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Invitation to Mathematical Psychology: Models and Benefits of Formal Theorizing

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ABSTRACT – In most areas, psychological phenomena tend to be explained only through textual constructions. Several authors, however, point to the need for theories that have a more formal nature, based on mathematical reasoning. In order to encourage broader access to its applications, we present the models and advantages of a mathematical psychology approach to the study of behavior. We review the limitations of verbal theorizing, then a common taxonomy in mathematical psychology follows, that classifies formal models as descriptive, process characterization, and explanatory. As well succeeded cases, we examine the mathematical psychology of decision making, of helping behavior, of memory, and of romantic relationships. Finally, we discuss the potential benefits and uses of this approach. Welcome to mathematical psychology.

KEYWORDS: mathematical psychology, formal theorizing, quantitative modeling

Convite à Psicologia Matemática: Modelos e Benefícios da Teorização Formal

RESUMO – Na maior parte das áreas os fenômenos psicológicos tendem a ser explicados apenas por meio de construções textuais. Diversos autores, no entanto, apontam para a necessidade de teorias que tenham uma natureza mais formal, baseada em raciocínio matemático. A fim de incentivar acesso mais amplo às suas aplicações, apresentamos os modelos e vantagens da abordagem da psicologia matemática para o estudo do comportamento. Revisamos as limitações da teorização verbal, apresentando em seguida uma taxonomia, comum na psicologia matemática, que classifica os modelos de dados como descritivos, explicativos e de caracterização. Como casos bem sucedidos, examinamos a psicologia matemática da tomada de decisão, do comportamento de ajuda, da memória e dos relacionamentos românticos. Por fim, discutimos os benefícios e usos potenciais da abordagem. Bem-vindo(a) à psicologia matemática.

PALAVRAS-CHAVE: psicologia matemática, teorização formal, modelagem quantitativa

From the classic definitions of William James and Wilhelm Wundt to their contemporary adaptations, psychology is broadly understood as the science of behavior and mental processes. Therefore, in addition to seeking to describe and predict the behavior of humans and other animals, we also seek to explain it. This is similar to how in physics it is possible to predict the "behavior" of an object thrown in the air, in addition to identifying which variables can influence its trajectory. In psychology, however, well-defined physical properties are often not evaluated.

Traditionally in the various subareas of psychology, when a theoretical research question is asked, the answer is developed through verbal theorization, that is, the phenomenon is explained using only textual constructions (Adner et al., 2009). Although this method is dominant and efficient so that anyone can interpret and understand the expected relationships between the relevant variables of a model, at least two problems can be derived (McGrath, 2011). First, predictions are often not clearly defined, both in terms of the direction and magnitude of the phenomenon.

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Second, certain behaviors that can be expected from certain variables, or from relationships between variables, are not easily described through verbal theorizing. On the other hand, in recent years psychology has gone through a crisis of replicability and criticism of its methodological and analytical practices (Nelson et al., 2018), which require greater investment in the underlying processes of theorization.

The main objective of the present work is to provide an introduction to the approach of the so-called mathematical psychology, describing the types of mathematical models that can be used in different areas of psychology. It is also intended to summarize the main advantages and benefits of formal theorizing, as an alternative invitation to traditional verbal theorizing, without the need for an exhaustive understanding of the mathematical models and techniques themselves (see, e.g., Coombs et al., 1970; Hunt, 2006). Thus, it is also sought to demonstrate that the mathematical description of psychological phenomena is not so complex and can be easily learned by any student or researcher in psychology without advanced training in arithmetic, algebra or geometry.

Mathematical Psychology: Modeling and formal theorizing

Mathematics, as a field of general knowledge, is concerned with the study of structures and patterns resulting from a series of axioms or assumptions (Devlin, 2012). Mathematics can also be considered a form of language, responsible for communicating the dynamics and, unequivocally, magnitude, direction and meaning of variables, as well as relationships between variables (Pasquali, 2001). Psychologists who use mathematics as a primary tool to describe their phenomena of interest are known as mathematical psychologists (Townsend, 2008).

The approach of mathematical psychology is at least as old as scientific psychology itself (Van Zandt & Townsend, 2012). Among the first mathematical psychologists, many being recognized as the creators of psychophysics, it is possible to identify Ersnt Weber (1795–1878), Gustav Fechner (1801–1887), Hermann von Helmholtz (1821–1894), Franciscus Donders (1818-1889), among others. However, the systematization of the approach took place only in the 1950s, after the initial development of three specific theories: (i) the signal detection theory; (ii) information theory and its applications in cognitive psychology; and (iii) the mathematical theory of learning. In the following decade, two of the most influential book series in the field (i.e., Luce et al., 1963-1965a; 1963-1963b) and the most influential journal in the field (i.e., Journal of Mathematical Psychology) were published, consolidating the approach. As manuals in the area reveal (e.g., Batchelder et al., 2016), in modern mathematical psychology the most diverse topics are studied, from the most basic psychological processes to complex dynamics between groups, as well as the development of Artificial Intelligences that simulate emotions.

In practical terms, one of the main foundations of mathematical psychology is the use of formal theorizing. Formal theorizing, using formal logic and mathematics, contrasts substantially with verbal theorizing (Doignon & Falmagne, 1991). While verbal theorization allows flexible understanding of a phenomenon due to the diversity of natural languages (e.g., Portuguese), formal theorization involves a mathematical and logical description of the phenomena of interest (Devlin, 2012). Thus, for those phenomena that can be clearly measured, formal theorization tends to be more objective (that is, less dependent on different perceptions and judgments) and provide possibilities for more robust hypothesis testing to assess the predictive power of a model. On the other hand, it should be noted that although, in principle, any area of study in psychology can be studied using the approach of mathematical psychology, some topics will be more favorable than others. In addition, the quality of theories involves issues beyond the theory itself, such as the most appropriate way to operationalize a variable of interest. In this way, verbal and formal theorizations are understood as complementary ways of understanding, defining and studying a phenomenon.

As an example of a possible application of mathematical psychology, imagine that we develop a theory of helping behavior that verbally proposes that people in uncertain situations tend to be less helpful to others. A formal theorization about the same phenomenon, however, would need to propose a mathematical model about what percentage is expected to be observed at each possible level of uncertainty. Hypothetically, some researcher in the field could say that the probability of emitting a helping behavior decreases according to a logistic function of the uncertainty in the situation, being dependent on two psychological factors (or parameters): (i) α (alpha), defined as the fundamental tendency not to help; and (ii) ψ (psi), defined as the importance of uncertainty. Such theorization can be described by the following equation:

$$\text{Probability of help} = \frac{1}{1+\exp\left(\alpha+\psi\text{Uncertainty}\right)}$$

Figure 1 presents a graphic representation of such a model, revealing its simplicity. For the line drawn in Figure 1, α is equal to –3 and ψ is equal to 6. These values allow us to conclude, for example, that when the uncertainty is equal to 0, the probability of helping will be close to 95%. This value is easily calculated by applying the theory formula:

$$\text{Probability of help} = \frac{1}{1+\exp\left(\alpha+\psi\text{Uncertainty}\right)} = \frac{1}{1+\exp\left(-3+6\times 0\right)} = \frac{1}{1+\exp\left(-3\right)} = 0.95$$

Note that, in addition to empirically defining two constructs (fundamental tendency not to help and the importance of uncertainty) fundamental to the theory, verbal theorization makes a specific prediction about what to observe in a given context (that is, what behavior is expected in the presence or absence of uncertainty).

It can also be observed, in Figure 1, that quadrants 1 and 3 present the non-support for the theory, whether it is proposed verbally or formally. Pragmatically, the values observed in these quadrants would indicate cases where uncertainty is low (or high) and there is low (or high) probability of help. In this example, the biggest difference is precisely which observations constitute evidence in favor of the theory. Taking into account verbal theorizing, any value observed in quadrants 3 and 4 would indicate evidence favorable to the theory. For formal theorizing, however, only values above or very close to the inverted "S" shaped line would indicate evidence about the validity of the theory. That is, in the case where α is equal to -3 and ψ is equal to 6, if the uncertainty is equal to 0 but the probability of help is very different from 95%, we can reject the theory. Such an example allows us to identify that an intuitive understanding of the formulas can already be useful and sufficient for the application of mathematical psychology techniques in theorization.

In psychology, when taken as a whole, there seems to be a preference for using quantitative methods of data analysis (Mertens, 2014). But ironically, formal theorizing is still used on a small scale (Townsend, 2008). Here, it is necessary to distinguish the "quantitative psychologist" from the "mathematical psychologist". The quantitative psychologist is one who uses statistical and mathematical tools to test his hypotheses, but such tools do not present a specific psychological interpretation. For example, analyses such as Student's t-test, ANOVAs, regression analysis and, more recently, machine learning tools (Yarkoni & Westfall, 2017), can be used to test the same hypothesis about any scientific theory. The mathematical psychologist, on the other hand, is the one who develops formal models that are often specific to their research problem. For example, in the model represented in Figure 1, changing the statistical or mathematical procedure to test the problem would have the consequence of testing a theory different from the one proposed and, therefore, the lack of adequacy of the chosen procedure. However, it should be noted that many models, whether statistical or developed by mathematical psychologists, can be considered as an extension of regression analysis (e.g., Busemeyer et al., 2015). Thus, what characterizes a model as "statistical" or "psychological" is more related to the interpretability of the parameters as a psychological process, constructs or events, than to the mathematical form of the models themselves.

Given these considerations, it is important to clearly define what a mathematical model is and how it can be used. For this, it is necessary to recognize that models are intentionally more parsimonious and more abstract than the real system they seek to explain (Fum et al., 2007). As in the famous aphorism popularized by Box and Draper (1987, p. 74), "all models are wrong, but some are useful". Thus, by definition, models will always be simpler than reality and should never be too complex, given the inconsistency of replacing something that is not fully understood (reality) with

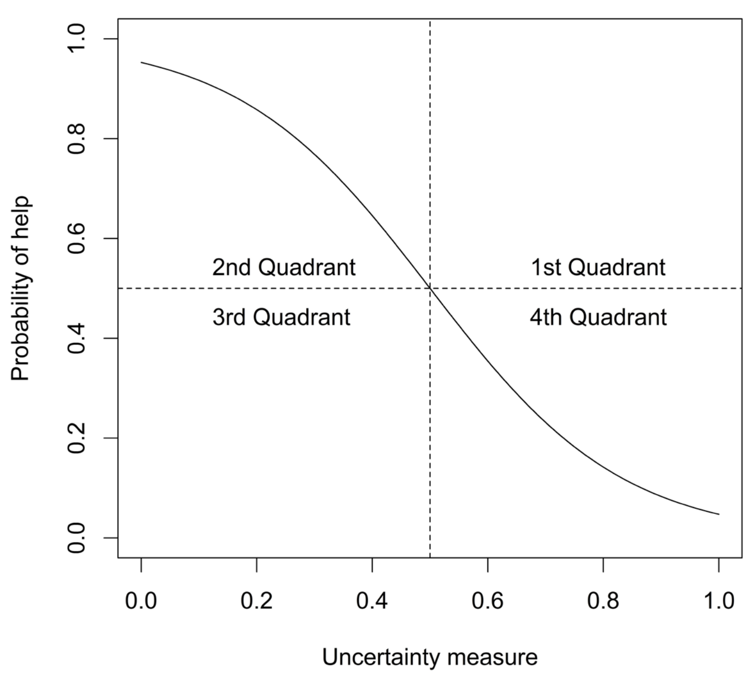

Figure 1. Quadrants of possible prediction values by verbal and formal models of helping behavior.

something that cannot be understood either (Norris, 2005). This obviously raises the problem of defining which details are necessary and which are minor for the problem studied. Although there is no single answer, some suggestions can be made depending on the purpose of the proposed model. Lewandowsky and Farrell (2010) recognized the existence of different typologies to name the purpose of the model. At the same time, they propose a specific typology that divides the process according to the objective of the model: to describe, characterize or explain a given phenomenon.

The Types of Models and Their Objectives

Descriptive

Descriptive models are those explicitly devoid of psychological content. Being "devoid of psychological content" means that while such models can predict and describe observed data, as well as set some limitations on underlying psychological processes, they do not specify how such processes contribute to the observed outcome. Our example presented in Figure 1 is a case of a descriptive model. Although the parameters can be interpreted as psychological constructs, the model does not tell us what psychological processes are taking place while individuals are deciding whether or not to help. For example, the model used in Figure 1 does not define the relationship between the fundamental tendency not to help and effective behavior. Thus, descriptive models cannot define which other variables inherent to them can influence the observed result.

In order to further discuss descriptive models, one can use a very common example in the learning literature: what is the best way to describe the acquisition of knowledge over time (Heathcote, Brown & Mewhort, 2000)? At the top of Figure 2 there is, on the *x* axis, the number of attempts

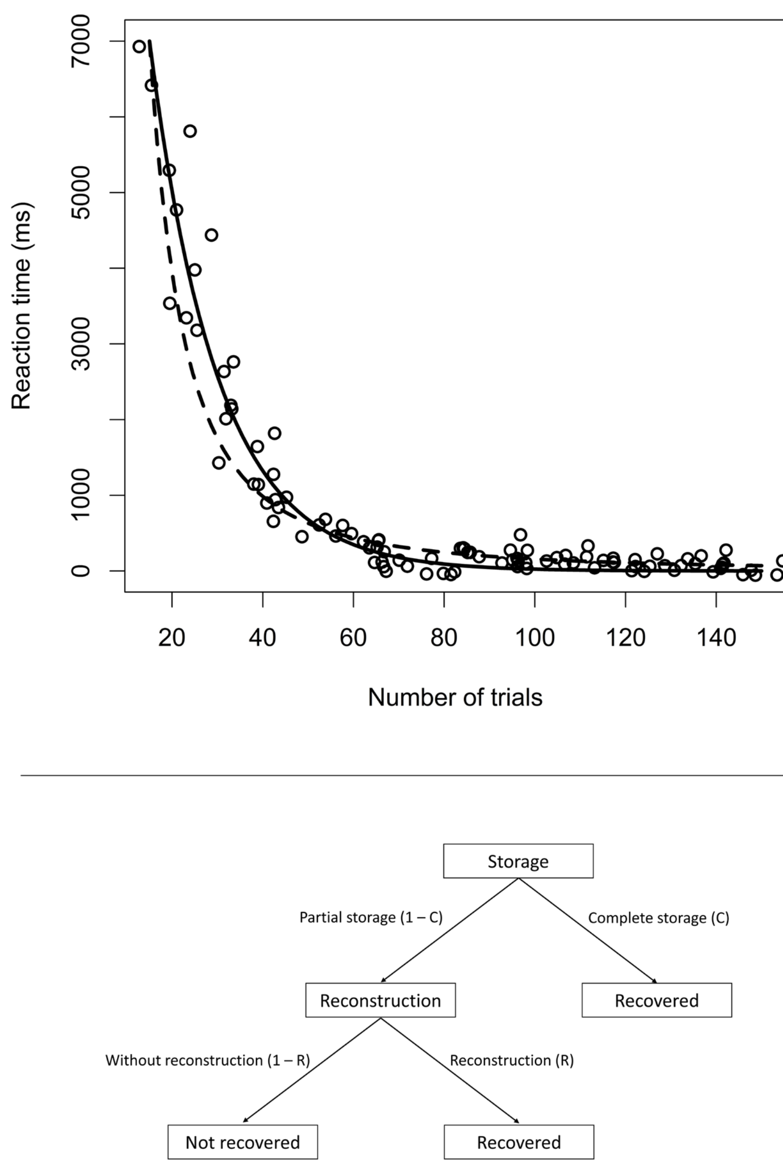

Figure 2. Two types of models. At the top, data plotted according to two competing descriptive models of learning. At the bottom, characterization model on short-term memory retrieval.

and, on the y axis, the reaction time for the response of already knowing or not a stimulus. It can be seen that, as the number of trials increases, the reaction time for the response decreases. The dots represent the answers themselves, while the lines, solid and dotted, represent predictions made by two different theories. Although the difference between the lines is visually subtle, they have virtually opposite theoretical implications. The model represented by the solid line says that the rate of learning is constant, but that there is less and less information to learn. The model represented by the dotted line says that the learning rate is variable, but that the amount of information to learn is constant. In this example, both models "describe" the data equally well. However, they are devoid of psychological content: there is no information available that allows concluding which competing model is the most adequate or most plausible to describe the learning process. It is therefore necessary to gather other evidence to conclude which model best describes the underlying processes.

Characterization

The second type of model includes those that categorize a psychological process and some main aspects stand out. They define which are the relevant unobservable processes, or else the relationships between observable and unobservable variables, and then proceed with the necessary measures. Another aspect is that this type of model does not attempt to explain how the underlying processes that influence the way in which hypothetical mental constructs generate certain outcomes occur, but only the order in which they occur.

As an example of a characterization model, we can mention the multinomial tree processing model for short-term memory retrieval, proposed by Schweickert (1993). The bottom part of Figure 2 shows the characterization of the short-term memory retrieval process, which can be retrieved in two cases: as a consequence of the complete storage of information or as a consequence of the reconstruction of a partially stored memory. In this case, the probability of recovering the stored memory completely is equal to C. On the other hand, the probability of the memory being recovered, given that it was only partially stored, is equal to $(1 - C) \times R$. In addition, the model proposes that there is also the possibility that memory will not be recovered. This only occurs when, after the memory has only been partially stored, it is also not rebuilt. The probability of this happening is equal to $(1 - R) \times (1 - C)$. But the model does not identify which processes influence the probability of information being correctly stored, nor those that influence partial memory reconstruction.

Explanatory

The third and last type of model is so named because it allows, taking into account current techniques, the best way to make inferences about psychological constructs. Like characterization models, it deals with psychological constructs, but it goes a step further by providing detailed explanations of them. As seen, Schweickert's (1993) multinomial tree-processing model identifies the stage of memory reconstruction, but does not give any indication of exactly how this process might take place. An explanatory model, however, was proposed by Lewandowsky and Farrell (2000), in which the reconstruction process occurs through a constant flow of exchanges between a response and a storage system, until the memory is completely restored. Due to the complexity of the mathematical model, it will not be presented here (see Kahana, 2020, for a more detailed explanation). However, it remains evident that the authors' proposal on how the reconstruction process takes place was all described using a mathematical model.

One might ask, in view of the revised models: since it is possible to specify them at the level of complexity, why are they not all elaborated as the type of explanation? First, it is not always possible to specify a process in the minute detail that an explanatory model requires. Thus, less complex models are a valuable alternative for research in a certain area to continue to develop. Not coincidentally, descriptive models are much more popular in psychology. Furthermore, there are cases in which a simpler characterization model may be preferred to one with a very detailed explanation.

Using a traditional t-test to compare the average difference in test performance between a control and an experimental condition is generally more practical than developing an explanatory model with individual estimates. However, in the case of public policies on education, for example, using a simpler model, but based on data collected from a good research design, can be more efficient. After all, a simple and robust result is better than using an explanatory model that is more susceptible to threats to the validity of the results. However, descriptive-level modeling in psychology has already allowed researchers to identify many principles for the field as a whole (e.g., Brown et al. 2007).

Benefits of Formal Theorizing

Lewandowsky and Farrell (2010) pointed out at least six distinct benefits of formal theorizing over verbal theorizing. The first two benefits are related to the interpretability of research data. Since data never "speaks for itself", it is necessary to use models to interpret it. The more accurate the model, the better the data can be interpreted. Furthermore, verbal theorization, by itself, does not allow the establishment of adequate parameters for quantitative analyses. Thus, if one wishes to pursue a more quantitative approach in psychology, treating the problem from the beginning as a mathematical problem will be more beneficial than changing the data to fit the method of analysis.

Another couple of benefits involve comparing alternative models. There are always numerous alternative models that explain the collected data equally well (as shown in Figure 2).

When verbal theorizing is used, the lack of precision can make it very difficult to identify competing models and explanations for a phenomenon. At the same time, formal theorization also makes it possible to establish the best criteria for quantitative assessment of model comparisons, which can be complemented, as in verbal theorization, by the judgment of experts.

Finally, the last pair of benefits proposed by Lewandowsky and Farrell (2010) involve the limitations of verbal theorizing. Even quite intuitive verbal theories can be incoherent and poorly specify their consequences. This occurs because of what precisely makes natural language so rich: subjectivity in the interpretations of its meanings. Thus, when using models developed by verbal theorization, it is often not possible to guarantee that all the assumptions of a theory have been identified and tested.

The Influence of Formalization on Psychology

The study field of decision-making processes is probably the best example of the success and application of mathematical psychology approaches (Baron, 2007). This field has a multidisciplinary nature, but with contributions coming mainly from psychology and economics (Fischhoff & Broomell, 2020). The models that explain such decision-making processes are generally divided into two types: normative and descriptive (Baron, 2007). Normative models are those concerned with identifying the best decision to make, assuming that the decision maker is fully informed, is able to calculate with perfect accuracy, and wants to maximize utility. These models depart from an econometric tradition and are heavily based on formal theorizing from utility theory (Simon, 1959). In this tradition, “utility maximization” by individuals is regarded as the definition of rational behavior. Descriptive models, on the other hand, are those that describe observed behaviors, with the general assumption that decision-making agents behave according to some consistent rules. They are more influenced by psychological theories and generally come from verbal theorizing (Janis & Mann, 1977). In these cases, rational behavior is defined as the maximization of utility, given that there are psychological biases and the influence of the environment during the decision process. A special type of descriptive model is one that involves social and situational variables in the decision process, thus being called social decision-making processes (e.g., Edwards, 1977).

The greatest contribution to understanding social decision-making processes came from the prospect theory of Kahneman and Tversky (1979). This theory, which earned Kahneman a Nobel Prize in economics (Altman, 2004), started from utility theories, but with less emphasis on rationality assumptions. This was due to the fact that the authors intended to create a descriptive model of decision making, while the data showed that people did not always maximize utility as expected (Stanovich, 2015).

Kahneman and Tversky found, from a series of studies mixing methodological paradigms from both social psychology and mathematical psychology, three regularities in the social decision-making process: (i) people perceive losses that are proportional to gains as of greater relative magnitude; (ii) people pay more attention to changes in their utility states (i.e., how good a certain object is at a given moment) rather than absolute utility values; and (iii) subjective probability estimates are influenced by cognitive biases. The impact of this theory opened doors for the investigation of heuristics in social decision-making processes and for the creation of the area of behavioral economics (Kahneman, 2003). In addition, in a more modest way as a whole, but very strong for the area of mathematical psychology, it configured a rescue of formal theorizing methods and their importance for psychology.

Despite the success of prospect theory, more recent research shows that it can often not describe people’s behavior as well (Yukalov & Sornette, 2008). This is an argument widely developed, for example, by Gigerenzer and Murray (2015), in addition to authors of the Minskyian and resource-rational approach, as well as other less popular ones (see Lieder & Griffiths, 2020, or Millroth & Collsiöö, 2020, for further discussion). Among these alternatives to prospect theory, quantum models of information processing are the most innovative and contemporary (Bruza et al., 2015). Such models are characterized by using the mathematical language developed in quantum physics to address the theoretical issues of psychology. However, the models do not need to assume the existence of quantum processes in the brain or mind. The authors who defend such models only propose that the observed consequences of cognitive processes are mathematically better described and explained by the same type of mathematics used to describe and explain quantum phenomena in physics (for more details, Busemeyer & Bruza, 2012).

A basic feature of prospect theory that can be solved by quantum models is related to the presupposition of preference transitivity (Regenwetter et al., 2011). Such an assumption says, for example, that if an individual prefers to drink water instead of drinking beer and also prefers to drink beer instead of drinking juice, he should prefer drinking water instead of drinking juice. However, the breach of this assumption is constantly identified in research on preference and decision making (e.g., Kocher & Sutter, 2005; Smaldino & Epstein, 2015). The transitivity assumption is fundamental for prospect theory, as it depends on the classical logic that composes traditional models of rational choice. In the mathematical theory that underlies the quantum models, the expected consequence is that people’s preferences are not fixed and that they depend, for example, on the order in which certain events occur (Bruza et al, 2015). Thus, quantum models make extra predictions when compared to prospect theory models, depending on what limitations the research design imposes on the decision process.

FINAL CONSIDERATIONS

Despite the advantages, benefits and success presented in relation to the use of the mathematical psychology approach, obviously its use in psychology, in general, is still quite shy. Falmagne (2005) pointed out as one of the main reasons for this the decline in mathematics education and also, in some cases, the precarious effectively scientific training in many psychology courses. Although the author says that these reasons are mostly based on anecdotal evidence, in support of his argument he pointed to the fact that statistics textbooks for psychologists present less and less mathematics (e.g., "Statistics without mathematics", by Dancey & Reidy, 2006). In addition, many universities have discontinued teaching and research programs specifically focused on mathematical psychology, merging them with others. Townsend (2008) agreed with such questions by examining the past and estimating the future of the approach. In any case, researchers working with mathematical modeling and formal theorizing as a whole have still made significant contributions to psychology (Hunt, 2006; Townsend, 2008).

As an example of use that has a more generalized interest, one can cite the model of interactions of romantic partners by Gottman et al. (2002), inspired by the theory of the general system of families suggested by Von Bertalanffy (1968). In this model, it is assumed that each person has certain personality traits that influence the probability of occurrence of positive characteristics in their speech. In turn, the affective quality of speech is also influenced by the affectivity expressed in the previous social exchange with the person with whom he communicates. Furthermore, it is also assumed that messages that express negative affect have greater influences on interaction than messages that express positive affect. The model has several theoretical and applied implications, in the clinic, for example, although its influence still seems small (Amato, 2007).

As pointed out by Luce (1995; 1997), there are at least six major barriers that need to be overcome so that mathematical approaches and formal theorizing are better recognized in psychology. First, such approaches need to be taught in more psychology departments. Second, there must be a greater effort to develop methods that facilitate the use of more complex techniques, such as the use of quantum models. Third, more focus should be placed on developing higher quality measures and better statistical methods for evaluating such measures. In addition, excessive use of variables based on theories that have been previously refuted should be reduced. Then, existing contributions should serve as a basis for building more reliable models. The last barrier is related to the fact that, many times, models are tested without taking into account the best measurement level of the variables that compose it (nominal, ordinal, interval or ratio).

Finally, such problems can make formal theorizing, and more specifically, mathematical modeling, seem even more difficult, if not impossible. However, such problems actually make the modeling process more important than ever. (Increasingly) elaborating systems are a fact of the world and those involved with the human brain and behavior are probably among the most complex of all (Srivastava, 2009). For example, modeling techniques allow to unite and constrain neural and cognitive-behavioral models simultaneously, generating theoretically and empirically more plausible inferences (Turner et al., 2013). As we sought to promote in this work, despite the practical difficulties, the cost-benefit relationship can be very advantageous for research in psychology and especially in Brazil, where few original theories are produced and most models are frankly imported from foreign literature.

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