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LINEAR AND NONLINEAR SEMIDEFINITE PROGRAMMING

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ABSTRACT. This paper provides a short introduction to optimization problems with semidefinite constraints. Basic duality and optimality conditions are presented. For linear semidefinite programming some advances by dealing with degeneracy and the semidefinite facial reduction are discussed. Two relatively recent areas of application are presented. Finally a short overview of relevant literature on algorithmic approaches for efficiently solving linear and nonlinear semidefinite programming is provided.

Keywords: Semidefinite programming, nonlinear semidefinite programming, Euclidean completion matrices.

1 INTRODUCTION

Semidefinite programming (SDP) is one of the most vigorous and fruitful research topics in optimization the last two decades. The intense activity on this area has involved researchers with quite different mathematical background reaching from nonlinear programming to semialgebraic geometry. This tremendous success of the semidefinite programming model can be explained by many factors. First, the existence of polynomial algorithms with efficient implementations that made the SDP model tractable in many situations. Second, the endless list of quite different and important fields of applications, where SDP has proved to be a useful tool. Third, the beauty and depth of the underlying theory, that links in a natural way different and usually unrelated areas of mathematics.

There are many and excellent survey papers [138, 137, 133, 58, 38, 92, 95, 87] and books [101, 28, 37, 18, 16, 29, 86, 19] covering the Semidefinite Programming model with algorithms and special applications. The previous list of references is by no means complete, but only a short overview on a incresing and large set of items. A special mention in the literature on Semidefinite Programming deserves the *Handbook of Semidefinite Programming* [141] edited by H. Wolkowicz, R. Saigal and L. Vandenberghe in 2000, that covered the principal results on

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the area during the 1990's. After the publication of the mentioned Handbook the research activity in Semidefinite Programming continued growing and new areas of development were added. In particular the interaction with algebraic geometry and the exploration of the close relationship between semidefinite matrices and polynomial optimization gave rise to important new results and to an even higher level of research activity. As recent as 2012 it appeared a new Handbook on Semidefinite, Conic and Polynomial Optimization edited by M.F. Anjos and J.B. Lasserre [9]. This new Handbook provides in 30 chapters a complete update of the research activity on the area in the last decade.

Our main intention in this short review is to motivate researchers to become involved in this amazing area of research. We focus on readers with a basic background in continuous Optimization, but without a previous knowledge in Semidefinite Programming. Our goal is to provide a simple access to some of the basic concepts and results in the area and to illustrate the potential of this model by presenting some selected applications. A short overview on the theoretical and algorithmic results in the case of nonlinear semidefinite programming is also given. We suggest to readers interested in a more detailed exposition of the semidefinite model to revise the above mentioned Handbooks and the references therein.

The paper is divided into two sections. The first one is devoted to the (linear) Semidefinite Programming and the second one to the case of nonlinear Semidefinite Programming.

2 THE LINEAR SEMIDEFINITE PROGRAMMING

The linear semidefinite programming can be intended as linear programming over the cone of positive semidefinite matrices. In order to formulate the problem in details let us fix some notations. In the sequel we denote with $\mathbb{S}^m$ the linear space of $m \times m$ real symmetric matrices equipped with the inner product

$$\langle A, B\rangle := Tr(AB) = \sum_{i,j=1}^{m} A_{ij} B_{ij}$$

where $A = (A_{ij}),\ B = (B_{ij}) \in \mathbb{S}^m$.

On this linear space we consider the positive semidefinite order, i.e. $A \succeq B$ iff $A - B$ is a positive semidefinite matrix. The order relations $\succ$ and $\prec$, $\preceq$ are defined similarly.

The primal semidefinite programming problem is then defined as follows:

$$(SDP-P) \qquad \begin{array}{ll} \min & \langle C, X\rangle \\ & \mathcal{A}X = b \\ & X \succeq 0 \end{array}$$

where $C \in \mathbb{S}^m$, $b \in \mathbb{R}^n$ are given data, and $\mathcal{A} : \mathbb{S}^m \to \mathbb{R}^n$ is a linear operator. In general $\mathcal{A}$ is written as

$$\mathcal{A}X = \begin{pmatrix} \langle A_1, X\rangle \\ \vdots \\ \langle A_n, X\rangle \end{pmatrix}$$

where $A_1, \ldots, A_n \in \mathbb{S}^m$ are also data of the problem.

Let us denote the set of positive semidefinite matrices as follows

$$\mathbb{S}^m_+ = \{A \in \mathbb{S}^m \mid A \succeq 0\}$$

The set $\mathbb{S}^m_+$ is a full-dimensional, convex closed pointed cone, such that (SDP-P) is a convex problem. Its boundary is the set of semidefinite matrices having at least a zero eigenvalues and its interior is the cone of positive definite matrices. The cone $\mathbb{S}^m_+$ is also self-dual, i.e. its polar cone

$$(\mathbb{S}^m_+)^* = \{B \in \mathbb{S}^m \mid \langle A, B\rangle \geq 0, \ \forall A \in \mathbb{S}^m_+\}$$

coincides with $\mathbb{S}^m_+$. This property allows to calculate the Lagrange dual of the (SDP). Let the optimal value of (SDP-P) be denoted as follows.

$$\begin{aligned} p^* &= \inf_{X \in \mathbb{S}^m_+} \sup_{y \in \mathbb{R}^n} \left\{ \langle C, X\rangle - \sum_{i=1}^n y_i(\langle A_i, X\rangle - b_i) \right\} \\ &= \inf_{X \in \mathbb{S}^m_+} \sup_{y \in \mathbb{R}^n} \left\{ \langle C - \sum_{i=1}^n y_i A_i, X\rangle + b^T y) \right\} \end{aligned}$$

Interchanging "sup" and "inf" we obtain the dual with corresponding optimal value d^*.

$$d^* = \sup_{y \in \mathbb{R}^n} \inf_{X \in \mathbb{S}^m_+} \left\{ \langle C, X\rangle - \sum_{i=1}^n y_i(\langle A_i, X\rangle - b_i) \right\} \tag{1}$$

Taking into account the selfduality of $\mathbb{S}^m_+$ the following expression is then obtained:

$$\inf_{X \in \mathbb{S}^m_+} \{\langle C - \sum_{i=1}^n y_i A_i, X\rangle + b^T y)\} = \begin{cases} b^T y, & C - \sum_{i=1}^n y_i A_i \in \mathbb{S}^m_+ \\ -\infty, & \text{else} \end{cases}$$

Consequently the dual problem to (SDP-P) can be written as:

$$(SDP - D) \qquad \begin{array}{ll} \max & b^T y \\ & \mathcal{A}^* y + Z = C \\ & Z \succeq 0 \end{array}$$

where $\mathcal{A}^* : \mathbb{R}^n \to \mathbb{S}^m$ denotes the adjoint operator of $\mathcal{A}$ defined as

$$\mathcal{A}^* y = \sum_{i=1}^n y_i A_i$$

This pair of primal and dual problems has the same structure of primal-dual problems in linear programming with the standard form. The only difference is that the cone defining the inequalities is now $\mathbb{S}^m_+$ instead of the cone of vectors with nonnegative components. This SDP model contains as special cases many other optimization problems as linear programming, convex quadratic programming, second order cone programming, etc. A rich list of semidefinite representable sets and problems can be found, for instance, in [18].

Many of the theoretical and algorithmic results from LP can be carried over to the SDP case. A first trivial one is the weak duality, since from the feasibility of X for (SDP-P) and (y, Z) for (SDP-D) it follows that

$$\langle C, X\rangle - b^T y = \langle \mathcal{A}^* y, X\rangle - b^T y + \langle Z, X\rangle = y^T \mathcal{A} X - b^T y + \langle Z, X\rangle = \langle Z, X\rangle \geq 0$$

where the last inequality is again a consequence of the self duality of $\mathbb{S}^m_+$.

From the above weak duality results it follows straightforwardly that the Karush-Kuhn-Tucker system (2) provides sufficient optimality conditions for the pair (SDP-P) and (SDP-D).

$$\begin{array}{rcl} \mathcal{A}X &=& b \\ \mathcal{A}^* y + Z &=& C \\ \langle Z, X\rangle &=& 0 \\ Z, X &\succeq& 0 \end{array} \tag{2}$$

Since the cone $\mathbb{S}^m_+$ is nonpolyhedral, the SDP is a convex but nonlinear optimization problem. In consequence not every nice duality properties of LP can be extended to SDP. For instance, there are solvable primal and dual pairs having a strictly positive duality gap. There are also primal dual problems with zero duality gap that are not both solvable, see for instance [141]. Such examples are impossible in LP and imply also that the above conditions (2) are not necessary for optimality.

The usual way to state strong duality results in the SDP setting is to require the Slater's Constraint Qualification (Slater-CQ). This can be intended for SDP problems as *strict feasibility*. For (SDP-P) it means the existence of a positive definite feasible point $X \succ 0$. Analogously for (SDP-D) it means the existence of a feasible solution (y, Z) with $Z \succ 0$. Under the strict feasibility assumptions the following strong duality results are known.

Theorem 2.1. *Let consider the dual problems (SDP-P) and (SDP-D) with optimal values*

$$\begin{array}{rcl} p^* &=& \inf\{\langle C, X\rangle \mid \mathcal{A}X = b,\ X \in \mathbb{S}^m_+\} \\ d^* &=& \sup\{b^T x \mid \mathcal{A}^* y + Z = C,\ Z \in \mathbb{S}^m_+,\ y \in \mathbb{R}^n\} \end{array}$$

1. *If the problem (SDP-P) is strictly feasible and p^* is finite, then $p^* = d^*$ and the dual optimal value d^* is attained.*

2. *If the problem (SDP-D) is strictly feasible and d^* is finite, then $p^* = d^*$ and the primal optimal value p^* is attained.*

3. *If both problems (SPD-P) and (SDP-D) are strictly feasible, then $p^* = d^*$ and both optimal values are attained.*

In particular the last strong duality result implies that under the Slater-CQ the above conditions (2) actually characterize primal-dual optimal points. The complementarity condition $\langle Z, X\rangle = 0$ in (2) can be equivalently replaced, see e.g. [37], by the usual matrix multiplication, such that the optimality conditions take the form:

$$\begin{array}{rcl} \mathcal{A}X & = & b \\ \mathcal{A}^* y + Z & = & C \\ ZX & = & 0 \\ Z, X & \succeq & 0 \end{array} \tag{3}$$

The notion of strict complementarity and degeneracy can be extended to the SDP setting. For instance, strict complementarity means $X + Z \succ 0$, see e.g. [7, 22]. However, not all the properties related to these concepts in LP can be carried over to SDP. In particular the classical theorem of Goldman and Tucker [54] on the existence of primal-dual strict complementarity solutions does not hold for SDP, see for instance [37] also for a discussion on maximal complementary solutions. In fact, the study of nondegeneracy in SDP requires a deeper analysis of the geometry of the semidefinite cone [114].

The Slater-CQ is a generic condition [7, 44]. It is also a crucial condition for the stability of most of the efficient solutions methods for SDP. The Slater-CQ holds also in many applications, for instance for the basic SDP relaxations of the max-cut problem. More details on SDP relaxations of the max-cut problem and other combinatorial problems can be found in [53, 140, 87, 40, 39]

There are however many SDP instances arising for instance also by relaxations of hard combinatorial problems where the Slater-CQ is not fulfilled, see for example [146, 82, 82, 83, 32]. A prevailing approach to get equivalent instances satisfying the Slater-CQ is the skew-symmetric embedding, see [41, 37]. This technique uses homogenization of the problem and increases the number of variables.

Another general approach to deal with the lack of strict feasibility bases on the so called facial reduction and extended duals [24, 25, 26, 122, 123, 135, 115]. Let us discuss this second approach, since it uses geometric properties of the semidefinite cone and provides in general smaller regularized problems.

A cone $F \subseteq \mathbb{S}^m$ is a face of $\mathbb{S}^m_+$, denoted by $F \trianglelefteq \mathbb{S}^m_+$ (and $F \triangleleft \mathbb{S}^m_+$ in case $F \neq \mathbb{S}^m_+$), if

$$A, B \in \mathbb{S}^m_+, (A + B) \in F \Rightarrow A, B \in F.$$

Obviously $\{0\} \triangleleft \mathbb{S}^m_+$. If $\{0\} \neq F \triangleleft \mathbb{S}^m_+$, then F is called a proper face of $\mathbb{S}^m_+$. If $F \trianglelefteq \mathbb{S}^m_+$ the conjugate or complementary face of F, denoted by F^c, is defined as $F^c = F^\perp \cap \mathbb{S}^m_+$. Moreover, if A is in the relative interior of a face $F \trianglelefteq \mathbb{S}^m_+$, then $F^c = \{A\}^\perp \cap \mathbb{S}^m_+$. Detailed results on the facial structure of $\mathbb{S}^m_+$ can be found, for instance in [113]. The following characterization of the faces of the semidefinite cone is known.

Theorem 2.2. *A cone $F \neq \{0\}$ is a face of $\mathbb{S}^m_+$ if and only if*

$$F = \{A \in \mathbb{S}^m \mid A = PWP^T, W \in \mathbb{S}^k_+\}$$

for some $k \in \{1, \dots, m\}$*, and* $P \in \mathbb{R}^{m\times k}$ *with rank* k*.*

Let us consider the dual set of feasible slack variables $\mathcal{F}_D = \{Z \in \mathbb{S}^m_+ \mid Z = C - \mathcal{A}^* y\}$. The corresponding minimal face is defined as

$$f_D = face(\mathcal{F}_D) = \bigcap\{H \trianglelefteq \mathbb{S}^m_+ \mid \mathcal{F}_D \subset H\}$$

The face $face(\mathcal{F}_D)$ is the smallest face of $\mathbb{S}^m_+$ containing $\mathcal{F}_D$. Using the order $\preceq_{f_D}$ derived from the cone f_D, i.e. $A \preceq_{f_D} B \Leftrightarrow B - A \in f_D$, a regularized dual problem can be defined.

$$(SDP_{reg} - D) \qquad d^*_{reg} = \sup\{b^T y \mid \mathcal{A}^* y \preceq_{f_D} C\} \tag{4}$$

The above regularized problem is equivalent to $(SDP - D)$, see [24, 25], in the sense that the feasible set remains the same

$$\mathcal{A}^* y \preceq_{f_D} C \Leftrightarrow \mathcal{A}^* y \preceq C$$

The Lagrangian dual of $(SDP_{reg} - D)$ can be easily calculated as

$$(SDP_{reg} - P) \qquad p^*_{reg} = \inf\{\langle C, X\rangle \mid \mathcal{A}X = b, X \succeq_{f_D^*} 0\} \tag{5}$$

where the dual cone is given by

$$f_D^* = \{Y \in \mathbb{S}^m \mid \langle Y, X\rangle \geq 0, \ \forall Y \succeq_{f_D} 0\}$$

The following theorem [24] provides then a strong stability result for the regularized dual problem.

Theorem 2.3. *If the original problem optimal value* d^* *in (1) is finite, then* $d^* = d^*_{reg} = p^*_{reg}$ *and the optimal value* p^*_{reg} *is attained.*

Recently a backward stable preprocessing algorithm has been developed that bases on the above semidefnite facial reduction and can provide equivalent regular reformulations to problems without the Slater-CQ, e.g. [34]. In particular the following auxiliary problem is considered

$$\begin{aligned} \min_{\delta, D} \quad & \delta \\ & \left\| \begin{bmatrix} \mathcal{A}D \\ \langle C, X\rangle \end{bmatrix} \right\|_2 \leq \delta \\ & \left\langle \frac{1}{\sqrt{n}} I, D \right\rangle = 1 \\ & D \succeq 0 \end{aligned} \tag{6}$$

This auxiliary problem can be written as a Semidefinite Programming, where in particular the first constraint is a second order cone constraint that can be also written as a semidefinite one, see for instance [18]. The problem (6) and its dual satisfy the Slater-CQ [34]. Consequently, using interior point methods an optimal solution (δ^*, D^*) in the relative interior of the optimal solution set can be obtained. In the most interesting case we get a description of the minimal face as

$$f_P = Q\,\mathbb{S}^{\bar{m}}_+\,Q^T$$

for some matrix $Q \in \mathbb{R}^{m\times\bar{m}}$ with $Q^T Q = I_{\bar{m}}$ and $\bar{m} < m$. A regularized reduction is then obtained, since the original semidefinite program (SDP-D) can be equivalently formulated as reduced problem satisfaying the Slater-CQ, see [34].

Theorem 2.4. *Let the feasible set $\mathcal{F}_D$ be nonempty and (δ^*, D^*) be a solution of the auxiliary problem (6). If $\delta^* = 0$ and*

$$D^* = [P\ Q]\begin{bmatrix} \Delta_+ & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} P^T \\ Q^T \end{bmatrix},$$

where $[P\ Q]$ is orthogonal, $Q \in \mathbb{R}^{m\times\bar{m}}$ and $\Delta_+ \succ 0$, then (SDP-D) is equivalent to the following problem

$$\begin{array}{ll} \sup & b^T y \\ & P^T(\mathcal{A}^* y - C)P = 0 \\ & Q^T(\mathcal{A}^* y - C)P = 0 \\ & Q^T(\mathcal{A}^* y - C)Q \preceq 0 \end{array}$$

The above remarkable result shows a way to identify hidden linear equality constraints into degenerated SDP problems. This procedure is well established in linear programming as part of general preprocessing steps, but it is not usual in nonlinear problems (as the SDP model). The facial reduction procedure to obtain regularized and reduced problems have been sucessfully used in different application in order to take advantage of degeneracy, see for instance [82, 13, 15, 14, 32].

In the seminal work [101] it was shown that the function $\log(\det(X))$ is a self-concordant barrier function. As a consequence SDP instances can be solved in polynomial time using a sequence of barrier subproblems. In [6] another fundamental approach based on the potential function methods was presented. Strong numerical result were also early reported for min-max eigenvalue problems [60, 63].

There is a long list of quite different algorithmic approaches for solving the SDP problem, see for instance [6, 18, 101, 103, 66, 72, 99, 59, 75, 95, 96, 31, 30, 81, 55, 117, 147, 116], among many others. The previous list is by far incomplete and we do not intend to describe here all the diverse ideas to deal with the efficient solution of semidefinite programming. Instead we point out to the excellent surveys in the algorithmic sections of the already mentioned Handbooks [141, 9]. There are many software tools available for solving general SDP problems, for instance SeDumi [130], SDPNAL [147], SDPT3 [136, 134], SDPA [145, 144] and PENNON [71, 75, 77], among others. A useful tool for modelling with SDP and for using the existing SDP-software is the program YALMIP [93]. For a detailed survey about software tools for SDP see [94]. There are also some available implementations of solvers for particular structured SDP problems, we just mention in this direction GloptiPoly [62, 61] for the so called generalized problem of moments [84, 85, 86] and SOSTOOL [121, 120] for solving sum of squares optimization programs [110, 111, 19].

We present in the rest of this section two selected areas of application of the semidefinite programming model.

2.1 Polynomial Lyapunov functions

One example of an important mathematical problem is the search of general methods to prove that a real n-variable polynomial $p(\mathbf{x}) \in \mathbb{R}[\mathbf{x}]$ is nonnegative, i.e.

$$p(\mathbf{x}) \geq 0, \ \forall \mathbf{x} \in \mathbb{R}^n.$$

This problem is connected with the famous 17^{th} Hilbert problem. It is NP-hard and has not a general computable solution. On the other hand, it is known that a single polynomial p is a nonnegative polynomial if and only if it can be written as a sum of squares of rational functions, and so, clearing denominators $hp = f$ for some sum of squares polynomials h, f ([11],[42]).

Hence, the general question can be transformed into a more restricted but more accesible question: When a given polynomial can be decomposed in a sum of squares of other polynomials? This last question can be answered in a computable way using SDP and the idea was first appeared in [27]. See also [118] for an extense survey of this and other methods to tackle the problem.

If $n = 1$ the ring $\mathbb{R}[x]$ of real polynomials of a single variable has the fundamental property that every nonnegative polynomial $p \in \mathbb{R}[x]$ is a sum of squares of some other polynomials. But for $n > 1$ not every nonnegative polinomial can be decomposed in a sum of square, but when it does the question is strongly related with a SDP problem. The following definitions and results can be seen in [86].

Let $\mathbb{R}[\mathbf{x}]$ denote the ring of real polynomials in the variables $\mathbf{x} = (x_1, \ldots, x_n)$. A polynomial $p \in \mathbb{R}[\mathbf{x}]$ is a sum of squares (in short SOS) if p can be written as:

$$p(\mathbf{x}) = \sum_{j \in J} p_j(\mathbf{x})^2, \quad \forall \mathbf{x} \in \mathbb{R}^n,$$

for some finite family of polynomials $\left\{p_j, \ j \in J\right\} \subset \mathbb{R}[\mathbf{x}]$. Notice that necessarily the degree of p must be even, and also, the degree of each p_j is bounded by half of that of p.

For a multi-index $\alpha \in \mathbb{N}^n$, let

$$|\alpha| = \sum_{i=1}^{n} \alpha_i,$$

and define the vector:

$$\mathbf{v}_d(\mathbf{x}) = (\mathbf{x}^\alpha)_{|\alpha| \leq d} = \left(1, x_1, \ldots, x_n, x_1^2, x_1x_2, \ldots, x_{n-1}x_n, x_n^2, \ldots, x_1^d, \ldots, x_n^d\right),$$

of all the monomials $\mathbf{x}^\alpha$ of degree less or equal to d which has dimension $s(d) = \binom{n+d}{d}$. Those monomials form the canonical basis of the vector space $\mathbb{R}[\mathbf{x}]^d$ of n-variables polynomials of degree at most d.

Proposition 2.5. *A polynomial* $p \in \mathbb{R}[\mathbf{x}]^{2d}$ *has a sum of square (SOS) decomposition if and only if there exists a real symmetric and positive semidefinite matrix* $Q \in \mathbb{R}^{s(d)\times s(d)}$ *such that*

$$p(\mathbf{x}) = \mathbf{v}_d(\mathbf{x})^{\mathrm{T}} Q \mathbf{v}_d(\mathbf{x}), \ \forall \mathbf{x} \in \mathbb{R}^n. \tag{7}$$

Therefore, given a SOS polynomial $g \in \mathbb{R}[\mathbf{x}]^{2d}$, the identity $g(\mathbf{x}) = \mathbf{v}_d(\mathbf{x})^{\mathrm{T}} Q \mathbf{v}_d(\mathbf{x})$ provides linear equations that the coefficients of the matrix Q must satisfy. Hence writing:

$$\mathbf{v}_d(\mathbf{x})^{\mathrm{T}} \mathbf{v}_d(\mathbf{x}) = \sum_{\alpha \in \mathbb{N}^n} B_\alpha \mathbf{x}^\alpha,$$

for appropriate $s(d) \times s(d)$ real symmetric matrix B_α, checking whether the polynomial $g(\mathbf{x}) = \sum_\alpha g_\alpha \mathbf{x}^\alpha$ is SOS reduces to solving the SDP (feasibility) problem:

$$\begin{gathered} \text{Find } \ Q \in \mathbb{R}^{s(d)\times s(d)}, \ \text{ such that:} \\ Q^{\mathrm{T}} = Q, \quad Q \succeq 0, \quad \langle Q, B_\alpha \rangle = g_\alpha, \quad \forall \alpha \in \mathbb{N}^n. \end{gathered} \tag{8}$$

a tractable convex optimization problem for which efficient software packages are available.

There are amazing ideas related to the above connection between positive polynomials and SDP. Using the so called moment problem nice hierarchies of tractable problems have been proposed to deal with, for instance, global optimization, see [86] and the references therein. Extending the idea to SOS-convexity [5, 4] new tractable relaxations have been proposed to problems in control. A last example is the new interest in classical Lyapunov's method for determining the stability of dynamical systems, specially by using SDP for finding polynomic Lyapunov's functions in polynomial differential equations.

In 1892 Lyapunov introduced his famous stability theory for nonlinear and linear systems. To be specific but no very technical, we recall that a dynamical system described by a homogeneous system of equations:

$$\dot{x} = f(x), \quad \text{where } \dot{x} = \frac{dx}{dt}, \quad f : \mathbb{R}^n \to \mathbb{R}^n, \quad f(0_n) = 0_n,$$

has a *stable* equilibrium point at $x = 0_n$ if any solution $x(t, x_0)$ corresponding to an initial condition x_0 in some neighborhood of 0_n, remains close to 0_n for all $t > 0$. In the particular case when $x(t, x_0)$ converges to 0_n if $t \to +\infty$, the equilibrium is called *asymptotically stable.*

It is well known that stability can be certified if there exists a Lyapunov's function $V = V(x)$ such that,

$$\begin{gathered} V : \mathbb{R}^n \to \mathbb{R}, \\ V(x) > 0, \ \forall x \neq 0_n, \quad V(0) = 0, \\ \frac{dV(x(t))}{dt} = \nabla V(x(t))\dot{x}(t) = \nabla V(x(t)) f(x(t)) \leq 0, \ \forall t > 0. \end{gathered}$$

and also asymptotical stability if furthermore the last inequality is strict.

For a long time a computable general method to find Lyapunov's function were available only for the linear case:

$$\dot{x} = Ax,$$

for example, in the form of a quadratic function:

$$V(x) = x^{\mathrm{T}} P x,$$

satisfying

$$\frac{d}{dt} V(x) = x^{\mathrm{T}} \left[A^{\mathrm{T}} P + PA \right] x = -x^{\mathrm{T}} Q x,$$

where P, Q are symmetric, positive definite $n \times n$ matrices. The matrix algebraic equation:

$$A^{\mathrm{T}} P + PA = -Q$$

is known as the Lyapunov algebraic equation. More about this important equation and its role in system stability and control can be found in [50].

There is no general procedure for finding the Lyapunov functions for nonlinear systems. In the last few decades however, advances in the theory and practice of convex optimization and in particular in semi-definite programming (SDP) have rejuvenated Lyapunov theory. The approach has been used to parameterize a class of Lyapunov functions with restricted complexity (e.g. quadratics, pointwise maximum of quadratics, polynomials, etc...) and then to pose the search of a Lyapunov function as a convex feasibility problem (see, for example [110], [109]).

Expanding on the concept of sum of squares decomposition of polynomials, this technique allows the formulation of semi-definite programs that search for polynomial Lyapunov functions for polynomial dynamical systems [3]. Sum of squares Lyapunov functions along with many others SDP based techniques, have also been applied to systems that undergo switching, e.g. ([124], [108], [119], [112]).

Perhaps so far it is not clear for the reader how the SDP problems arise in the context of dynamical systems stability and in the Lyapunov's function finding. But searching for a polynomic Lyapunov function for a polynomial dynamical system is reduced to find the coefficients of a n-variable polynomial $p(x_1, \ldots, x_n)$ of some degree d such that the following *polynomial* inequalities hold:

$$\begin{aligned} p(x) &> 0, \ \forall x \neq 0, \\ \frac{d}{dt} p[x(t)] &= \nabla p(x) f(x) < 0. \end{aligned}$$

In this case we have *two* polynomial inequalities, but the solution of the problem (7) which find a matrix Q representing p as a quadratic form of $\mathbf{v}_d(\mathbf{x})$ is not unique. In fact, it can be shown that the whole solution matrix set of the equations given by $p(\mathbf{x}) = \mathbf{v}_d(\mathbf{x})^{\mathrm{T}} Q \mathbf{v}_d(\mathbf{x})$ is a linear space. When this linear space intersects the positive semi-definite matrix cone then $p(\mathbf{x})$ is SOS.

In practice the general method searchs for a representation Q of $p(\mathbf{x}) = \mathbf{v}_d(\mathbf{x})^{\mathrm{T}} Q \mathbf{v}_d(\mathbf{x})$ and another quadratic representation R of

$$\frac{d}{dt} p\,[x(t)] = \nabla p(x) Q f(x) = \mathbf{v}_d(\mathbf{x})^{\mathrm{T}} R \mathbf{v}_d(\mathbf{x}).$$

Then, a sufficient condition for the dynamical system stability is that the following matrix inequality system holds:

$$\begin{cases} \mathbf{v}_d(\mathbf{x})^{\mathrm{T}} Q \mathbf{v}_d(\mathbf{x}) > 0 \\ \mathbf{v}_d(\mathbf{x})^{\mathrm{T}} R \mathbf{v}_d(\mathbf{x}) \leq 0 \end{cases}, \ \forall \mathbf{x} \in \mathbb{R}^n,$$

or equivalently, if the following $2s(d) \times 2s(d)-$matrix is positive (semi-)definite:

$$\tilde{Q} = \begin{pmatrix} Q & 0 \\ 0 & -R \end{pmatrix}.$$

In this case the feasibility problem (8) can be formulated in terms of the searching of matrix $\tilde{Q}$. In usual control theory language this is called a LMI (linear matrix inequality) problem (see [109]), and there exists efficient software to solve it, for instance SOSTOOL [121, 120].

Extensions of this kind of results to the so called switched and hybrid systems are developed in ([119]). A dynamical system is called switched/hybrid system if it can be written in the following form:

$$\dot{x} = f_i(x), \quad i = 1, \ldots, N, \tag{9}$$

where x is the continuous state, i is a discrete state and $f_i(x)$ is the vector field describing the dynamics of the $i-th$ mode/subsystem. Depending on how the discrete state i evolves, the system (9) is categorized as a switched system, if for each $x \in \mathbb{R}^n$ only one i is possible, or as a hybrid system, if for some $x \in \mathbb{R}^n$ multiple i are possible.

In the case of switched system, the system is in $i-th$ mode at time t if

$$x(t) \in X_i = \left\{ x \in \mathbb{R}^n : g_{ik}(x) \geq 0, \ k = 1, \ldots, m_i^X \right\}. \tag{10}$$

Additionaly, the state space partition $\{X_i\}$ must satisfy $\bigcup_i X_i = \mathbb{R}^n$ and $int(X_i) \cap int(X_j) = \emptyset$, for $i \neq j$. A boundary S_{ij} between X_i and X_j is defined analogously by

$$S_{ij} = \left\{ x \in \mathbb{R}^n : h_{ij0}(x) = 0, \ h_{ijk}(x) \geq 0, \ k = 1, ..., m_{ij}^S \right\}. \tag{11}$$

The stability analysis of switched polynomial system is based again in SOS decomposition, using piecewise polynomial Lyapunov functions. A typical result follows:

Theorem 2.6. *Consider the switched system (9)-(11). Assume there exists polynomials $V_i(x)$, $c_{ij}(x)$ with $V(0) = 0$ if $0 \in X_i$, and sum of squares $a_{ik}(x) \geq 0$ and $b_{ik}(x) \geq 0$, such that*

$$V_i(x) - \sum_{k=1}^{m_i^X} a_{ik}(x) g_{ik}(x) \quad > \quad 0, \ \forall x \neq 0, \ \forall i = 1, \ldots, N,$$

$$\begin{aligned}\frac{\partial V_i}{\partial x} f_i(x) + \sum_{k=1}^{m_i^X} b_{ik}(x) g_{ik}(x) &< 0, \ \forall x \neq 0, \ \forall i = 1, \ldots, N, \\ V_i(x) + c_{ij}(x) h_{ij0}(x) - V_j(x) &= 0, \ \forall i \neq j\end{aligned}$$

then the origin of the state space is asymptotically stable. A Lyapunov function that prove this is the piecewise polynomial function $V(x)$, defined by:

$$V(x) = V_i(x), \ \textit{if}\ x \in X_i.$$

The SOS polynomials a_{ik}, b_{ik} at X_i are computed using constrained feasibility SDP and LMI methods.

2.2 Euclidean Distance Matrices

Let us discuss some SDP relaxations of the Euclidean Distance Matrix Completion problem. A complete survey on the topic is provided in the recent Handbook [9]. We present some of the problems and results in [82, 83] and encourage the readers to look for details in the survey and the references therein.

A matrix $D \in \mathbb{S}^n$ is called an *Euclidean Distance Matrix* (EDM), if there exist vectors $p_1, \ldots, p_n \in \mathbb{R}^r$, such that

$$D_{ij} = \|p_i - p_j\|_2^2, \quad \forall i, j = 1, \ldots, n$$

The smallest dimension r, where the above representation is possible is called the embedding dimension of D, denoted by $embdim(D)$. Let us denote the set of all Euclidean Distance Matrices by $\mathcal{E}^n$.

From the vectors $p_1, \ldots, p_n \in \mathbb{R}^r$ we can define the so called Gramm matrix $Y \in \mathbb{S}^n$ as

$$Y_{ij} = p_i^T p_j, \quad \forall i, j = 1, \ldots, n$$

It holds then the relation

$$D_{i,j} = \|p_i - p_j\|_2^2 = p_i^T p_i + p_j^T p_j - 2p_i^T p_j = Y_{ii} + Y_{jj} - 2Y_{ij}, \quad \forall i, j = 1, \ldots, n \quad (12)$$

Given a Matrix $Y \in \mathbb{S}^n$ the row vector formed with its diagonal is a mapping that shall be denoted by $diag(Y)$. The adjoint operator of this mapping shall be called $Diag(d) = diag^*(d)$ and is obtained as the diagonal matrix with the vector d along the diagonal. Further the row vector with all entries equal to one should be noted by $\mathbf{e}$. The last expression in the above relationship (12) can be intended as a mapping $\mathcal{K} : \mathbb{S}^n \to \mathbb{S}^n$, i.e.

$$\mathcal{K} := diag(Y)\mathbf{e}^T + \mathbf{e}\, diag(Y)^T - 2Y$$

Using the linear map $\mathcal{K}$ the set of Euclidean Distance Matrices can be described as image of the cone of semidefinite constraints, i.e. $\mathcal{K}(\mathbb{S}^n) = \mathcal{E}^n$. There is an explicit representation of the Moore-Penrose generalized inverse of $\mathcal{K}$ as follows:

$$\mathcal{K}^\dagger(D) = -\frac{1}{2}\left[I_n - \frac{1}{n}\mathbf{e}\mathbf{e}^T\right][D - Diag(diag(D))]\left[I_n - \frac{1}{n}\mathbf{e}\mathbf{e}^T\right].$$

The range spaces of $\mathcal{K}$ and $\mathcal{K}^\dagger$ are called the *hollow space* and the *centered space* (denoted as $\mathbb{S}^n_H$ and $\mathbb{S}^n_C$), respectively, and can be described as follows:

$$\mathbb{S}^n_H = \{D \in \mathbb{S}^n \mid diag(D) = 0\}, \quad \mathbb{S}^n_C = \{Y \in \mathbb{S}^n \mid Y\mathbf{e} = 0\}.$$

The following relations are then useful

$$\mathcal{K}(\mathbb{S}^n_C) = \mathbb{S}^n_H, \quad \mathcal{K}^\dagger(\mathbb{S}^n_H) = \mathbb{S}^n_C, \quad \mathcal{K}(\mathbb{S}^n_+ \cap \mathbb{S}^n_C) = \mathcal{E}^n, \quad \mathcal{K}^\dagger(\mathcal{E}^n) = \mathbb{S}^n_+ \cap \mathbb{S}^n_C$$

$$D \in \mathcal{E}^n \Longleftrightarrow \mathcal{K}^\dagger(D) \in \mathbb{S}^n_+, \quad embdim(D) = \operatorname{rank}(\mathcal{K}^\dagger(D))$$

If we restrict $\mathcal{K}$ and $\mathcal{K}^\dagger$ to the subspaces $\mathbb{S}^n_C$ and $\mathbb{S}^n_H$, respectively, then $\mathcal{K}$ is bijection and $\mathcal{K}^\dagger$ its inverse. Moreover, the restriction $\mathcal{K} : \mathbb{S}^n_+ \cap \mathbb{S}^n_C \longrightarrow \mathcal{E}^n$ is also a bijection and $\mathcal{K}^\dagger : \mathcal{E}^n \longrightarrow \mathbb{S}^n_+ \cap \mathbb{S}^n_C$ its inverse. So far, the problem of deciding whether a given Matrix $D \in \mathbb{S}^n$ is an Euclidean distance matrix with embedding dimension not greater than r can be stated as follows:

$$\begin{array}{rl} \text{Find} & Y \in \mathbb{S}^n_+ \cap \mathbb{S}^n_C \\ \text{such that} & \mathcal{K}(Y) = D \\ & \operatorname{rank}(Y) = r \end{array}$$

or equivalently as

$$\begin{array}{rl} \min & 0 \\ \text{subject to} & \mathcal{K}(Y) = D \\ & Y\mathbf{e} = 0 \\ & Y \succeq 0 \\ & \operatorname{rank}(Y) = r \end{array}$$

Deleting the last rank constraint we obtain an instance of the (SDP-P), where the Slater-CQ fails (due to the condition $Y\mathbf{e} = 0$).

Consider now that for a matrix $D \in \mathbb{S}^n$ with zero diagonal and nonnegative elements some entries are known and other are not specified. Let us further assume that every specified principal submatrix of D is an Euclidean distance matrix with embedding dimension less or equal to r. The *Euclidean Distance Matrix Completion* (EDMC) problem consists in finding the not specified entries of D, in such a way that D is an Euclidean distance matrix. In order to specify the problem mathematically, let us associate to D a 0-1 matrix $H \in \mathbb{S}^n$ such that $H_{ij} = 1$ for the specified entries of D and $H_{ij} = 0$ otherwise. Using the Hadamard component-wise product ($(A \circ B)_{ij} = A_{ij}B_{ij}$) the (EDMC) problem for D can be then written as

$$\begin{array}{rl} \text{Find} & \Gamma \in \mathcal{E}^n \\ \text{such that} & H \circ \Gamma = H \circ D \end{array} \tag{13}$$

The *low dimensional Euclidean Distance Matrix Completion* adds the constraint that the embedding dimension should not be smaller than r, i. e.

$$\begin{array}{rl} \text{Find} & \Gamma \in \mathcal{E}^n \\ \text{such that} & H \circ \Gamma = H \circ D \\ & embdim(\Gamma) = r \end{array}$$

This problem can be equivalently written as

$$\begin{array}{rl} \min & 0 \\ \text{subject to} & H \circ \mathcal{K}(Y) = H \circ D \\ & Y\mathbf{e} = 0 \\ & Y \succeq 0 \\ & \operatorname{rank}(Y) = r \end{array}$$

and it is NP-hard. The relaxation obtained by deleting the rank constraint is a tractable SDP problem, but the solutions usually has too large values for rank(Y) and there are many different heuristics to improve this relaxation [83].

Another idea is to take advantage of the degeneracy (in the sense that the Slater-CQ fails) and to reduce the dimension of the problem using a proper semidefinite facial reduction. Given a subset $\alpha \subset \{1, \dots, n\}$ and a matrix $Y \in \mathbb{S}^n$ let us denote by the principal submatrix of Y formed from the rows and columns with index in α as $Y[\alpha]$. Based on this notation we can define for a fixed matrix $\bar{D} \in \mathcal{E}^k$, with $|\alpha| = k$ the set

$$\mathcal{E}^n(\alpha, \bar{D}) = \{D \in \mathcal{E}^n \mid D[\alpha] = \bar{D}\}$$

For instance, if the fixed entries of the matrix D in the above low dimensional (EDMC) problem are exactly those from the matrix formed by the first k rows and columns, where the specified submatrix is $\bar{D} \in \mathcal{E}^k$ with $embdim(\bar{D}) = r$, then the low dimensional (EDMC) problem can be intended as to find one element in the set

$$\{Y \in \mathbb{S}^n \mid Y \in \mathcal{K}^\dagger(\mathcal{E}^n(1:k, \bar{D})), \operatorname{rank} Y = r\}$$

Here we write MATLAB notation $1 : k = \{1, \dots, k\}$ for simplicity.

Theorem 2.7. *Let $D \in \mathcal{E}^n$, with embedding dimension r. Let $\bar{D} = D[1:k] \in \mathcal{E}^k$ with embedding dimension t, and $B = \mathcal{K}^\dagger(\bar{D}) = \bar{U}_B S \bar{U}_B$ for some $\bar{U}_B \in \mathbb{R}^{k \times t}$ with $\bar{U}_B^T \bar{U}_B = I_t$ and $S \in \mathbb{S}^t_+$ positive definite. Then*

$$\operatorname{face} \mathcal{K}^\dagger(\mathcal{E}^n(1:k, \bar{D})) = (U \mathbb{S}_+^{n-k+t+1} U) \cap \mathbb{S}^n_C = (UV)\mathbb{S}_+^{n-k+t}(UV)^T$$

where

$$U_B = \left[\bar{U}_B \ \frac{1}{\sqrt{n}}\boldsymbol{e}\right] \in \mathbb{R}^{k \times (t+1)},$$

$$U = \begin{bmatrix} U_b & 0 \\ 0 & I_{n-k} \end{bmatrix} \in \mathbb{R}^{n \times (n-k+t+1)} \quad \textit{and} \quad \left[V \ \frac{U^T \boldsymbol{e}}{\|U^T \boldsymbol{e}\|}\right]$$

is an square orthogonal matrix of dimension $(n - k + t + 1)$.

This remarkable result provides a reduction of the size of the (EDM) completion problem. Instead of working with matrices in $\mathbb{S}^n$, the problem is now stated with smaller matrices in $\mathbb{S}^{n-k+t}$.

There is a natural way to associate a weighted undirected Graph $G = (N, E, \omega)$ to the (EDMC) problem defined by a matrix $D \in \mathbb{S}^n$ (with zero diagonal, nonnegative elements and specified and unspecified entries). In fact, taking the nodes set $N = \{1, \ldots, n\}$, the edge set $E = \{ij \mid i \neq j, D_{ij}\text{is specified}\}$ and the weights $\omega_{ij} = \sqrt{D_{ij}}$, for all $ij \in E$. In this setting the matrix H used in (13) correspond just to the adjacency matrix of G. Moreover, a specified principal submatrix in D can be interpreted as a clique in G. So far, the above result deals with the case of a single clique. It shows in particular, the equivalence to appropriated faces and opens the possibility to reduce the problem using information of a clique.

In [82] the above result is extended in many ways, first considering two (or more) disjoint cliques and then describing the faces associated to intersecting cliques. A deeper insight of the subsequent reduction of the problem can be taken from [82, 83], where this procedure is applied to the so called Sensor Network Localization Problem and numerical examples with the solution of large instances are discussed. The same technique of semidefinite facial reduction over cliques for EDMC problems have been successfully applied to other areas, see for instance [15, 12].

3 NONLINEAR SEMIDEFINITE PROGRAMMING

Let us consider in this section the following nonlinear semidefinite programming (NLSDP) model

$$(NLSDP) \qquad \begin{array}{ll} \min & f(x) \\ & G(x) \preceq 0 \end{array}$$

where the mappings $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ and $G : \mathbb{R}^n \longrightarrow \mathbb{S}^m$ are in general smooth and nonlinear. Equality constraints can be also included in the (NLSDP) model, but for simplicity of presentation, we have chosen the above simple (NLSDP). In particular, all statements discussed in this section can be adapted to the case with equalities.

The (SDP) model of the previous section is already a nonlinear convex optimization problem. However in some important application problems, see e.g. [102, 100, 106], it is helpful to incorporate non convex and nonlinear functions into the model resulting in the above (NLSDP). More recently NLSDP has been used for modelling in new different applications areas like magnetic resonance tissue quantification [8], truss design and structural optimization [2, 17, 67, 73], material optimization [1, 74, 79, 129, 127, 128, 57, 78], passive reduced order modelling [48], fixed-order control design [10], finance [80, 88] and reduced order control design for PDE systems [91], among others.

The optimality conditions of first and second order for NLSDP are widely characterized, see for instance [21, 23, 35, 125, 47]. An important effort research is recently devoted to the study and characterization of stability for solutions of nonlinear semidefinite programming (or in general conic) problems, see for instance [107, 20, 33, 70, 49, 98, 97].

We present briefly the optimality condition for the model (NLSDP) and refer to [65] for a detailed discussion of the key differences to the usual case of nonlinear programming.

The Lagrangian $\mathcal{L} : I\!R^n \times \mathbb{S}^m \rightarrow I\!R$ of (NLSDO) is defined by $\mathcal{L}(x, Y) := f(x) + \langle G(x), Y \rangle$. and its gradient with respect to x can be written as

$$\nabla_x \mathcal{L}(x, Y) = \nabla f(x) + DG(x)^*[Y] \tag{14}$$

Here $DG(x)[.] : \mathbb{R}^n \longrightarrow \mathbb{S}^m$ is defined as

$$DG(x)[d] = \sum_{i=1}^{n} d_i G_i(x), \quad \text{with } G_i(x) = \frac{\partial G(x)}{\partial x_i}$$

and the adjoint $DG(x)^*[.] : \mathbb{S}^m \longrightarrow \mathbb{R}^n$ is then

$$DG(x)^*[Y] = \nabla_x \langle G(x), Y \rangle = (\langle G_1(x), Y \rangle, \dots \langle G_n(x), Y \rangle)^T.$$

The Mangasarian-Fromovitz constraint qualification is satisfied at the feasible point x if there exists a vector $d \in \mathbb{R}^n$ such that $G(\bar{x}) + DG(x)[d] \prec 0$.

Theorem 3.1. *If $\bar{x}$ is a local minimizar of (NLSDP) where the Mangasarian-Fromovitz constraint qualification holds true, then there exist matrices $\bar{Y}$, $\bar{S} \in \mathbb{S}^m$ such that*

$$\begin{aligned} G(\bar{x}) + \bar{S} &= 0, \\ \nabla f(\bar{x}) + DG(\bar{x})^*[Y] &= 0, \\ \bar{Y}\bar{S} &= 0, \\ \bar{Y}, \ \bar{S} &\succeq 0. \end{aligned} \tag{15}$$

A point $(\bar{x}, \bar{Y}, \bar{S})$ satisfying (15) is a stationary point of (NLSDP). For simplicity let us consider only the case that the above $\bar{Y}$ and $\bar{S}$ and unique and satisfy *strict complementarity*, i.e. $\bar{Y}+\bar{S} \succ 0$. In the following we state second order sufficient conditions due to [125]. Let us then consider a strict complementary stationary point $(\bar{x}, \bar{Y}, \bar{S})$. In this case the cone of critical directions at $\bar{x}$ can be written as follows, see e.g. [20, 65],

$$C(\bar{x}) := \{h \mid U_1^T DG(\bar{x})[h] U_1 = 0\}, \tag{16}$$

where $U = [U_1, U_2]$ is an unitary matrix that simultaneously diagonalizes $\bar{Y}$ and $\bar{S}$. Here also, U_2 has $r := \text{rank}(\bar{S})$ columns and U_1 has $m - r$ columns. Moreover the first $m - r$ diagonal entries of $U^T \bar{S} U$ are zero, and the last r diagonal entries of $U^T \bar{Y} U$ are zero.

Let us denote the Hessian of the Lagrangian by

$$\nabla_x^2 \mathcal{L}(x, Y) = \nabla^2 f(x) + D^2 G(x)^*[Y] \tag{17}$$

where $D^2 G(x)^*[Y] = \nabla_x^2 \langle G(x), Y \rangle$

The *second order suffcient condition* is satisfied at $\bar{x}$, $\bar{Y}$ if

$$h^T (\nabla_x^2 \mathcal{L}(\bar{x}, \bar{Y}) + \mathcal{H}(\bar{x}, \bar{Y})) h > 0 \qquad \forall\, h \in C(\bar{x}) \setminus \{0\} \tag{18}$$

Here $\mathcal{H}$ is a nonnegative matrix related to the curvature of the semidefinite cone in $G(\bar{x})$ along direction $\bar{Y}$ (see [125]) and is given by its matrix entries

$$\mathcal{H}_{i,j} := -2\langle \bar{Y}, G_i(\bar{x})G(\bar{x})^{\dagger} G_j(\bar{x})\rangle,$$

where $G(\bar{x})^{\dagger}$ denotes the Moore-Penrose pseudo-inverse of $G(\bar{x})$.

Theorem 3.2. *Let $(\bar{x}, \bar{Y}, \bar{S})$ be a stationary point of (NLSDP) satisfying strict complementarity. If the second order sufficient condition holds true, then $\bar{x}$ is a strict local minimizer.*

The following very simple example of [43] shows that the classical second order sufficient condition, i.e.

$$h^T(\nabla_x^2 \mathcal{L}(\bar{x}, \bar{Y}))h > 0 \qquad \forall\, h \in C(\bar{x}) \setminus \{0\}$$

is generally too strong in the case of semidefinite constraints, since it does not exploit curvature of the non-polyhedral semidefinite cone.

$$\begin{array}{ll} \min\limits_{x\in\mathbb{R}^2} & -x_1 - (x_2 - 1)^2 \\ s.t. & \begin{bmatrix} -1 & 0 & -x_1 \\ 0 & -1 & -x_2 \\ -x_1 & -x_2 & -1 \end{bmatrix} \preceq 0 \end{array} \tag{19}$$

It is a trivial task to check that the constraint $G(x) \preceq 0$ is equivalent to the inequality $x_1^2 + x_2^2 \leq 1$, such that $\bar{x} = (0, -1)^T$ is the global minimizer of the problem.

The first order optimality conditions (15) are satisfied at $\bar{x}$ with associated multiplier

$$\bar{Y} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 2 \end{bmatrix}.$$

Strict complementarity condition also holds true. The Hessian of the Lagrangian at $(\bar{x}, \bar{Y})$ for this problem can be calculated as

$$\nabla_{xx}^2 \mathcal{L}(\bar{x}, \bar{Y}) = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}.$$

It is negative definite, and the stronger second order condition is not satisfied.

The orthogonal matrix

$$U = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

simultaneously diagonalizes $\bar{Y}$, $G(\bar{x})$ and the Moore-Penrose pseudoinverse matrix at $\bar{x}$ is then given by

$$G(\bar{x})^\dagger = \frac{-1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \tag{20}$$

Consequently the matrix associated to the curvature becomes

$$\mathcal{H}(\bar{x}, \bar{Y}) = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}.$$

Finally, the cone of critical directions has the form $h = (h_1, 0)^T$ with $h_1 \in \mathbb{R}$ and then the weaker second order sufficient condition holds true.

The most developed general algorithmic approach for NLSDP is the one due to Kočvara and Stingl, see [71, 72, 73, 75, 76, 126]. It bases on generalized augmented Lagrangians designed for the semidefinite constraint and solves a sequence of unconstrained minimization problems driven by a penalty parameter. There are other approaches for dealing with general NLSDP, for instance, sequential semidefinite programming [46, 48, 36, 56, 51, 52, 131, 139, 43], bundle methods [104, 105], partially augmented Lagrangian approach [10, 45, 106], interior point trust region [89, 90, 91], predictor-corrector interior point [64], augmented Lagrangian [106, 132], successive linearization [68] and primal-dual interior point methods [142, 143, 69] among others. There is not a definitive answer to the question of which is the most convenient approach for solving NLSDP in general, which explains the intense research activity going on in this area.

4 CONCLUDING REMARKS

The various recent developments in SDP connecting to new areas of mathematics are in our opinion a strong evidence, that this topic remains a promising research area. It will be for sure in the next years a beautiful source of new interesting applications as well as theoretical results.

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