Theoretical model for the transmittance in a left-handed metamaterial of different geometries

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Abstract— **In this work, a theoretical model for the transmittance in a left-handed metamaterial of different geometries is presented. The proposed unit cells are a combination of conducting wires of rectangular cross-section with square-ring and hexagonal resonators. In this new dielectric compound, dependence of transmittance on the thickness of the resonators are analyzed for the particular case of normal incidence in the GHz range. It is found that the transmittance for the hexagonal resonator shows a significant increase compared to the square geometry resonator.**

Index Terms— Transmittance, metamaterials, unit cells.

I. INTRODUCTION

In 1968, The Russian physicist Victor Veselago proposed to consider the possibility of materials with negative electric permittivity, ε , and magnetic permeability, μ , leading to a negative refractive index $n = \sqrt{\varepsilon\mu}$ [1]. Since then, the search for micro structures capable of generating negative values for ε and μ became a very active area of work since John Pendry and colleagues introduced in 1999, the open split-ring resonators (SRR) [2]. The electromagnetic properties of the SRR were deeply analyzed in Refs. [3] and [4]. Its possible applications to communications have been studied by many researchers in various areas, for example, in microwave filters [5] - [10], power dividers [11] - [14] and antennas [15] - [22], and others several potential applications. Studies of the modes of propagation of electromagnetic waves using metamaterials as waveguides have also been reported [23] - [31]. Now well, making the first target materials had to wait until the early 2000s when a compound with negative refractive index operating in a frequency band in the range of microwaves were demonstrated [32] - [34].

This work deals with the electromagnetic response of a medium composed of unit cells which are constructed by the combination of rectangular cross-sectional wires and SRRs with square or hexagonal geometries. The transmittance of an electromagnetic wave in the range of microwaves with normal incidence in a left-handed metamaterial is numerically analyzed. In Section II, the theoretical model for the transmittance of an electromagnetic wave with normal incidence in left-handed metamaterials is concisely presented. Section III contains the results and a brief discussion of these.

II. THEORETICAL MODEL
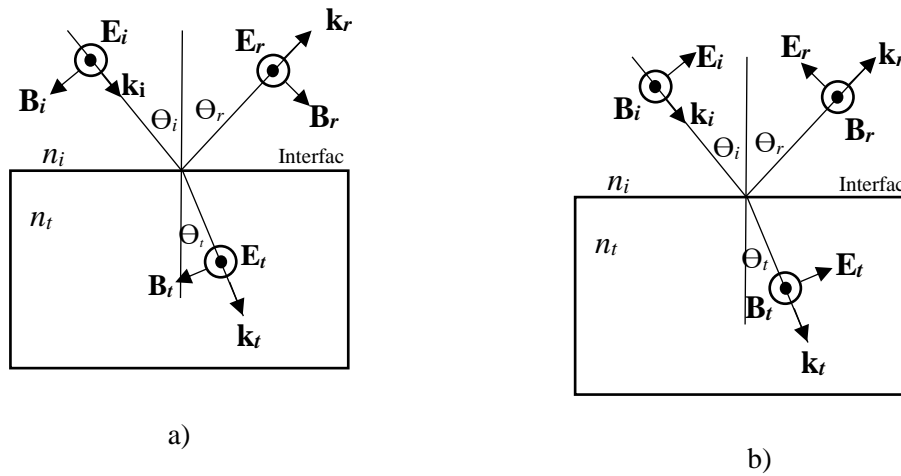
A. Transmittance at an interface

a)

b)

Fig 1. a) Incident wave whose **E** field is normal to the plane of incidence (TE polarization) and b) incident wave whose **E** field is in the plane of incidence (TM polarization).

Fig. 1 shows the vectors of electric field and magnetic induction field for the TE (Fig. 1a) and TM (Fig.1b) cases when an electromagnetic wave is incident on an interface separating two media. For the case of normal incidence ($\Theta_i = 0$), it is known that the transmittance T is given by,

$$T = \frac{4 n_t n_i}{(n_t + n_i)^2} \qquad (1)$$

where n_i and n_t are the refractive index of the first and second medium, respectively. If $n_i = 1$ and $n_t = n$, Eq. (1) reduces to,

$$T = \frac{4n}{(n+1)^2} \qquad (2)$$

B. Transmittance in a metamaterial.

The refractive index of a natural medium depends on the effective electric permittivity and magnetic permeability, and is given by $n = \sqrt{\varepsilon_{eff}\mu_{eff}}$, where $\varepsilon_{eff} = \frac{\varepsilon}{\varepsilon_0}$ and $\mu_{eff} = \frac{\mu}{\mu_0}$, with ε and μ the electrical permittivity and magnetic permeability of the medium, respectively, and ε_0 and μ_0 the electric permittivity and magnetic permeability of the vacuum. The basic idea of a metamaterial is that this is constructed by doping a dielectric subtract with unitary cells formed by SRRs and thin wires, as shown in Fig. 2a. The incident electromagnetic wave is polarized in such a way that, the magnetic vector $\vec{H}$ oscillates perpendicular to the SRR plane and the electric field vector $\vec{E}$ oscillates parallel to the wires. The electrical permittivity ε (and therefore its electric properties) of the entire cubic cell is provided by the thin wires, while the magnetic permeability μ (and therefore its magnetic properties) is provided by the resonators SRRs. If a lossless regime is assumed, which means that both, the thin wires

and the SRRs of the cubic cell are made of excellent conductors, the magnetic permeability of a square or hexagonal SRR as shown in Fig. 2a and b, is given by [35],

$$\mu_{eff} = 1 - \frac{F\omega^2}{\omega^2 - \omega_{LC}^2} \tag{3}$$

F is known as the filling factor and gives the number of rings inside the metamaterial unit cell. For the resonators of square and hexagonal geometries it can be shown that the filling factor is given, respectively, by,

$$F_s = \frac{b^2 t - (b-2h)^2 t}{a_x a_y a_z} \tag{4}$$

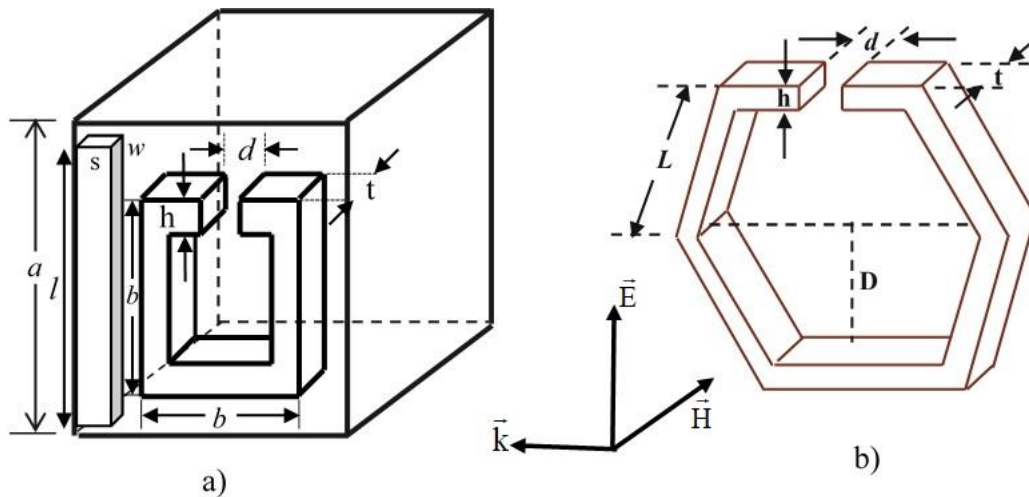$$F_h = \frac{9\pi t L^2}{2\sqrt{3} a_x a_y a_z} \tag{5}$$

Fig 2. a) Metamaterial cell made with rectangular cross section thin wires and open split-ring resonators SRR, *a* is the cell length, s, *w* and *l* are the width, thickness and length of the wire, respectively. SRR of square geometry: *b* is the edge length of the SRR, *d* it is the split-ring gap, h is the inner width, and t is the thickness. The maximum coupling of the fields **E** and **H** are obtained with the polarization shown at the same figure. b) SRR of hexagonal geometry: *L* is the edge length, *d* is the split-ring gap, h is the inner width, t is the thickness and D =$L/(2\tan(30^0))$ is the apothem.

In Eq. (3), ω_{LC} plays as the resonance frequency of the resonators and it has been calculated for the resonators as [35],

$$\omega_{Ls} = \frac{c}{b}\sqrt{\frac{d}{kh}} \quad \text{and} \quad \omega_{Lh} = c\sqrt{\frac{d}{3kLDh}} \tag{6}$$

where k is the dielectric constant of the surrounding environment and c the speed of light in the free space.

The effective electrical permittivity of the wire with rectangular cross-sectional is determined by using the Drude-Lorentz model and is given by [36],

$$\varepsilon_{eff} = 1 - \frac{\omega_p^2}{\omega^2 + \Gamma^2} \tag{7}$$

where the plasma frequency, ω_p, and the damping parameter, Γ, were obtained as,

$$\omega_p^2 = \frac{c^2}{a^2\left[\left[ln\left(\frac{a}{w}\right)\right]^2 + \left[ln\left(\frac{a}{s}\right)\right]^2\right]^{\frac{1}{2}}} \tag{8}$$

$$\Gamma = \frac{\varepsilon_0 a^2 \omega_p^2 \left[\left[ln\left(\frac{a}{w}\right)\right]^2 + \left[ln\left(\frac{a}{s}\right)\right]^2\right]^{\frac{1}{2}}}{2sw\sigma\left[ln\left(\frac{a}{w}\right) + ln\left(\frac{a}{s}\right)\right]} \tag{9}$$

The transmittance given by Eq. (2) is totally determined by the geometric parameters of the metamaterial.

III.   RESULTS AND DISCUSSION

To analyze the predictions obtained by the previous models, comparative graphs of the permeability and permittivity, Eqs. (3) and (7), and the refractive index and the transmittance given by Eq. (2) for the metamaterial cells in Fig. 2a and b, are shown in Fig. 3. The dimension for the cells is $a = 1.00$ cm, and the parameter specifications of the wire and the resonators are: $w = 1.50$ mm, s = 2.10 mm, $b = 0.70$ cm, $d = 0.890$ µm, h = 1.50 mm, $k = 3.05$, $L = 3$ mm and σ = 5,813 ×10^7 Ω^{-1}m^{-1} (copper conductivity). The reader should be aware that for the respective value of a, there are specifics values for ω_p, ω_{Ls}, ω_{Lh}, and the filling factors F_s and F_h.

Some important remarks are necessary. The metamaterial cell in our proposed model resonates at certain specific frequencies for a value of the dielectric constant k (see Eq. (6)), and the behavior of the metamaterials as left-handed materials (LHM), i.e., with a negative refractive index (see Eqs. (3) and (7)), is exhibited in a frequency narrow band in both geometries. It shows the possibility of building antennas or sensors in the GHz range based on these unitary cells of metamaterials. In particular, recent studies on plasmonic resonances in sub-terahertz fishnet metamaterial and antenna applications based on hexagonal resonators have been reported [37] - [39]. Designs with rectangular or hexagonal resonators can facilitate the experimental realization of these and other potential applications. Nevertheless, widening the frequency interval for this is not a simple task and experimental work must be done to compare with the theoretical predictions. Regarding applications such as absorbers or waveguides, analysis of transmittance properties is important. This has to do not only with the coupling between the incident electromagnetic waves and the unit cell of the metamaterial, but also with the coupling between cells. A recent work, for example, has been reported where the effect on the absorption performance of the multi-mode resonant coupling between unit cells of a metamaterial is

investigated [40]. As shown in Table 1, the filling factors increase as the resonator thickness is increased and, because the filling factor is greater for the hexagonal resonator than it is for the square resonator, the hexagonal resonator shows a better transmittance compared to the latter. For both geometries it is noticeable that the transmittance increases approximately linearly and then drops off abruptly. The range of linearity is greater the smaller the thickness of the resonator. From the results, it seems to be evident that, due to the non-homogeneity of the structure, the transmittance depends on the geometry and parameters of the unit cell and the SRRs, however, an analysis must be done about the dependence of the transmittance on the length and cell number of the structure. On the other hand, it is known that transmission losses are practically impossible to avoid in a LHM, so that, a theoretical estimation of losses should be an important problem to face. Finally, a particular polarization of the incident wave on the cell must be considered to define the behavior of the metamaterial as an LHM, and in this case, the magnetic field should be along the axis of the SSR and the electric field should be oriented along the wires. However, the non-homogeneity of a structure could change the polarization of the electric field, which must be considered in absorption calculations, for example. Therefore, more work should be done about the influence of the anisotropy of these cells on the propagation properties of electromagnetic waves in structures built with them. On the other hand, it can be seen in Fig. 3 that the hexagonal geometry resonator has a slightly wider frequency range in the order of GHz where the transmittance increases considerably compared to the square geometry resonator.
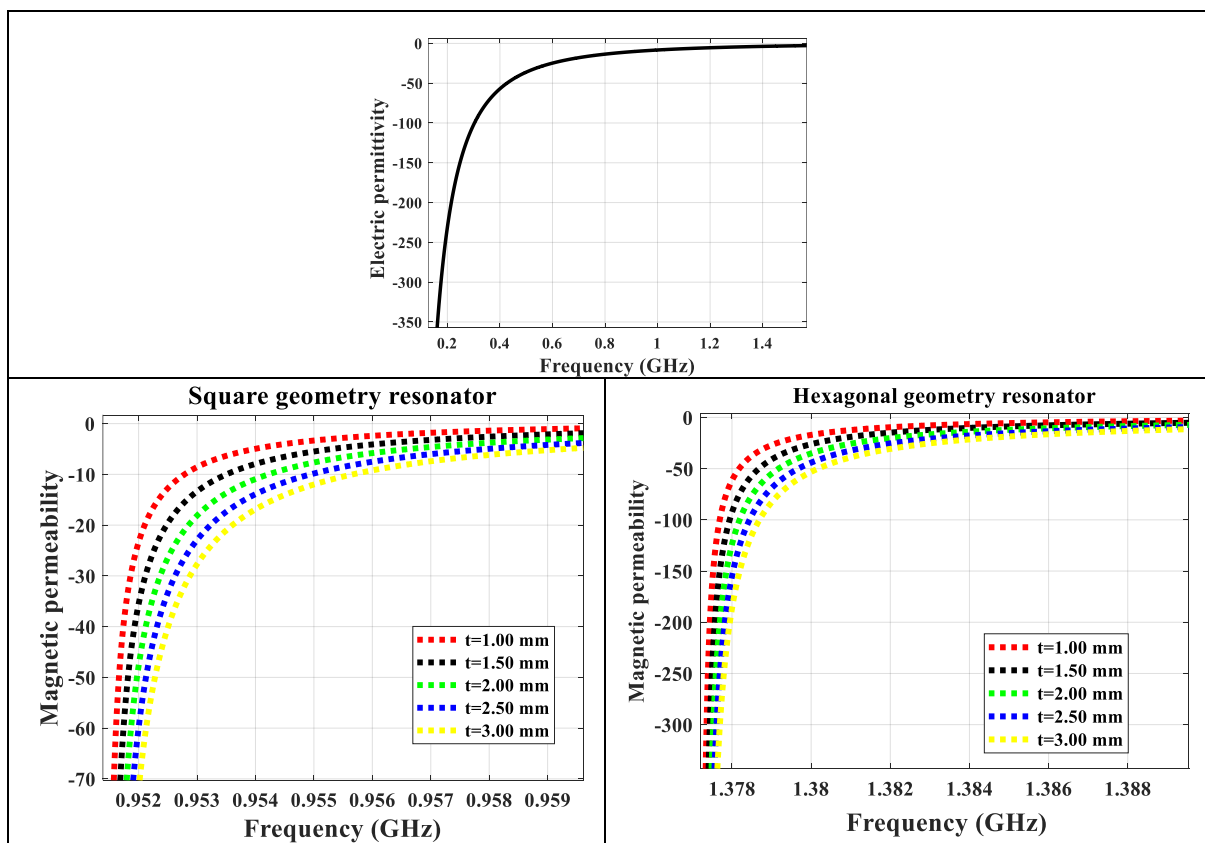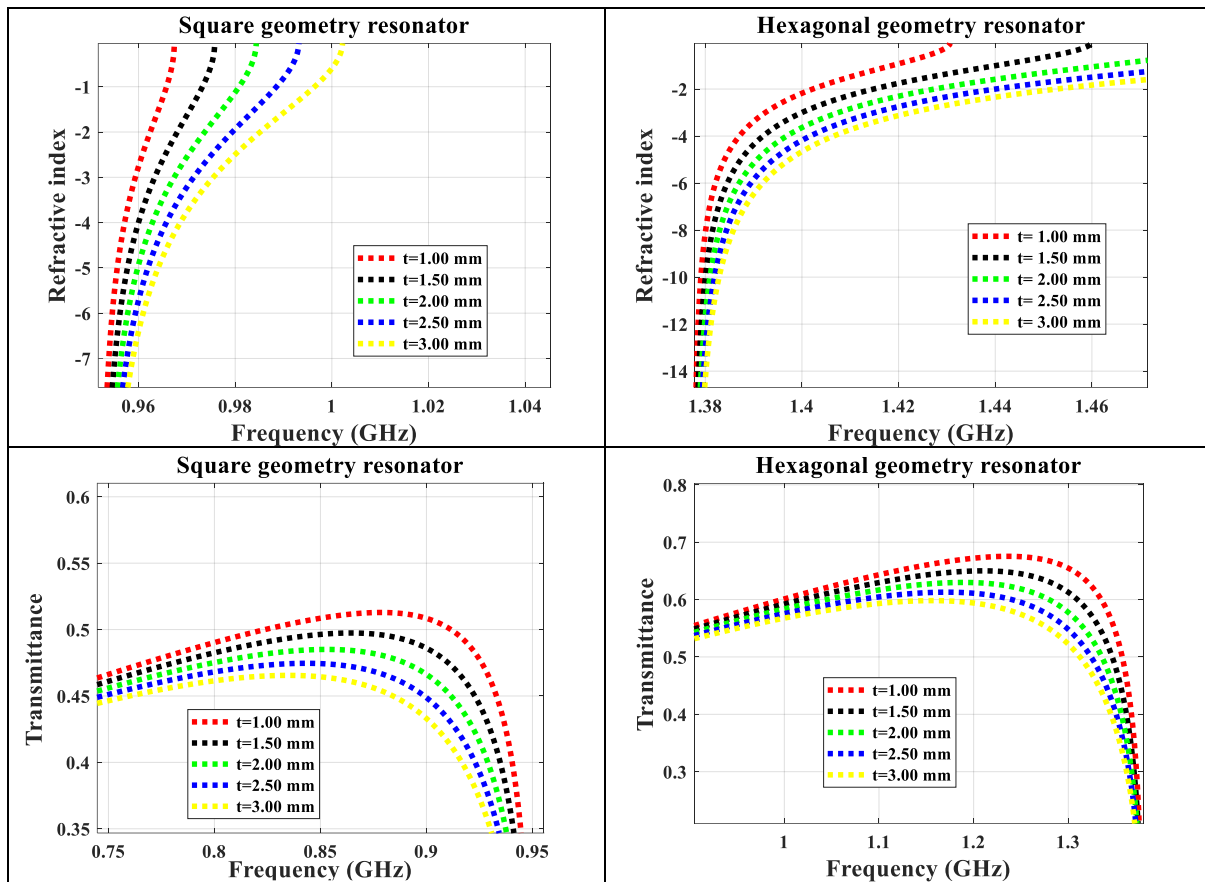
Fig 3. The graphs show the behavior of electric permittivity, magnetic permeability, refractive index and dependence of transmittance as a function of the frequency for different thicknesses t of the resonators shown in Fig 2.

TABLE I. FILLING FACTOR DEPENDING ON THE THICKNESS OF THE RESONATORS

Thickness of the resonator t(mm)	Filling factor for the resonators of square geometry F_s	Filling factor for the resonators of hexagonal geometry F_h
1.0	0.0330	0.0735
1.5	0.0495	0.1102
2.0	0.0660	0.1469
2.5	0.0825	0.1836
3.0	0.0990	0.2204

IV. CONCLUSIONS

A theoretical model has been presented for the transmittance in left-handed metamaterials of different geometries, rectangular and hexagonal. The behavior of the transmittance of the metamaterials depending on the thickness of the resonator has been analyzed. It was found that the hexagonal resonator shows a better transmittance compared to the square resonator. The metamaterial of the proposed theoretical model behaves as a left-handed metamaterial in the frequency range considered where the refractive index takes negative values. The proposed theoretical model can be implemented and applied

in antennas or sensors in the GHz range where the thickness of the metamaterial is the influencing factor for device resonance.

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