Modeling Random Corrosion Processes via Polynomial Chaos Expansion

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Polynomial Chaos Expansion (PCE) is widely recognized as a flexible tool to represent different types of random variables/processes. However, applications to real, experimental data are still limited. In this article, PCE is used to represent the random time-evolution of metal corrosion growth in marine environments. The PCE coefficients are determined in order to represent data of 45 corrosion coupons tested by Jeffrey and Melchers (2001) at Taylors Beach, Australia. Accuracy of the representation and possibilities for model extrapolation are considered in the study. Results show that reasonably accurate smooth representations of the corrosion process can be obtained. The representation is not better because a smooth model is used to represent non-smooth corrosion data.
Random corrosion leads to time-variant reliability problems, due to resistance degradation over time. Time variant reliability problems are not trivial to solve, especially under random process loading. Two example problems are solved herein, showing how the developed PCE representations can be employed in reliability analysis of structures subject to marine corrosion. Monte Carlo Simulation is used to solve the resulting time-variant reliability problems. However, an accurate and more computationally efficient solution is also presented.
Keywords: *metal corrosion, polynomial chaos, marine corrosion, structural reliability*

Introduction

Mass loss of mild steel due to marine immersion corrosion is a random phenomenon whose statistics change over time. Therefore, the corrosion mass loss of individual coupons can be seen as functions of exposure time (Jeffrey and Melchers, 2001) and the phenomenon itself is considered a stochastic process.

Modeling and simulation of this kind of process can be made using Polynomial Chaos Expansion (PCE). PCE is a widely known tool, which can be applied to represent different types of random variables/processes and which is suitable for responses with smooth probability density functions or devoid of significant nonlinearities (Acharjee and Zabaras, 2007).

However, PCE applications to real, measured data are still limited. In this article, PCE is used to represent the random time-evolution of metal corrosion growth in marine environments.

The plan of the paper is as follows. First, the fundaments of Polynomial Chaos expansion are presented, focusing on practical implementation. The PCE is then used to represent data of 45 corrosion coupons tested by Jeffrey and Melchers (2001) at Taylors Beach, Australia. Accuracy of the representation and possibilities for model extrapolation are discussed. Two numerical examples are then presented, illustrating use of the developed stochastic model of corrosion growth in time-variant reliability analysis of structures subject to marine corrosion. The article is finished with some concluding remarks.

Nomenclature

$C(\xi, t)$	*= corrosion loss as function of time*
$f_X(x)$	*= Probability Density Function (PDF)*
$F_X(x)$	*= Cumulative probability Distribution Function (CDF)*
M	*= stochastic dimension of chaos polynomial*
p	*= order of chaos polynomial*
P_f	*= probability of failure*
u	*= coefficient of PCE for random variable*
$u(t)$	*= coefficient of PCE for random process*
X	*= random variable*
$X(t)$	*= random process of time*

Greek Symbols

β	*= reliability index*
ϕ	*= standard normal probability density function*
Φ	*= standard normal cumulative probability distribution func.*
ξ	*= standard Gaussian random variable*
$\boldsymbol{\xi}$	*= vector of standard Gaussian random variables*
Ψ	*= orthogonal basis of random variables*

Polynomial Chaos Expansion

The Cameron-Martin theorem (Cameron and Martin, 1947) shows that any second order random variable X can be represented by a series expansion as follows:

$$X(\boldsymbol{\xi}) = \sum_{i=1}^{\infty} u_i \Psi_i(\boldsymbol{\xi}) \tag{1}$$

where u_i are coefficients to be determined, $\Psi_i(\boldsymbol{\xi})$ are orthogonal functions of random variables, forming a complete polynomial basis, and vector $\boldsymbol{\xi}$ represents the stochastic dimension of the problem. This is the so-called polynomial chaos expansion, presented by Xiu and Karniadakis (2002) based on a generalization of ideas presented previously by Wiener (1938), Ogura (1972), Askey and Wilson (1985) and Ghanem and Spanos (1991).

For practical purposes, the infinite series in Eq. (1) must be truncated. Adopting n_{chaos} terms in the summation, one has:

$$X(\boldsymbol{\xi}) = \sum_{i=1}^{n_{chaos}} u_i \Psi_i(\boldsymbol{\xi}) \tag{2}$$

Functions $\Psi_i(\boldsymbol{\xi})$ satisfy the following orthogonality conditions:

$$\begin{aligned} &\Psi_1(\boldsymbol{\xi}) \equiv 1 \\ &E[\Psi_i(\boldsymbol{\xi})] = 0 \quad \text{for } i > 1 \\ &E\left[\Psi_i(\boldsymbol{\xi})\Psi_j(\boldsymbol{\xi})\right] = 0 \quad \text{for } i \neq j \end{aligned} \tag{3}$$

Orthogonal polynomials of an important class are members of the so-called Askey-scheme polynomials (Askey and Wilson, 1985). Each subset of these orthogonal polynomials has different weighting functions in their orthogonality relationship (Xiu and Karniadakis, 2002) and some of these weighting functions are identical to the probability distribution function of certain random variables. This

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paper focus on the family of multivariate Hermite polynomials, which are orthogonal with respect to the Gaussian measure; hence $\boldsymbol{\xi}$ are independent standard Gaussian random variables.

In classical form, one-dimensional Hermite polynomials are given by:

$$h_n(\xi) = (-1)^n e^{\xi^2/2} \frac{d^n[e^{-\xi^2/2}]}{d\xi^n} \tag{4}$$

Multidimensional or multivariate Hermite polynomials are obtained as products of one-dimensional polynomials. In order to generate them, the following integer sequences are useful:

$$\begin{aligned} \boldsymbol{\alpha} &= \{\alpha_1, \alpha_2, \dots, \alpha_p\} \quad \text{with } \alpha_j > 0 \\ \boldsymbol{i} &= \{i_1, i_2, \dots, i_p\} \quad \text{with } i_j > 0 \end{aligned} \tag{5}$$

Multidimensional Hermite polynomials associated with the sequences $(\boldsymbol{\alpha}, \boldsymbol{i})$ are then obtained as:

$$\Psi_{\boldsymbol{i},\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \prod_{k=1}^{p} h_{\alpha_k}(\xi_{i_k}) \tag{6}$$

For practical computational purposes, polynomials of finite dimensions have to be used. The polynomial basis obtained using M Gaussian random variables is denoted by $\Gamma_p(\xi_1, \xi_2, \dots, \xi_M)$ and is called "homogeneous chaos of dimension M and order p". This base is generated as follows: for each set of M integers = $\{\alpha_1, \alpha_2, \dots, \alpha_M\}$, with M varying from 0 to p, the following base vector is obtained:

$$\Psi_{\boldsymbol{\alpha}}(\boldsymbol{\xi}) = \prod_{k=1}^{M} h_{\alpha_k}(\xi_k). \tag{7}$$

It can be shown that the dimension of $\Gamma_p(\xi_1, \xi_2, \dots, \xi_M)$ is obtained as the binomial factor:

$$\dim[\Gamma_p] = \binom{M+p-1}{p}. \tag{8}$$

Chaos polynomials of dimension up to $M = 4$ and order up to $p = 4$ are presented in Ghanem and Spanos (1991). A recursive algorithm to generate multi-dimensional Hermite polynomials is presented by Sudret and Der Kiureghian (2000).

Practical Implementation

The formulation presented above allows the representation of any second-order random variable. The polynomial chaos expansions (PCE) are obtained by requiring the cumulative distribution functions (CDF) of the representation (F_X) to match the empirical CDF of the experimental points at each discrete time step. Hence, for each point of time, the coefficients of the PCE are determined by minimizing the following error measure:

$$error_j = \sum_{i=1}^{n_{samp}} \left| F_X(x_i) - \frac{i}{n_{samp}+1} \right| \tag{9}$$

where the cumulative distribution function (CDF) of the representation is given by:

$$F_X(x) = P[X(\boldsymbol{\xi}) \le x] = P\left[\sum_{i=1}^{n_{chaos}} u_i \Psi_i(\boldsymbol{\xi}) \le x\right] \tag{10}$$

In the general case, this probability is not known analytically, but it can be approximated by means of large samples of random vector $\boldsymbol{\xi}$.

The extension of the polynomial chaos expansion to approximate stochastic processes (instead of only random variables) can be done in a discrete fashion, by determining the coefficients in specific points of the continuous (t) and replacing them by functions of the continuous. In other words, the values obtained for a specific coefficient u_i are replaced by a function of t, $\tilde{u}_i(t)$. Thus, the PC expansion is given by:

$$X(t, \boldsymbol{\xi}) = \sum_{i=1}^{n_{chaos}} \tilde{u}_i(t) \Psi_i(\boldsymbol{\xi}) \tag{11}$$

Common functions are used to approximate the coefficients u_i over t:

$$\tilde{u}_i(t) = p_1^i + p_2^i \cdot t^{q_i} \tag{12}$$

The coefficients p_j^i and the exponents q_i are determined by minimizing the difference between moments of the approximation and the experimental data. First and second-order moments were used in this paper. Equations (9) and (10) could have been used instead, but convergence difficulties were encountered. Since both the coefficients and the exponents are optimization variables, the objective function may have many local minima, so a good starting point is required for the optimization process.

Insight about the (corrosion) random process is used to determine a proper initial value for the exponents q_i. It is known, for instance, that mean corrosion growth in time is a less-than-linear function of time (Jeffrey and Melchers, 2001). Therefore, initial values $q_i = 0.75$ are adopted. Initial values for the coefficients p_j^i are calculated by least squares, considering the constant u_i values found in the first part of the process (using Eqs. (9) and (10)). Given the initial points, the optimization problem is solved by using the *fmincon* function of MATLAB, with lower and upper bounds equal to $[0.25, 1.50]$ for q_i and $[-10^8, +10^8]$ for p_j^i.

Application of the PCE to Represent Experimental Corrosion Data

The developed algorithm was applied to an experimental corrosion dataset contained in the research report of Jeffrey and Melchers (2001), which corresponds to corrosion mass loss of coupons tested at Taylors Beach, Australia. The exposure time varied from 1 to 4.21 years. It is worth noting that the Taylors Beach dataset was chosen for being the largest dataset of the aforementioned report, but other corrosion dataset can be used as well. Table 1 presents the selected data, incorporating the information of no corrosion in the initial time. The experimental data are also presented as dots in Figs. 1 to 4.

Table 1. Corrosion dataset.

Years exposed	Corrosion loss (μm)								
0.00	0	0	0	0	0	0	0	0	0
1.00	146	140	147	133	148	150	156	157	132
1.49	119	151	152	154	151	152	182	191	154
2.00	189	186	189	202	180	185	197	194	202
3.03	308	245	290	258	311	281	277	150	258
4.21	349	308	316	383	351	376	348	382	364

Polynomial chaos expansion (PCE) representations were obtained for stochastic dimensions 1 and 2 (called PCE1 and PCE2, respectively), for orders ranging from 1 to 10. For stochastic dimension 1, the best representation was obtained for PCE order equal to 4 (with 5 PCE coefficients), while fifth order polynomials (with 21 PCE coefficients) led to the best approximation for stochastic dimension 2.

The continuous representation was first obtained by leaving the data for exposure time 4.21 years out, to serve as validation data. Figures 1 and 2 show the PCE1 and PCE2 representations, respectively, *versus* the experimental and the validation data. One observes that, although the extrapolation results appear reasonable, the small amount of experimental data makes extrapolation validation almost prohibitive. Validation errors were evaluated for PCE1 and PCE2, in order to verify the extrapolation capabilities of the two models. Results are shown in Table 2. It can be observed that results for PCE1 are good in terms of the mean, but bad in terms of standard deviation. Nevertheless, it can be observed that PCE1 presents better results than PCE2. The errors in standard deviation are discussed further in the following paragraph.

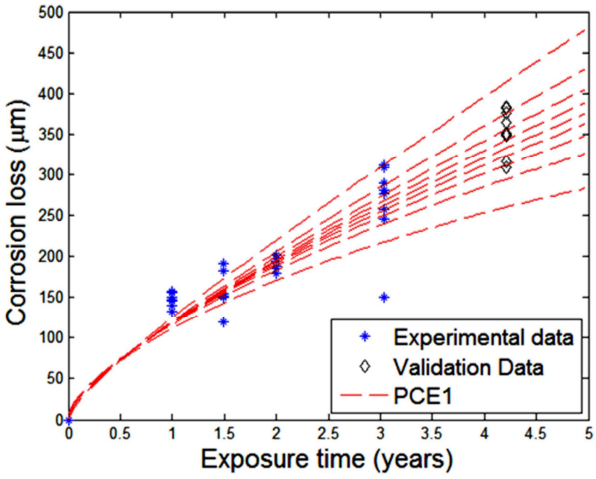

Figure 1. PCE1 representation versus experimental and validation data.

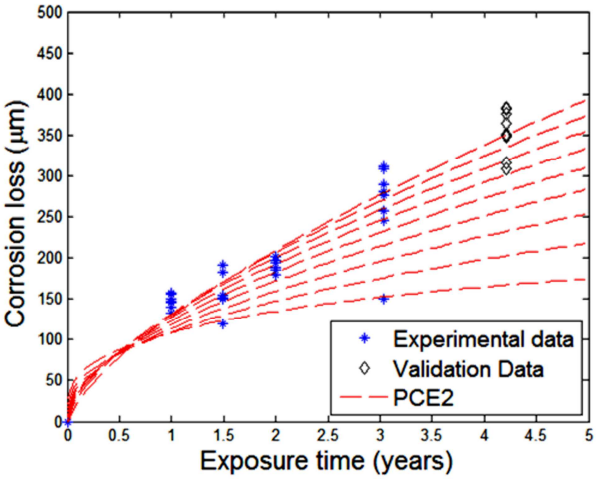

Figure 2. PCE2 representation versus experimental and validation data.

After verifying the extrapolation capabilities, the PCE1 and PCE2 representations were constructed again, this time considering all the experimental data available. Tables 3 and 4 show the errors obtained after the interpolation of the PCE coefficients in time, considering the mean and the standard deviation, respectively. The errors in mean are less than 5% for all but the first time point (one year). The errors are still quite significant in terms of the standard deviations. For the third time point (2 years), the errors are very large due to the unusual concentration of the experimental data points, which leads to very small standard deviation of the experimental results. Looking at the fourth column of Table 4 (experimental standard deviations), one observes that the standard deviations oscillate with respect to time, what makes it difficult to obtain smooth and accurate representations of the process. Hence, the errors in standard deviation are bound to be large for any smooth representation of the process.

Table 2. Validation errors for exposure time 4.21 years.

	Validation error (%)	
	Mean	**Standard deviation**
PCE1	5.30	66.95
PCE2	22.96	130.09

Table 3. Approximation errors in mean corrosion loss.

Exposure time (years)	**Mean**			**Error (%)**	
	PCE1	**PCE2**	**Experimental**	**PCE1**	**PCE2**
1.00	116.311	121.440	145.444	20.031	16.504
1.49	156.168	161.841	156.222	0.035	3.597
2.00	194.391	200.478	191.556	1.480	4.658
3.03	265.173	271.822	264.222	0.360	2.876
4.21	339.407	346.435	353.000	3.851	1.860
			Average	5.151	5.899

Table 4. Approximation errors in standard deviation.

Exposure time (years)	**Standard deviation**			**Error (%)**	
	PCE1	**PCE2**	**Experimental**	**PCE1**	**PCE2**
1.00	8.527	6.886	8.434	1.101	18.360
1.49	11.656	9.707	19.321	39.672	49.758
2.00	14.616	12.705	7.259	101.352	75.021
3.03	20.042	18.628	45.592	56.041	59.142
4.21	25.707	25.091	25.451	1.006	1.418
			Average	39.834	40.740

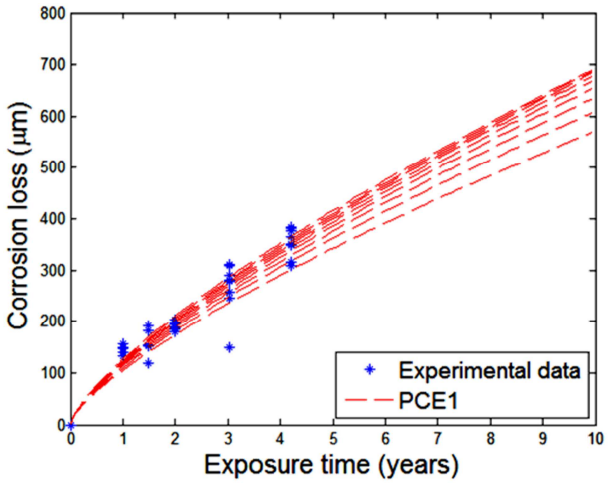

Figure 3. Final PCE1 representation and extrapolation to 10 years.

Figures 3 and 4 show the final PCE1 and PCE2 representations obtained, respectively, as well as the experimental data points. The PCE representations are extrapolated for up to 10 years (period

assumed in the following reliability analysis), in order to further explore the extrapolation capabilities of the PCE representations. It is observed that the predictions (extrapolations) by the two representations are different. In contrast to the large errors observed in Table 4, the visual fitting of the models to the experimental data is observed to be good.

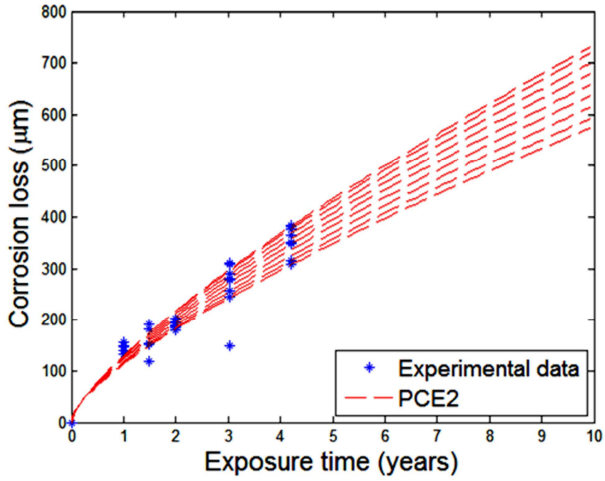

Figure 4. Final PCE2 representation and extrapolation to 10 years.

From the results in Tables 3 and 4 one observes that the PCE1 representation is better than the PCE2 representation, mainly for the standard deviation. In addition, the computational cost of PCE1 is smaller than of PCE2 (with sixteen coefficients less). Thus, the 4th order PCE1 representation is described in further detail. The polynomial basis for this case is (considering Hermite polynomial chaos):

$$\Psi_1(\xi) = 1;\ \ \Psi_2(\xi) = \xi;\ \ \Psi_3(\xi) = \xi^2 - 1;$$

$$\Psi_4(\xi) = \xi^3 - 3\xi;\ \ \Psi_5(\xi) = \xi^4 - 6\xi^2 + 3. \qquad (13)$$

Figure 5 shows the coefficients for PCE1 and their respective polynomial adjustments.

The approximating functions obtained for each PCE coefficient are as follows:

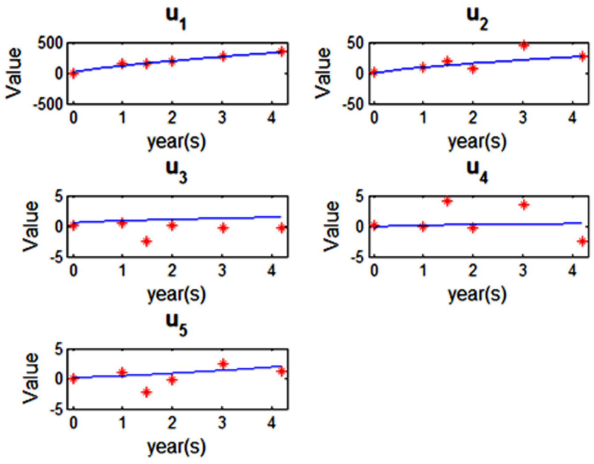

Figure 5. PCE coefficients (dots) and approximating polynomials (lines).

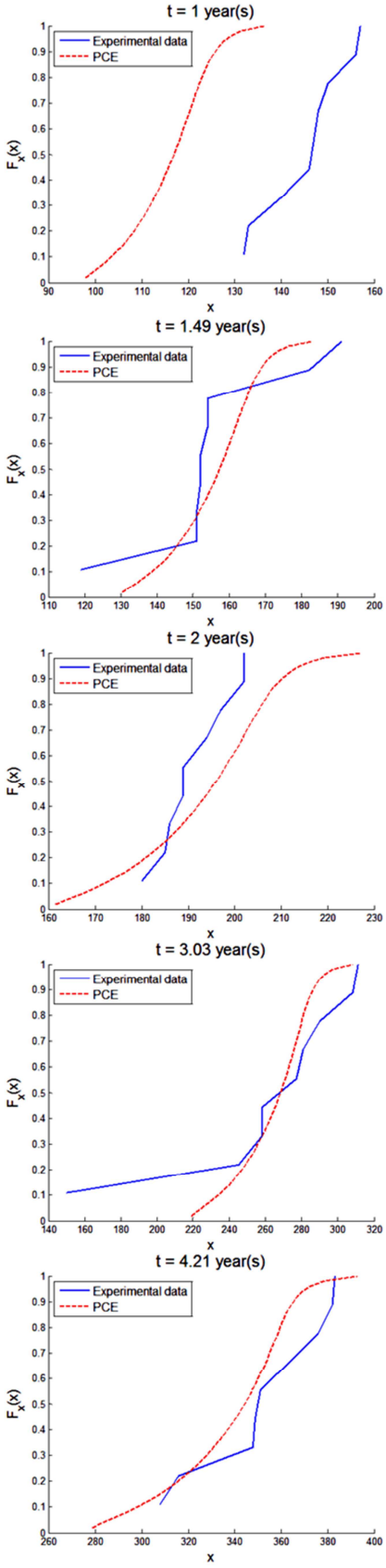

Figure 6. Comparison of empirical CDF (solid lines) and CDF obtained from PCE representation (dotted lines).

$$\begin{aligned}
\tilde{u}_1(t) &= +3.98 + 112.61 \cdot t^{0.76} \\
\tilde{u}_2(t) &= -1.26 + 10.41 \cdot t^{0.70} \\
\tilde{u}_3(t) &= +0.53 + 0.36 \cdot t^{0.69} \\
\tilde{u}_4(t) &= -0.21 + 0.36 \cdot t^{0.40} \\
\tilde{u}_5(t) &= +0.24 + 0.26 \cdot t^{1.36}
\end{aligned} \tag{14}$$

The interpolation of coefficients using predetermined functions is very important to improve extrapolation and interpolation capabilities. On one hand, the use of high order functions can lead to over fitting and poor results. On the other hand, low order functions may not be sufficient to represent the underlying behavior. Thus, it is interesting to find a compromise between complexity and interpolation/extrapolation capabilities. In the present paper functions composed of a constant and a non-linear term were used (Eq. (12)). The optimal exponents and parameters found are presented in Eq. (14).

Figure 6 compares the empirical CDF of the experimental data with the CDF of the PCE1 representation for the 5 time points for which data is available. The (mean-square) difference between these CDFs is used to guide the solution for the unknown coefficients, following Eq. (10). The fitting of the two CDFs is much better before the time-interpolation of the coefficients. After interpolation, the fit is not very good for $t = 1$ and for $t = 2$ years, but it is still reasonable for the other times.

Looking at Figs. 3 and 4, the PCE representations and extrapolations obtained herein are considered good, especially considering the small amount of data (45 points in total, 9 for each time). Also, recall that the smoothing necessary for a continuous representation of the data leads to errors that cannot be reduced without loss of extrapolation capabilities.

Example Problems: Time-Variant Reliability Analysis

Resistance degradation due to corrosion leads to time-variant reliability problems. These problems are not trivial to solve, especially when random load processes are involved. Two examples are presented herein, in order to illustrate how the developed PCE representations can be employed in life prediction of structures subject to marine corrosion.

Solution of time-dependent reliability problems under resistance degradation and under random loading is not trivial (Beck and Melchers, 2004; Beck and Melchers, 2005; Beck, 2008). The most straightforward solution is brute Monte Carlo simulation. An alternative formulation is presented herein.

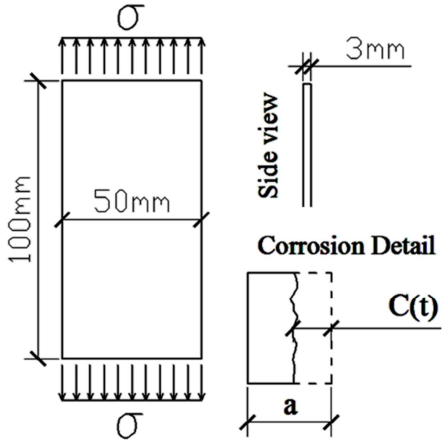

Figure 6. Plate subject to tensile end loading and to corrosion.

The problem consists in estimating the probability of failure of a plate with thickness $a = 3\ mm$, width $b = 50\ mm$ and length equal to $100\ mm$, subject to tensile end loading σ and to corrosion loss $C(\xi,t)$, according to Fig. 7. For the sake of brevity, $C(\xi,t)$ is often written only as a function of time, i.e. $C(t)$.

The plate material has random yield stress Y with lognormal distribution and parameters $Y \approx \text{LN}(\mu_Y,\sigma_Y) = \text{LN}(1.25,0.0875)$ MPa, which corresponds to a coefficient of variation of 7%.

Two different load cases are studied: with time-dependent and time-independent loading. For both cases, the failure probability over a period of 10 years is calculated. Details about the two load cases, as well as the methodologies used to calculate the failure probabilities are presented in the following subsections. The equations are developed based on Beck (2008).

All problems were solved by computer codes implemented in MATLAB®, running on a computer with an Intel Core 2 Duo – E7500, 2.93 GHz processor, with 3.21 GB of usable RAM, using parallel processing (2 CPU cores).

Random loading and limit state function

Two cases of random loading are considered. In the first case, the tensile loading is considered a random variable; in the second case, tensile loading is a random process of time: $S = S(t)$. In both cases, the probability distribution function of tensile loading is that of a standard normal distribution, hence $S \sim \text{N}(\mu_S,\sigma_S) = \text{N}(0,1)$. The limit state equation is written as:

$$g\left(Y, S(t), C(t)\right) = Y \cdot b \cdot \left(a - C(t)\right) - S(t) \cdot a \cdot b = 0 \tag{15}$$

where, for the random variable load case, $S(t) = S$. Both sides of Eq. (15) are further divided by $a.b$. The failure probability to be evaluated in both cases is:

$$P_f(t_D) = P\left[\min_{0 \le t \le t_D}\left[g\left(Y, S(t), C(t)\right)\right] < 0\right] \tag{16}$$

The generation of stochastic load process samples is explained later. The random variable load problem is solved first.

Monte Carlo simulation

The most straightforward method to evaluate failure probabilities over time (Eq. (16)) is crude Monte Carlo Simulation (MCS). In this case, n_{samp} random samples of random variables Y, S and ξ are generated according to their probability distribution functions. For each sample of ξ a complete time history of $C(t)$ is obtained, following Eqs. (11), (13) and (14).

A failure indicator function $I()$ is used, which takes values of 1 for failure or 0 for survival depending on the value of the limit state function $g()$, for a given set of values (Y_i, S_i and ξ_i) and a specific time t_p:

$$I\left(Y_i, S_i, \xi_i, t_p\right) = \begin{cases} 1, & \text{if } g\left(Y_i, S_i, \xi_i, t_p\right) < 0 \\ 0, & \text{otherwise.} \end{cases} \tag{17}$$

The indicator function is evaluated at each discrete point of time (t_p) starting from an initial time ($t_{ini} = 0$) until the final time ($t_D = 10$ years), with a time increment of dt years, for each sample point (Y_i, S_i and ξ_i) of the random space, with $i = 1,2,...,n_{samp}$. For the random load process, $S_i = S_i(t_p)$. The number of discretization points in time is:

$$t_{steps} = \frac{t_D - t_{ini}}{dt} + 1. \tag{18}$$

A vector *ind* with dimension t_{steps}, containing the number of failures over time is computed:

$$ind\left(t_p\right)=\sum_{i=1}^{nsamp} I\left(Y_i,S_i,\xi_i,t_p\right), \text{ for } p=1,2,\ldots,t_{steps}. \tag{19}$$

Importantly, if failure occurs at a time t_p, the indicator function will be equal to one for all later times (first passage problem).

The mean value estimate of failure probabilities over time is then given by:

$$\overline{P_f}=\frac{ind}{n_{samp}} \tag{20}$$

Resistance integration solution for random variable loading

Consider a resistance *R* per unit area, function of *Y* and *C*(*t*). For each sample (Y_i, ξ_i, t_p), with I = 1,2,..., n_{samp} and p = 1,2,...,t_{steps}, one has r_{ip} (a scalar number):

$$r_{ip}=\frac{Y_i\cdot\left(a-C\left(\xi_i,t_p\right)\right)}{a} \tag{21}$$

The limit state equation for each t_p is rewritten as a function of *S* and R_p (the latter being the random variable resistance for a given time):

$$g\left(R_p,S\right)=R_p-S=0 \tag{22}$$

Recalling that $S\approx N(0,1)$, the probability of failure given an occurrence r_{ip} can now be calculated as:

$$P_f\left(t_p\mid r_{ip}\right)=P\left[\left\{r_{ip}<S\right\}\right]=1-P\left[\left\{S\le r_{ip}\right\}\right]=1-\Phi\left(\frac{r_{ip}-\mu_S}{\sigma_S}\right)=1-\Phi\left(r_{ip}\right) \tag{23}$$

That is, for each occurrence r_{ip}, it is possible to evaluate analytically the conditional probability of failure, using the cumulative distribution function of the random variable *S*, i.e., Φ(). Thus, for each time (t_p) the failure probability can be calculated by integrating the conditional probability over the resistance:

$$P_f\left(t_p\right)=\int_{R_p} P_f\left(t_p\mid r_p^i\right)\cdot f_R\left(r_p^i\right)dR_p \tag{24}$$

Equation (24) is solved by Monte Carlo Simulation, but using a much smaller number of samples than required for the time-variant Monte Carlo simulation.

Results for Random Variable Loading

The problem is first solved for the random variable loading, using Monte Carlo simulation and a sample of n_{samp} = 30000. This is large enough to calculate a failure probability of the order 10^{-3}-10^{-4}.

For the resistance integration solution, two distinct solutions were computed: with n_{samp} = 500 and with n_{samp} = 1000. These values were defined comparing the obtained results with the Monte Carlo solution, trying to use the smaller possible sample size. Results for *dt* = 0.25 are shown in Fig. 8.

The MC solution required about 99 seconds to run, while the resistance integration solutions took about 10 seconds and 12 seconds to run, respectively. These differences are not significant for the current problem, but they could become significant for a more relevant, real engineering problem.

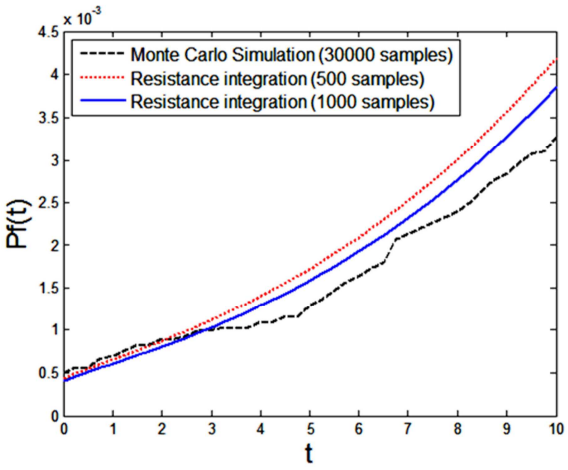

Figure 7. Failure probability over time for random variable load case.

Stochastic Process Loading

In this case, the tensile loading was considered a continuous Gaussian stochastic process *S*(*t*) with zero mean and unit variance. An exponential (Beck, 2008) autocorrelation function is considered (leading to a broadband process, with λ = 1 and cutoff frequency $\omega_b = 10$).

A spectral representation is used to obtain samples of this load process (Borgman, 1969; Grigoriu, 2000). In this representation, the power spectrum density function $G(\omega)$ of the process is discretized in a finite number m_{spec} of frequency components:

$$\int_{\omega_a}^{\omega_b} G(\omega)\,d\omega \approx \sum_{k=1}^{m_{spec}} G(\omega_k)\,\Delta\omega_k \tag{25}$$

where ω_a and ω_b are the lower and upper (truncation) frequencies, $\Delta\omega_k = \frac{\omega_b-\omega_a}{m_{spec}}$ is the frequency interval and ω_k is the k^{th} frequency component. The discretized process representation is then obtained as:

$$\sigma_{m_{spec}}(t)=\sum_{k=1}^{n} G(\omega_k)\,\Delta\omega_k\,\left(V_k\cos(\omega_k\,t)+W_k\sin(\omega_k\,t)\right) \tag{26}$$

where $\sigma_{m_{spec}}(t)$ is the m_{spec}-dimensional approximation of process $\sigma(t)$, and V_k and W_k are independent Gaussian random variables with zero means and unit variances.

Adopting sample period equal to the total period of simulation (t_D = 10 years), the number of discretization points of the power spectral density (m_{spec}) is:

$$m_{spec}=\frac{\omega_b t_D}{2\pi}=\frac{10\cdot10}{2\pi}\cong 15.91 \tag{27}$$

The number $m_{spec}=16$ is adopted, which corresponds to a period t_D = 10.05 years, slightly larger than required. The number of points of discretization in time was set as t_{steps} = 200, for which a smooth representation of the load process is achieved. One realization of the stochastic process σ(*t*) is shown in Figure **8** 9.

The moments of order 0, 2 and 4 of the power spectral density function of the process are as follows:

$$\begin{aligned}\lambda_0&=1\\ \lambda_2&=5.7975\\ \lambda_4&=220.786\end{aligned} \tag{28}$$

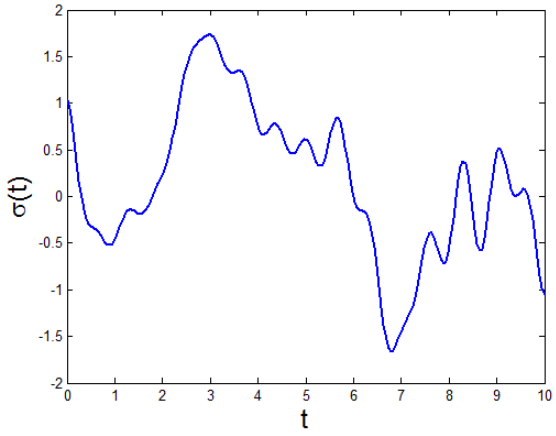

Figure 8. One realization of the stochastic process S(t), with t_{steps} = 200.

From these moments, the regularity factor α is evaluated as $\alpha = \lambda_2 / \sqrt{\lambda_0 \lambda_4} \cong 0.39$, i.e., the regularity factor is much smaller than 1, as expected for broadband processes. The average angular frequency is equal to $\omega_0 = \sqrt{\lambda_2 / \lambda_0} \cong 2.4078$, which leads to a mean crossing rate $\upsilon_0^+ = \omega_0 / 2\pi \cong 0.3832$.

Solution by Linear Approximation of the Barrier in Each Time Interval

As the load is a stationary Gaussian stochastic process, the up-crossing rate for a barrier level that varies linearly with time can be obtained analytically. For a stationary Gaussian process and a barrier $r(t) = a_r + b_r.t$, the up-crossing rate is given by (Cramer and Leadbetter, 1967):

$$\upsilon^+\left(r^i, t\right) = \omega_0 \cdot \phi\left[\frac{a_r - b_r t - \mu_S}{\sigma_S}\right] \cdot \Psi\left[\frac{b_r}{\omega_0 \sigma_S}\right] \tag{29}$$

where $\Psi[x] = \phi[x] - x\Phi[-x]$.

Consider again a resistance R per unit area, function of Y and $C(t)$. For each sample (Y_i, ξ_i), with $i = 1,2,...,n_{samp}$, a function of time $r_i(t)$ is obtained:

$$r_i(t) = \frac{Y_i \cdot \left(a - C(\xi_i, t)\right)}{a}. \tag{30}$$

This function is evaluated at the t_{steps} time discretization points. For each instant t_p, with $p = 2,...,t_{steps}$, the coefficients of the linear approximation of the resistance are:

$$a_r = r_i\left(t_{p-1}\right)$$
$$b_r = \frac{r_i\left(t_p\right) - r_i\left(t_{p-1}\right)}{dt}. \tag{31}$$

The integral over time of the rate shown in Eq. (29) is zero for p = 1. For $p = 2,...,t_{steps}$ the integral is calculated in an incremental fashion:

$$\text{int}_p(r_i) = \text{int}_{p-1}(r_i) + \int_{t_{p-1}}^{t_p} \upsilon^+\left(r_i, t\right) dt. \tag{32}$$

In addition, the integral of the linear approximation can be easily calculated:

$$\int_{t_{p-1}}^{t_p} \upsilon^+\left(r_i, t\right) dt = dt \cdot \left(\frac{\upsilon^+\left(r_i, t_{p-1}\right) + \upsilon^+\left(r_i, t_p\right)}{2}\right). \tag{33}$$

At time t_{ini} = 0, the initial failure probability is calculated as:

$$P_{f_0}\left(r^i\right) = 1 - P\left[\left\{S_i\left(t_1\right) \le r_i\left(t_1\right)\right\}\right] = 1 - \Phi\left(\frac{r_i\left(t_1\right) - \mu_S}{\sigma_S}\right) = 1 - \Phi\left(r_i\left(t_1\right)\right) \cdot \tag{34}$$

The probability of failure at the time t_1, given the occurrence of $r_i(t_p)$, is equal to the initial failure probability:

$$P_f\left(t_1 \mid r_i\right) = P_{f_0}\left(r_i\right) \cdot \tag{35}$$

Thus, the conditional failure probability at each time is:

$$P_f\left(t_p \mid r_i\right) = P_{f_0}\left(r_i\right) + \left(1 - P_{f_0}\left(r_i\right)\right) \cdot \left(1 - \exp\left(-\text{int}_p(r_i)\right)\right) \cdot \tag{36}$$

Finally, the failure probability can be calculated by integrating the conditional probability over the resistance:

$$P_f\left(t_p\right) = \int_{R_p} P_f\left(t_p \mid r_i\right) \cdot f_R\left(r_i\right) dR_p \cdot \tag{37}$$

This integral is solved by Monte Carlo Simulation, but the sample size (n_{samp}) required in this case is much smaller than that required for the time-variant Monte Carlo solution.

Results for Stochastic Process Loading

The problem was solved by Monte Carlo method with sample size n_{samp} = 30000 which is large enough to accurately calculate the required probability of failure.

The solution by linear approximation of the barrier was computed for n_{samp} = 500 and for n_{samp} = 1000. Results for t_{steps} = 200 (dt = 0.05) are shown in Fig. 10. As expected, one observed a significant increase in failure probabilities when loading is a stochastic process instead of a single random variable.

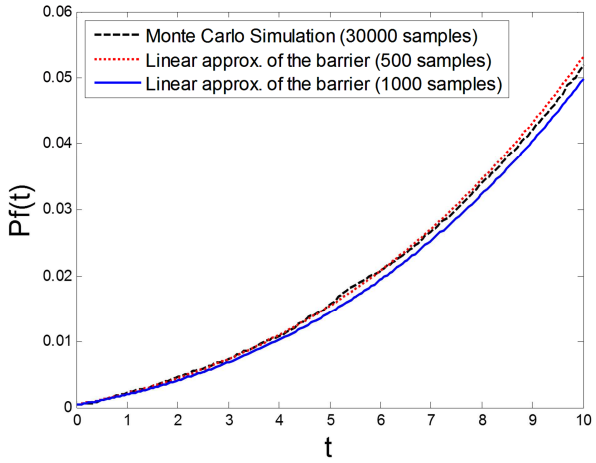

Figure 9. Failure probability over time, stochastic load process.

The time-variant Monte Carlo solution took about 162 seconds to run, while the resistance integration solutions took about 15 seconds and 16 seconds, respectively. Hence, the resistance integration solution is computationally much more efficient than the crude time-variant Monte Carlo simulation.

Concluding Remarks

This article addressed modeling of stochastic corrosion process in marine environments by Polynomial Chaos Expansions (PCE). It was shown that PCE is a flexible tool to represent different kinds of random experimental data.

One and two-dimensional PCE representations were obtained herein for the results of 45 corrosion coupons tested by Jeffrey and Melchers (2001) at Taylors Beach, Australia. The experimental data was observed to be non-smooth with respect to the standard deviation of corrosion loss for different years. This limited the accuracy that could be achieved by the continuous and smooth PCE representations. Hence, errors of the PCE representation were significant with respect to standard deviation. However, the visual fitting of the PCE representations to the experimental data was shown to be good.

Moreover, it was shown that the PCE representations developed herein can be used to extrapolate the data for longer corrosion times. This is important in order to allow the lifetime safety management of structures subject to marine environments. It was shown that a compromise has to be found between the accuracy and extrapolation capacities of the model. Use of higher-order PCE representations could be used to increase accuracy, but this would limit the extrapolation capacity of the model. Moreover, it was shown that the extrapolation capacity of the model depends directly on insight about the corrosion process. In this article, good extrapolation was obtained by using smaller-than-one powers of time.

Two examples were presented showing application of the PCE representations to reliability problems with resistance degradation due to corrosion. Solution of such time-dependent reliability problems under random loading is not trivial, and generally only Monte Carlos simulation can be employed. An alternative formulation was presented herein, where conditional crossing rates are evaluated by linearizing the resistance variation between two time points. Conditional failure probabilities are evaluated from conditional crossing rates, and Monte Carlo simulation is performed only over the resistance random variables. This alternative formulation was shown to be very efficient, requiring many times less samples than the complete time-variant Monte Carlo simulation.

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