetic chart is an integration of the Shewhart chart and the Conforming Run Length (*CRL*) chart. The *CRL* is the number of conforming samples between two consecutive nonconforming samples (Bourke, 1991). According to the synthetic run rule, the control chart signals when a second point, not far from the first one, falls beyond the warning limits. Wu & Spedding (2000) introduced the $\overline{X}$ chart with the synthetic run rule and Davis & Woodall (2002) obtained its steady-state properties. The results of their studies motivated other researchers to consider the synthetic run rule as an alternative to enhance the performance of the control charts.

Khoo et al. (2010) proposed a synthetic double sampling chart, which combines the double sampling $\overline{X}$ chart and the *CRL* chart for monitoring the process mean. Wu et al. (2010) proposed a scheme comprising a synthetic chart and an $\overline{X}$ chart, denoted as the Syn-$\overline{X}$ chart, for monitoring the process mean. In this scheme, a nonconforming sample is the one with an $\overline{X}$ value larger than the upper warning limit $UWL = \mu_0 + w\sigma_{\overline{X}}$ or smaller than the lower warning limit $LWL = \mu_0 - w\sigma_{\overline{X}}$. The Syn-$\overline{X}$ chart signals when a sample point falls beyond the control limits or when *CRL*<*L*, where *L* is a specified positive integer.

Zhang et al. (2011) evaluated the performances of the synthetic chart when the process parameters are estimated. The synthetic $\overline{X}$ chart is the name they used for the synthetic chart proposed by Wu & Spedding (2000). They demonstrated that when the number of samples during Phase I is small, the performances of the synthetic chart with known parameters and with estimated parameters are quite different.

Haridy et al. (2012) proposed a combined scheme comprising a synthetic chart and an *np* chart, which has always a better overall performance than the individual synthetic chart and individual *np* chart. An optimal design of procedure for a synthetic chart able to monitor the mean based on the Median Run Length (MRL) was suggested by Khoo et al. (2012). Costa & Machado (2015) considered the Markov chain approach to obtain the properties of the synthetic and side-sensitive synthetic double sampling $\overline{X}$ chart. Sun et al. (2018) evaluated the performance of Synthetic exponential control charts with unknown parameters. Recently, Shongwe et al. (2019) proposed side-sensitive synthetic and runs-rules charts for monitoring autocorrelated processes with skipping sampling strategies

The growing interest in synthetic charts may be explained by the fact that many practitioners prefer waiting until the occurrence of a second point beyond the control limits before looking for an assignable cause.

In parallel to these studies dealing with the monitoring of a single quality characteristic, many researchers have been working with processes that require the monitoring of several quality characteristics. In 1947, Hotelling proposed the T^2 statistic to control the mean vector of multivariate processes. After that, more and more statistics have been proposed to control the mean vector and/or the covariance matrix of multivariate processes; recent and interesting works about this subject are: Leoni et al. (2015), Lee Ho & Costa (2015), Simões et al. (2016), Aparisi & Lee Ho (2017), Leoni & Costa (2017), Melo et al. (2017a, b) and Shokrizadeh et al. (2017).

Alt (1985) was the first researcher to propose a chart to control the covariance matrix of bivariate processes, the generalized variance $|\mathbf{S}|$ chart. The $|\mathbf{S}|$ chart is not simple to deal with once the monitoring statistic of the chart depends on the determinant of the sample covariance matrix. Moreover, the chart is slow in signaling changes in the covariance matrix. Costa & Machado (2008) proposed a simpler and more efficient statistic to control the covariance matrix of bivariate processes. Their VMAX statistic is the maximum of two variances from the sample observations of two quality characteristics.

Costa & Machado (2009), Machado & Costa (2008) and Machado et al. (2008, 2009) also worked with the VMAX statistic. Alternatively, Costa & Machado (2011) proposed the use of the RMAX statistic to control the covariance matrix of bivariate processes. The RMAX statistic is the maximum of two standardized ranges from the sample observations of two quality characteristics.

Recently, Costa & Faria (2017) proposed the S chart with variable charting statistic (VCS) to control the covariance matrix as an alternative to the use of the bivariate $|\mathbf{S}|$ chart and the trivariate VMAX chart. The idea of the S chart with variable charting statistic is working with only one characteristic per time. For example, considering the bivariate case, only one of the two characteristics X or Y is measured and only one charting statistic S_x or S_y is computed. The statistic in use and the position of the current sample point on the chart define the statistic for the next sample. The VCS chart is not only operationally simpler than the bivariate |S| and trivariate VMAX charts but also signals faster even with less measurements per sample.

Machado et al. (2018) adopted the attribute inspection for monitoring the covariance matrix of bivariate processes. They considered the use of three attribute charts, the np_{xy}, the np_w and the Max D charts, to control the covariance matrix of bivariate processes. In comparison with the generalized variance |S| chart, the three attribute charts signal faster, with smaller samples, all kind of disturbances, except when the two variables are highly correlated. To compete with the VMAX chart, the Max D chart needs larger samples, but no more than twice bigger. The attribute inspection has the advantage of being cheaper, faster and easier than the variable inspection.

In this article, we consider the use of the RMAX chart proposed by Costa & Machado (2011) to control the variability of two variables. In order to enhance the RMAX chart's performance two synthetic rules are investigated. The first synthetic chart signals when a second point, not far from the first one, falls beyond the warning limit. The second synthetic chart additionally signals when a sample point falls beyond the control limit. The proposed charts are addressed to users that prefer simple sample statistics, such as the sample ranges, and also the occurrence of a second point in the action region before investigating process disturbances.

The paper is organized as follows. In Section 3, we describe the RMAX and the synthetic RMAX charts. In Section 4, we investigate the performance of the proposed charts and compare it with the performance of the RMAX chart proposed by Costa and Machado[25] and the generalized variance $|\mathbf{S}|$ chart. In Section 5 we present an illustrative example. Conclusions are in Section 6.

2 Monitoring processes with the sample ranges

The sample range is the simplest measure of variation, it is the difference between the largest observation and the smallest observation in a subgroup. For the univariate case, the R chart is the standard chart for monitoring the process variance. With the usual sample size of 4 and 5, the R chart is slightly inferior to the S^2 chart in terms of efficiency in detecting process shifts. However, the R chart remains in use thanks to the practitioner's familiarity with sample ranges.

It is also well known that with small subgroup sizes, the lower control limit of the R chart is set to zero, that is, the Shewhart chart based on the sample ranges is unable to detect reductions in the process variance. To overcome this drawback Acosta-Mejia & Pignatiello (2008) modified the R chart to make it also sensitive to variance decreases.

Lee (2011) explored the idea of varying the R chart's parameters with the aim of increasing its power. Abujiya et al. (2016) considered the use of a Shewhart chart in combination with a CUSUM R chart for fast detection of changes in the variance. Costa (2017) adopted the double sampling scheme to enhance the performance of the R chart. In some situations, the proposed chart outperforms its competitor, that is, the variance chart with double sampling.

For the multivariate case, Costa & Machado (2011) proposed the RMAX chart to detect changes in the covariance matrix $\mathbf{\Sigma}$ of bivariate processes. According to the literature, no other control chart for monitoring the covariance matrix based on sample ranges was proposed since then. In this article, we propose two synthetic RMAX charts. In Section 3.1, we describe the RMAX chart proposed by Costa & Machado (2011) and, in Section 3.2, we introduce the synthetic RMAX charts.

2.1 The RMAX chart

In this section, we describe the RMAX chart proposed by Costa & Machado (2011) to detect changes in the covariance matrix $\mathbf{\Sigma}$ of bivariate processes. The sample points plotted on the RMAX chart are the larger value of two standardized sample ranges, $W_i = R_i / \sigma_i,\ i = 1, 2$, where R_1 and R_2 are, respectively, the sample ranges from the observations of the first and the second quality characteristics. The process is considered to start with the covariance matrix on target $(\mathbf{\Sigma} = \mathbf{\Sigma}_0)$,-see Equation 1.

$$\mathbf{\Sigma}_0 = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix} \tag{1}$$

The occurrence of the assignable cause changes the covariance matrix from $\mathbf{\Sigma}_0$ to $\mathbf{\Sigma}_1$, according to Equation 2.

$$\mathbf{\Sigma}_1 = \begin{pmatrix} a_1^2\sigma_1^2 & a_1 a_2 \sigma_{12} \\ a_1 a_2 \sigma_{12} & a_2^2\sigma_2^2 \end{pmatrix} \quad (2)$$

After the occurrence of the assignable cause, it is assumed that at least one a_i becomes larger than one, $i = 1, 2$. With the standard RMAX chart in use, an out-of-control signal is triggered by a sample point falling beyond the control limit. Costa & Machado (2011) obtained the properties of the RMAX chart, that is, a closed theoretical expression to obtain the false alarm risk α and the power of detection $P=1-\beta$, being α and β, respectively, the well-known Type I and Type II errors, see the Appendix 1.

They observed that the coefficient of correlation, $\rho = \sigma_{12} / (\sigma_1 \sigma_2)$, has minor influence on the properties of the RMAX chart. Based on that, we fixed ρ=0.5.

2.2 The synthetic RMAX chart

The standard RMAX chart combined with the synthetic rule, shortly the synthetic RMAX chart, signals when a sample point falls beyond the control limit or when a second point, not far from the first one, falls beyond the warning limit. In order to measure the distance between the two points beyond the warning limits, we attributively classify the samples as conforming and nonconforming - being conforming when their sample points fall below the warning limit and nonconforming when their sample points fall beyond the warning limit. The distance is measured by the *CRL*, the number of conforming samples between two consecutive nonconforming samples plus the ending nonconforming one; in other words, the first of the two consecutive nonconforming samples is the reference to compute the *CRL*. A *CRL* lower than or equal to a specified positive integer *L* (*CRL* < *L*) triggers a signal.

When the control limit goes to infinite, the synthetic RMAX chart only signals when a second point, not far from the first one, falls beyond the warning limit. In order to distinguish the two synthetic charts, the one with a control limit will be the synthetic RMAX chart and the other one without a control limit will be the pure synthetic RMAX chart.

Figure 1 shows the pure synthetic RMAX chart. The sample is classified as nonconforming when the value of the monitoring statistic RMAX falls beyond the warning limit *WL*. Samples 9 and 13 are nonconforming (Figure 1). In this case, *CRL* = 4 (13th sample – 9th sample = 4). As *CRL* < *L* (=5), the pure synthetic RMAX chart signals an out-of-control condition.

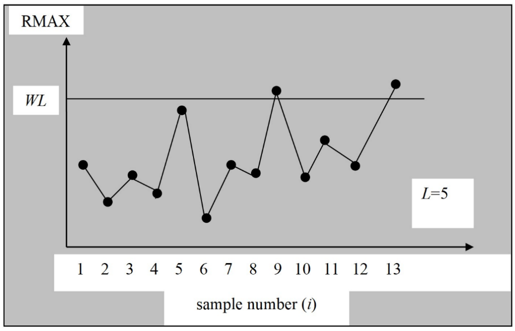

Figure 1. The pure synthetic RMAX chart.

Figure 2 shows the synthetic RMAX chart. The sample is also classified as nonconforming when the value of the monitoring statistic RMAX falls beyond the warning limit *WL*. Samples 3 and 9 are nonconforming (Figure 2). In this case, *CRL* = 6 (9^{th} sample – 3^{th} sample = 6). As *CRL* > *L* (=5), the synthetic RMAX chart does not signal at sample 9. However, the synthetic RMAX chart signals at sample 13, once the value of the monitoring statistic RMAX is beyond the control limit *CL*.

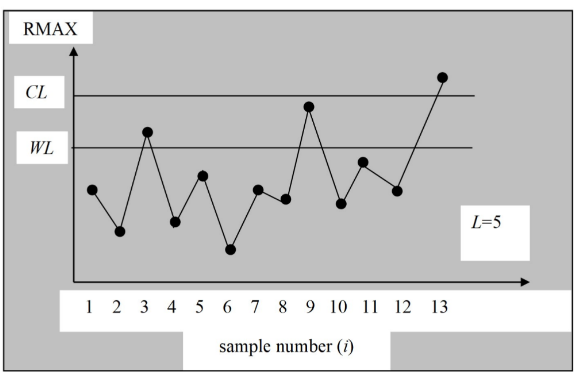

Figure 2. The synthetic RMAX chart.

3 The performance of the synthetic RMAX charts

In this section, we compare the speed with which the generalized variance $|\mathbf{S}|$ chart, the standard RMAX chart (*Std* chart), the pure synthetic RMAX chart (*PSyn* chart) and the synthetic RMAX (*Syn* chart) chart signal changes in the covariance matrix. According to Davis & Woodall (2002), the proper parameter to measure the performance of a synthetic chart is the steady-state average run length (*SSARL*), that is, the *ARL* value obtained when the process remains in-control for a long time before the occurrence of the assignable cause. When the process is in-control, the *SSARL* measures the rate of false alarms. A chart with a larger in-control *SSARL* ($SSARL_0$) has a lower false alarm rate than other charts. A chart with a smaller out-of-control *SSARL* has a better ability to detect process changes than other charts. The in-control *SSARL* is an input parameter, and the control limit *CL* and the warning limit *WL* are the adjusting parameters to obtain the desired in-control *SSARL* (Machado & Costa, 2014), see Figure 2. When the *PSyn* chart is in use *CL*=∞. Following the work of Machado & Costa (2014), we also considered the Markov chain approach to obtain the *SSARL*s of the synthetic RMAX charts.

The general transition matrix of the Markov chain in (3) is used to obtain the steady-state ARLs of the synthetic *RMAX* chart. In this transition matrix, $A = F_n(WL/a_1, WL/a_2)$, $B = F_n(CL/a_1, CL/a_2) - A$ and *C*=1-(*A*+*B*). The expression of $F_n(a,b)$ is in the Appendix 1.

$$\begin{array}{c|cccccccc} & 00..00 & 0..001 & 0..010 & 0..100 & ... & 010..0 & 100..0 & Signal \\ 00...00 & A & B & 0 & 0 & ... & 0 & 0 & C \\ 0..001 & 0 & 0 & A & 0 & ... & 0 & 0 & B+C \\ 0..010 & 0 & 0 & 0 & A & ... & 0 & 0 & B+C \\ ... & .. & ... & ... & ... & ... & ... & ... & ... \\ 001..0 & 0 & 0 & 0 & 0 & ... & A & 0 & B+C \\ 010..0 & 0 & 0 & 0 & 0 & ... & 0 & A & B+C \\ 100..0 & A & 0 & 0 & 0 & ... & 0 & 0 & B+C \\ Signal & 0 & 0 & 0 & 0 & ... & 0 & 0 & 1 \end{array} \quad (3)$$

The transient states describe the position of the last *L* sample points; “1” means the sample point fell in the warning region, and “0” means the sample point fell in the central region. For instance, the transient state (010..0) is reached when the second of the last *L* points falls in the action region and all others points fall in the central region. The events “0” and “1” occur with probabilities *A* and *B*, respectively. The synthetic is called the pure synthetic for the particular case where *C*=0.

The steady-state ARL is given by $\mathbf{S}'\mathbf{ARL}$, where **S**’ is the vector with the stationary probabilities of being in each nonabsorbing state and **ARL** is the vector of ARLs taking each nonabsorbing state as the initial state. $\mathbf{ARL} = (\mathbf{I} - \mathbf{R})^{-1}\mathbf{1}$, where **I** is an (*L*+1) by (*L*+1) identity matrix, **R** is the transition matrix given in (2) with the last row and column removed, and **1** is an (*L*+1) by one vector of ones. The vector $\mathbf{S}' = (1/D, B/D, ..., B/D)$, with *D*=1+*LB*, was obtained by solving the system of linear equations $\mathbf{S}'\mathbf{R}_{\mathbf{adj}} = \mathbf{S}$, constrained to $\mathbf{S}'\mathbf{1} = 1$. The matrix $\mathbf{R}_{\mathbf{adj}}$ is an adjusted version of **R**, where A and B in the first row are divided by (*A*+*B*) and the remaining As are switched by 1 s. The matrix $\mathbf{R}_{\mathbf{adj}}$ is as follows in (4):

$$\begin{pmatrix} A/(A+B) & B/(A+B) & 0 & 0 & ... & 0 & 0 \\ 0 & 0 & 1 & 0 & ... & 0 & 0 \\ 0 & 0 & 0 & 1 & ... & 0 & 0 \\ ... & ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 1 & 0 \\ 0 & 0 & 0 & 0 & ... & 0 & 1 \\ 1 & 0 & 0 & 0 & ... & 0 & 0 \end{pmatrix} \quad (4)$$

Tables 1 and 2 give the *SSARL* of the generalized variance $|\mathbf{S}|$ chart, the standard RMAX chart (*Std* chart), the pure synthetic RMAX chart (*PSyn* chart) and the synthetic RMAX (*Syn* chart). In these Tables, ρ=0.5 and $SSARL_0 \cong 370.4$. Comparing the charts for n=5, we conclude the RMAX charts in all versions have a better performance than the generalized variance |**S**| chart, except for some cases where the assignable cause increases the variability of both variables. In these cases, the generalized variance |**S**| chart is faster in signaling. But even for these cases, the advantage of the |**S**| chart in comparison with the synthetic chart, which has the better overall performance among all, is minimal. For example, when L=5, a_1=2.0 and a_2=2.0, the *SSARL*=1.91 for the synthetic chart and the *ARL*=1.85 for the |**S**| chart. If we adopt the sample interval of one hour, the |**S**| chart signals only three minutes before the synthetic chart. By the other hand, if L=5, a_1=1.50 and a_2=1.0, the *SSARL*=9.83 for the synthetic chart and the *ARL*=26.70 for the |**S**| chart. This means the synthetic chart signals almost 17 hours before the |**S**| chart. The conclusions are similar for n=3.

Table 1. The *SSARL*s of the |**S**|, *Std, Syn* and *PSyn* charts, n=5.

	Chart	|S|	*Std*	*Syn*	*PSyn*	*Syn*	*PSyn*	*Syn*	*PSyn*	*Syn*	*PSyn*
	L			2	2	3	3	5	5	7	7
	CL			5.50		5.50		5.50		5.50	
	WL	4.72	5.37	4.40	4.14	4.54	4.30	4.66	4.44	4.74	4.52
a_1	a_2										
1.00	1.00	370.40	370.38	370.37	370.28	370.35	370.32	370.34	370.27	370.34	370.29
1.25	1.00	74.13	47.12	41.74	49.24	40.29	43.23	39.23	39.20	38.87	38.12
1.50	1.00	26.70	11.85	10.35	13.12	10.00	11.35	9.83	10.43	9.85	10.52
1.75	1.00	13.32	5.22	4.70	6.06	4.60	5.42	4.60	5.18	4.64	5.48
2.00	1.00	8.11	3.15	2.93	3.78	2.90	3.50	2.92	3.44	2.95	3.82
2.50	1.00	4.23	1.82	1.78	2.24	1.78	2.17	1.79	2.17	1.80	2.68
1.25	1.25	21.90	26.15	21.18	22.54	20.37	19.93	19.94	18.52	19.90	18.34
1.50	1.50	5.61	6.64	5.49	6.36	5.35	5.75	5.37	5.57	5.44	5.77
1.75	1.75	2.74	3.10	2.76	3.38	2.74	3.18	2.78	3.18	2.82	3.45
2.00	2.00	1.85	2.00	1.89	2.38	1.89	2.31	1.91	2.32	1.92	2.65
2.50	2.50	1.28	1.31	1.30	1.63	1.31	1.62	1.31	1.62	1.31	2.11

Table 2. The *SSARL*s of the |**S**|, *Std. Syn* and *PSyn* charts. n=3.

	Chart	|S|	*Std*	*Syn*	*PSyn*	*Syn*	*PSyn*	*Syn*	*PSyn*	*Syn*	*PSyn*
	L			2	2	3	3	5	5	7	7
	CL			5.50		5.50		5.50		5.50	
	WL	6.56	4.94	3.68	3.62	3.84	3.78	4.00	3.94	4.08	4.03
a_1	a_2										
1.00	1.00	370.40	370.32	370.29	370.37	370.29	370.26	370.33	370.40	370.30	370.37
1.25	1.00	113.48	64.71	63.15	75.94	58.34	66.60	54.70	59.88	53.18	57.08
1.50	1.00	51.57	19.00	18.53	25.31	16.99	21.03	15.98	18.47	15.68	17.62
1.75	1.00	29.36	8.81	8.67	12.65	8.09	10.46	7.78	9.33	7.73	9.04
2.00	1.00	19.25	5.32	5.31	8.05	5.05	6.74	4.94	6.15	4.96	6.05
2.50	1.00	10.65	2.92	2.99	4.75	2.92	4.12	2.91	3.90	2.93	3.89
1.25	1.25	44.05	36.72	31.53	36.39	28.91	31.70	27.25	28.81	26.74	27.84
1.50	1.50	13.86	10.57	9.19	11.49	8.53	9.89	8.25	9.14	8.26	9.02
1.75	1.75	6.90	5.05	4.63	6.11	4.41	5.36	4.38	5.11	4.43	5.12
2.00	2.00	4.39	3.17	3.05	4.21	2.97	3.78	2.98	3.68	3.02	3.72
2.50	2.50	2.58	1.88	1.92	2.86	1.91	2.66	1.92	2.65	1.93	2.67

Tables 3 and 4 give the *SSARL* of the generalized variance $|\mathbf{S}|$ chart, the standard RMAX chart (*Std* chart), the pure synthetic RMAX chart (*PSyn* chart) and the synthetic RMAX (*Syn* chart). In these Tables, ρ=0.5 and $SSARL_0 \cong 700.0$. The synthetic chart is always the best option, except for *n*=5 and large disturbances (when a_1=a_2 and both ≥ 1.50). In these cases, the generalized variance $|\mathbf{S}|$ chart performs better.

Table 3. The *SSARL*s of the |**S**|, *Std. Syn* and *PSyn* charts. *n*=5.

	Chart	|S|	*Std*	*Syn*	*PSyn*	*Syn*	*PSyn*	*Syn*	*PSyn*	*Syn*	*PSyn*
	L			2	2	3	3	5	5	7	7
	CL			6.00		6.00		6.00		6.00	
	WL	5.47	5.60	4.37	4.29	4.52	4.45	4.66	4.59	4.74	4.67
a_1	a_2										
1.00	1.00	700.00	699.93	699.80	699.76	699.80	699.71	699.81	699.83	699.99	699.77
1.25	1.00	120.46	71.33	62.76	75.77	57.62	65.16	53.79	57.81	52.25	54.88
1.50	1.00	39.28	15.61	13.52	18.04	12.44	14.98	11.83	13.33	11.70	12.88
1.75	1.00	18.28	6.33	5.71	8.01	5.40	6.79	5.29	6.28	5.33	6.22
2.00	1.00	10.56	3.62	3.41	4.99	3.31	4.35	3.30	4.16	3.34	4.18
2.50	1.00	5.13	1.97	1.97	3.08	1.96	2.82	1.97	2.79	1.98	2.81
1.25	1.25	31.60	38.83	28.99	32.54	26.53	28.17	25.05	25.62	24.67	24.86
1.50	1.50	7.02	8.59	6.70	8.14	6.30	7.13	6.19	6.76	6.27	6.78
1.75	1.75	3.17	3.69	3.21	4.18	3.12	3.79	3.15	3.74	3.21	3.80
2.00	2.00	2.03	2.24	2.14	2.97	2.12	2.78	2.14	2.79	2.17	2.83
2.50	2.50	1.33	1.38	1.42	2.23	1.42	2.17	1.43	2.17	1.43	2.18

Table 4. The *SSARL*s of the |**S**|, *Std. Syn* and *PSyn* charts. *n*=3.

	Chart	|S|	*Std*	*Syn*	*PSyn*	*Syn*	*PSyn*	*Syn*	*PSyn*	*Syn*	*PSyn*
	L			2	2	3	3	5	5	7	7
	CL			5.50		5.50		5.50		5.50	
	WL	8.05	5.18	3.91	3.78	4.06	3.95	4.21	4.10	4.30	4.19
a_1	a_2										
1.00	1.00	700.00	699.98	699.97	699.77	699.87	699.82	699.87	699.80	699.98	699.74
1.25	1.00	188.79	99.36	92.17	117.71	86.27	101.87	81.45	90.12	79.29	85.03
1.50	1.00	78.82	25.53	23.23	34.05	21.56	27.74	20.38	23.86	19.98	22.50
1.75	1.00	42.24	10.94	10.05	15.62	9.46	12.67	9.12	11.09	9.06	10.66
2.00	1.00	26.45	6.28	5.88	9.43	5.63	7.76	5.52	6.97	5.53	6.82
2.50	1.00	13.74	3.25	3.16	5.24	3.09	4.48	3.09	4.21	3.11	4.20
1.25	1.25	66.19	55.16	45.10	53.22	41.76	45.69	39.41	40.80	38.57	39.01
1.50	1.50	18.38	13.97	11.37	14.64	10.62	12.37	10.24	11.23	10.22	10.98
1.75	1.75	8.49	6.18	5.29	7.20	5.06	6.21	5.01	5.84	5.06	5.83
2.00	2.00	5.14	3.68	3.33	4.73	3.24	4.19	3.26	4.05	3.30	4.09
2.50	2.50	2.85	2.06	2.00	3.05	1.99	2.82	2.00	2.80	2.02	2.83

The choice of *L* depends on the magnitude of the disturbance the practitioner is interested to detect. We defined nine cases of disturbances: cases a, b and f were considered small disturbances; cases c, d, g and h, moderate disturbances and cases e, i and j, large disturbances (see Table 5). Figure 3 shows the *SSARL*s for the synthetic chart with *n*=5 and

$SSARL_0$=370.4. For small disturbances, the synthetic chart reaches its best performance with the input parameter *L* = 5 or 6, except for case a where the best performance is reached for *L*=10. For moderate disturbances, the synthetic chart reaches its best performance with the input parameter *L* = 3 or 4. For large disturbances, the best choice is *L*=2. Similar conclusions can be drawn for other values of *n* and *SSARL*.

Table 5. Cases of disturbances.

case	(a_1;a_2)	case	(a_1;a_2)	case	(a_1;a_2)
a	(1.25;1.00)	e	(2.50;1.00)	h	(1.75;1.75)
b	(1.50;1.00)	f	(1.25;1.25)	i	(2.00;2.00)
c	(1.50;1.00)	g	(1.50;1.50)	j	(2.50;2.50)
d	(2.00;1.00)	-	-	-	-

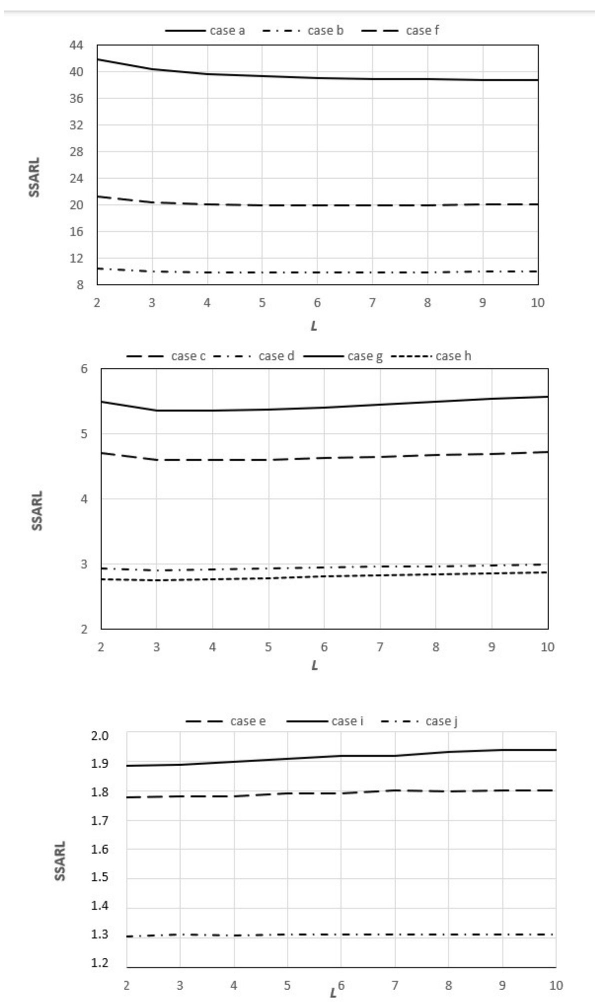

Figure 3. SSARLs for the synthetic chart (*n*=5 and $SSARL_0$=370.4).

Table 6 shows the influence of the control limit *CL* on the *SSARL* values for the *Syn* chart with *L*=5, n=3 and 5. We can notice that CL has a minor influence on the *SSARL* values.

Table 6. The *SSARL*s of the *Syn* chart, *L*=5.

			3			5	
	CL	5.40	5.50	5.60	5.40	5.50	5.60
	WL	4.02	4.00	3.98	4.89	4.66	4.58
a_1	a_2						
1.00	1.00	370.40	370.33	370.40	370.4	370.34	370.40
1.25	1.00	54.43	54.70	55.10	43.47	39.23	37.76
1.50	1.00	15.86	15.98	16.14	10.75	9.83	9.58
1.75	1.00	7.70	7.78	7.86	4.85	4.60	4.55
2.00	1.00	4.89	4.94	5.00	3.00	2.92	2.92
2.50	1.00	2.87	2.91	2.95	1.79	1.79	1.80
1.25	1.25	27.3	27.25	27.29	23.07	19.94	18.86
1.50	1.50	8.22	8.25	8.30	5.90	5.37	5.22
1.75	1.75	4.34	4.38	4.42	2.90	2.78	2.76
2.00	2.00	2,95	2.98	3.02	1.93	1.91	1.92
2.50	2.50	1.90	1.92	1.95	1.30	1.31	1.32

4 Illustrative example

In this section, we provide an example to illustrate the ability of the pure synthetic RMAX chart (*PSyn* chart) and the synthetic RMAX (*Syn* chart) chart in detecting shifts in the covariance matrix. To this end, we considered a bivariate process whose quality characteristics of interest. X_1 and X_2 , are normally distributed. When the process the covariance matrix is given by $\mathbf{\Sigma}_0 = \begin{pmatrix} 1 & 0.5 \\ 0.5 & 1 \end{pmatrix}$. We initially generate 5 samples of size n= 5 with the process in control. The remaining samples were simulated considering that the assignable cause changed the variability of X_1, that is, $a_1 = 1.50$. Table 7 presents the data of X_1 and X_2 , the sample ranges (R_1 and R_2) and the statistic RMAX.

Figure 4 shows the pure synthetic RMAX chart with design parameters *L*=5 and *WL*=4.52. Samples 9 and 12 are nonconforming (RMAX>*WL*). In this case. *CRL* = 4 (13th sample – 9th sample = 4). As *CRL* < *L* (=5), the pure synthetic RMAX chart signals an out-of-control condition at sample 12.

Table 7. Values of X_1, X_2 , R_1, R_2 and RMAX.

Sample		Observations					RMAX
		1	2	3	4	5	
1	X_1	0.53	-1.83	0.20	0.89	-0.80	2.71
	X_2	-0.27	-1.71	-0.10	-1.30	-1.55	
2		-1.63	-0.86	-0.25	0.78	-1.30	2.42
		-0.54	-0.51	-0.93	0.14	-0.93	
3		-0.27	0.12	-0.10	-0.73	-1.05	1.37
		-0.66	0.71	-0.18	0.22	-0.43	
4		-0.27	-0.68	-0.59	-1.60	-0.23	1.57
		-1.01	-0.31	-0.17	-1.38	0.18	
5		-0.07	0.77	2.02	-0.56	0.15	2.58

Table 7. Continued…

Sample	Observations					RMAX
	1	2	3	4	5	
	0.48	-0.27	0.07	-1.66	-0.39	
6	2.00	-1.44	0.45	-2.00	-0.04	3.99
	0.70	-1.25	-0.49	0.31	0.26	
7	-0.31	1.96	2.66	-0.87	0.21	3.52
	0.34	0.07	0.96	-0.54	-0.57	
8	-1.22	-1.02	-1.18	0.02	2.00	3.46
	-0.78	0.10	-1.90	1.56	-0.20	
9	-0.96	-0.40	0.71	3.59	0.97	4.54
	-0.17	-2.00	-0.90	0.63	-0.09	
10	0.46	1.52	0.28	0.31	0.51	1.24
	-0.76	0.20	-0.24	0.38	0.33	
11	-0.73	-3.28	0.87	-0.62	-0.97	4.15
	-2.36	-1.56	0.38	-0.98	-0.93	
12	0.84	0.48	2.81	-2.02	-0.23	4.83
	1.07	-0.09	0.04	-0.04	-0.53	
13	-1.40	2.30	4.41	-0.58	3.10	5.82
	-1.90	0.79	0.63	-1.20	-0.15	

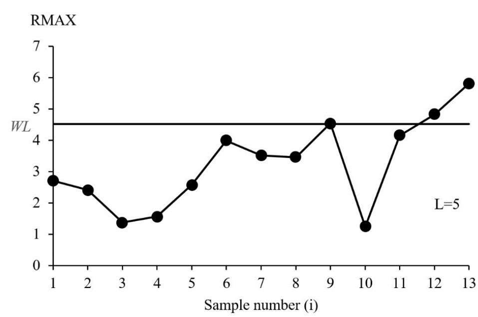

Figure 4. The pure synthetic RMAX chart – example.

Figure 5 shows the synthetic RMAX chart. Samples 12 and 13 are nonconforming (RMAX>*WL*). In this case, *CRL* = 1. As *CRL* < *L* (=5), the synthetic RMAX chart signals at sample 13. Even though *CRL* > 5, the synthetic RMAX chart would signal at sample 13, once the value of the monitoring statistic RMAX is beyond the control limit *CL*=5.51.

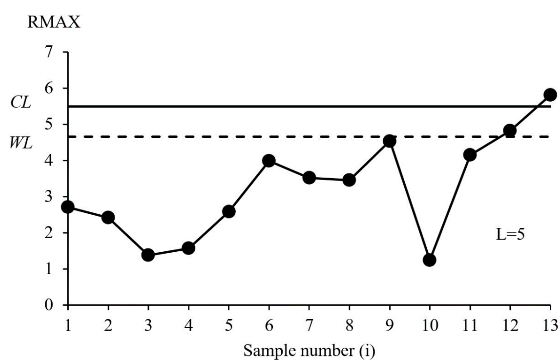

Figure 5. The synthetic RMAX chart – example.

5 Conclusions

The general conclusion related to the combined use of the standard RMAX chart with the synthetic run rules is the following: if the aim of the monitoring is to detect large changes in the covariance matrix, the standard RMAX chart doesn't need additional signal rules, such as the synthetic run rules. However, if the aim is to detect moderate or even small changes in the covariance matrix, the synthetic run rules really enhances the performance of the standard RMAX chart.

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Authors contribution

Marcela Machado and Antonio Costa conceived of the presented idea, developed the theory and performed the computations. Marcela Machado and Felipe Simões wrote the paper in consultation with Antonio Costa.

Appendix 1. The power of the control chart based on the sample ranges – the bivariate case.

Let X_1 and X_2 be two quality characteristics that follow a bivariate normal distribution. The monitoring statistic is given by $\mathrm{RMAX} = \max\{R_1, R_2\}$. The sample ranges are given by $R_i = \max[x_{i1}, x_{i2}, \ldots, x_{in}] - \min[x_{i1}, x_{i2}, \ldots, x_{in}]$, with $x_{ij} = X_{ij} / \sigma_i$, *i*=1,2. If RMAX falls beyond the control limit (CL), the control chart signals an out-of-control condition.

The in-control covariance matrix is given by $\mathbf{\Sigma}_0 = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$. The occurrence of the assignable cause changes the initial covariance matrix to $\mathbf{\Sigma}_1 = \begin{pmatrix} a_1^2\sigma_1^2 & a_1 a_2 \sigma_{12} \\ a_1 a_2 \sigma_{12} & a_2^2 \sigma_2^2 \end{pmatrix}$,

where $\rho = \dfrac{\sigma_{12}}{\sigma_1 \sigma_2}$, $a_k > 1,\ \ k \in \{1,2\}$ and $a_i \geq 1,\ \ i \neq k$ The *ARL* of the RMAX chart is given by:

$$ARL = \frac{1}{1 - F_n(CL/a_1, CL/a_2)} \quad \text{(A1)}$$

During the in-control period $a_i = 1$, and the ARL is the in-control *ARL*, also known as the ARL_0.

Let (x_1, x_2) be the pair of values of the two quality characteristics of each inspected item. Consider that $y_1 = \min(x_{11}, x_{12}, \ldots, x_{1n})$ and $y_2 = \min(x_{21}, x_{22}, \ldots, x_{2n})$. Consider also two cases. I_1 and I_2. I_1 is the case where (y_1, y_2). that is. one item of the sample has the minimum value of the two quality characteristics and I_2 is the case where $[(y_1, x_2 > y_2) \cap (x_1 > y_1, y_2)]$. that is. one item of the sample has the minimum value of the first quality characteristic and another item has the minimum value of the second quality characteristic.

Based on that.

$$F_n(w_1, w_2) = I_1 + I_2 = n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} D_1^{n-1} dF(y_1, y_2) + n\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (n-1) D_1^{n-2} D_2 dG(y_1, y_2) \quad \text{(A2)}$$

where

$$F(y_1, y_2) = \frac{1}{2\pi\sqrt{1-\rho^2}} \int_{-\infty}^{y_1}\int_{-\infty}^{y_2} \exp\left(\frac{-y_1^2 - 2\rho\, y_1 y_2 + y_2^2}{2(1-\rho^2)} \right) dy_1 dy_2 \quad \text{(A3)}$$

and

$$G(y_1, y_2) = \frac{1}{2\pi} \int_{-\infty}^{y_1} \int_{-\infty}^{y_2} \exp\left(\frac{-y_1^2 + y_2^2}{2} \right) dy_1 dy_2 \ . \quad \text{(A4)}$$

The probability D_1 is given by:

$$D_1 = \Pr[y_1 < x_1 \leq y_1 + w_1, y_2 < x_2 \leq y_2 + w_2]$$

$$= B(y_1 + w_1, y_2 + w_2, \rho) - B(y_1 + w_1, y_2, \rho) - B(y_1, y_2 + w_2, \rho) - B(y_1, y_2, \rho) \quad \text{(A5)}$$

In expression (A5), $B(x, y, \rho)$ is the distribution function of a bivariate normal with zero mean vector and a given correlation coefficient. The probability D_2 is given by:

$$D_2 = \Pr[y_1 < x_1 \leq y_1 + w_1 / y_2] \Pr[y_2 < x_2 \leq y_2 + w_2 / y_1]$$

$$= [\Phi(A_1) - \Phi(A_2)][\Phi(A_3) - \Phi(A_4)] \quad \text{(A6)}$$

According to the conditional distribution $x / y \sim N(\rho\, y; (1 - \rho^2))$. Then, it follows that:

$$A_1 = (y_1 + w_1 - y_2 \rho) \big/ \sqrt{1 - \rho^2}$$

$$A_2 = (y_1 - y_2 \rho) \big/ \sqrt{1 - \rho^2}$$

$$A_3 = (y_2 + w_2 - y_1 \rho) \big/ \sqrt{1 - \rho^2}$$

$$A_4 = (y_2 - y_1 \rho) \big/ \sqrt{1 - \rho^2}$$

where $\Phi(x)$ the standard normal cumulative distribution function.

The false alarm risk (*α*) of the control chart is a continuous decreasing function of *CL*, a grid search using expression (A1) allow us to obtain the value of *CL* that equates the ARL_0 to the inverse of the specified false alarm risk (α), that is, $ARL_0 = 1/\alpha$. During the out-of-control period, expression (A1) gives the *ARL* for different combinations of ρ , *n*, *CL*, a_1 and a_2 .

The subroutine DTWODQ, available on the IMSL Fortran library (Microsoft Fortran PowerStation 4.0, 1989), was used to compute the double integration in expressions (A2), (A3) and (A4).