Revista Brasileira de Ensino de Física, v. 30, n. 4, 4302 (2008)
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Sum of squares in discrete physical spaces

(Soma de quadrados em espaços físicos discretos)

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Recebido em 14/7/2008; Aceito em 9/9/2008; Publicado em 27/2/2009

Discretization of three-dimensional physical spaces can induce, on observable physical quantities, effects which are not present in the continuum. Consider, as an example, the problem of the radiation spectrum of a blackbody, studied in introductory courses in quantum mechanics. One sees that the Rayleigh assumption of continuous and uniform frequency distribution of standing waves inside a cubic cavity with perfectly reflecting inner walls can be validated by a heuristic type of reasoning. However, by means of number theory, one sees that there might exist frequencies for which it is not possible to have standing waves inside the cavity. Nevertheless, within the same context, one can argue that a more general criterion can be adopted to validate the hypothesis of continuity of the observables which are expressed as the square root of the sum of three integers of a threedimensional space $On_xn_yn_z$.
Keywords: space discretization, blackbody radiation, Rayleigh-Jeans formula.

A discretização de espaços físicos tridimensionais pode induzir em quantidades físicas observáveis efeitos que não existem no contínuo. Considere, por exemplo, o problema do espectro de radiação do corpo negro, estudada nas disciplinas introdutórias de mecânica quântica. Pode-se notar que a hipótese de Rayleigh de uma distribuição de freqüências contínua e uniforme para as ondas estacionárias dentro de uma cavidade com paredes internas perfeitamente refletoras pode ser validada por um tipo heurístico de raciocínio. Contudo, pode meio da teoria de números, observa-se que podem existir freqüências para as quais não é possível haver ondas estacionárias. Apesar disso, no mesmo contexto, pode-se argumentar que um critério mais geral pode ser usado para validar a hipótese da continuidade dos observáveis que são expressos como a raiz quadrada da soma de três inteiros de um espaço tridimensional $On_xn_yn_z$.
Palavras-chave: discretização espacial, radiação de corpo negro, fórmula de Rayleigh-Jeans.

1. Introduction

At the beginning of last century, Planck solved the problem of the radiation spectrum from a blackbody by setting forth the hypothesis that molecules represented by harmonic oscillators (resonators) carry quantized values of energy [1]. Subsequently, Einstein inserted the light quantum concept in a corpuscular scenario. In his seminal paper, Einstein showed that if one assigns the mean energy k_BT for the set of oscillators, the blackbody energy density $u(\nu,\ T)$ could be described, in the low frequency limit, by means of the well-known Rayleigh-Jeans relation

$$u(\nu, T) = \frac{8\pi\nu^2}{c^3}k_BT. \quad (1)$$

In order to obtain the pre-factor $\frac{8\pi\nu^2}{c^3}$ in Eq. (1), giving the spectral density of the radiating wave, Rayleigh estimated the density of normal modes in a resonant cavity having the shape of a cubic box. In doing so, he retained only wave vectors components being multiple integers of $\frac{\pi}{a}$, a being the length of the box edge. This implies that the count of normal modes is done in a discrete momentum space. We thus start by briefly recalling Rayleigh derivation of Eq. (1) and use this rather well known example to discuss, on the basis of number theory, some properties of observable physical quantities calculated in real spaces and bearing a discrete character. We provide a quantitative analysis by which it can be argued that Rayleigh method in counting normal modes in the cubic black box is indeed correct. We shall also see in brief what consequences a discrete Minkowski space may have on light cone events.

A pedagogical valence can thus be attributed to the present work if one considers that the examples chosen show that interdisciplinary concepts tend to give a deeper vision of the problem at hand and can at times bring in new ideas even when discussing well known

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topics.

2. The Rayleigh-Jeans formula

Let us consider a blackbody at a certain fixed temperature T. More specifically, let us consider a cubic box with perfectly reflecting internal walls, having a small hole, through which electromagnetic radiation can be absorbed or emitted. The box behaves like a perfectly absorbing body, *i.e.*, a blackbody, when we send radiation through the hole, since electromagnetic energy remains stored in the body itself. The emitted radiation, indeed, may be considered to be negligible, if the hole is sufficiently small, so that it would also be possible to observe the propertied of the emitted radiation without perturbing, in a significant way, the energy density inside the box.

Electromagnetic radiation, with wave vector $\boldsymbol{k}$ and frequency $\nu = \frac{\omega}{2\pi}$, can be described by means of the oscillating function $f(\mathbf{r}, t)$ as follows

$$f(\mathbf{r}, t) = f_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} = f_0 e^{i(k_x x + k_{yx} y + k_z z - \omega t)}. \qquad (2)$$

Because of multiple reflections on the internal walls, the electromagnetic wave $f(\mathbf{r}, t)$ can give rise to a standing wave, owing to constructive interference, only for the following values the wave vector components

$$\begin{cases} k_x a = n_x \pi, \\ k_y a = n_y \pi, \\ k_z a = n_z \pi, \end{cases} \qquad (3)$$

where $n_x,\ n_y,\ n_z$ are non-negative integers. From the dispersion relation for electromagnetic waves, we can set

$$k = \frac{2\pi\nu}{c} \Rightarrow \nu = \frac{ck}{2\pi} = \frac{c}{2a}\sqrt{n_x^2 + n_y^2 + n_z^2}. \qquad (4)$$

Before dwelling more deeply in the allowed values of k, given that the indices $n_x,\ n_y,\ n_z$ are non-negative integers, we first derive Eq. (1). The number of normal modes in the cavity for frequency intervals of amplitude dν can be calculated by starting from Eq. (4). Let us then represent all possible values of the quantity $\rho = \frac{2a\nu}{c}$ in the three-dimensional discrete space $On_xn_yn_z$, where Eq. (4) defines the equation of a sphere. The number dN of normal modes in the spherical shell of radius ρ and of thickness dρ (thus in the frequency interval $(\nu,\ \nu + d\nu)$) is given by

$$dN = \frac{2\left(4\pi\rho^2\right) d\rho}{8} = \frac{8\pi a^3 \nu^2 d\nu}{c^3}, \qquad (5)$$

where the factor 2 indicates the possible polarizations of electromagnetic radiation and the factor 8 accounts for the positively defined nature of the frequency, so that only one octant of the sphere (the one obtained in the portion of space where all components $n_x,\ n_y,\ n_z$ are positive). Therefore, the number of normal modes in the interval of frequency $(\nu,\ \nu + d\nu)$ is given by the following expression

$$\frac{dn}{d\nu} = \frac{1}{a^3}\frac{dN}{d\nu} = \frac{8\pi\nu^2}{c^3}. \qquad (6)$$

If we multiply the average energy k_BT of a single oscillator to the factor obtained in Eq. (6), we recover the Rayleigh-Jeans formula (1).

In carrying out the calculation, however, we had to assume a continuous and uniform distribution of points in the three-dimensional discrete space. The hypothesis of uniformity of the distribution of points in the discrete space $On_xn_yn_z$ does not need further consideration, since it is trivial to prove its validity. As for the continuity of the distribution of points, we could guess that this hypothesis is never rigorously fulfilled. We shall see in the following section how to validate this last hypothesis by means of a heuristic definition of continuity. Therefore, if the heuristic continuity condition is fulfilled by the system, one can affirm that no error has been committed in counting the normal modes in a cavity.

3. Three-dimensional discrete physical spaces

We need to give a heuristic type of definition of continuity for the discrete three-dimensional space $On_xn_yn_z$. From the physical point of view we can say that a uniform discrete three-dimensional space is continuous if, given the most accurate instrument available, the distance between any two nearest neighbor points cannot be resolved by this instrument. On the other hand, by using the conventional instrument chosen, if this distance takes on a finite value, then we necessarily define the distribution as non-continuous.

We now need to transfer this concept in the context of the preceding section. Therefore, we may say that, if the dimensionless quantity $\frac{\Delta\rho}{\rho}$ is much smaller than one, we can consider the distribution of frequencies in the space $On_xn_yn_z$ as continuous. The statement $\frac{\Delta\rho}{\rho} << 1$, on its turn, is fulfilled if the quantity $\Delta\rho$ may be considered infinitesimal ($\Delta\rho \to d\rho$), the quantity ρ being finite in the space $On_xn_yn_z$.

We therefore set

$$\frac{\Delta\rho}{\rho} = \frac{n_x dn_x + n_y dn_y + n_z dn_z}{\rho^2} << 1. \qquad (7)$$

It is possible to estimate the variation of the quantity $\Delta\rho$ in the spherical octant where the indices $n_x,\ n_y,\ n_z$ are positive. We represent, in Fig. 1, the quantity $\Delta\rho$ in terms of the spherical coordinates related to the three-dimensional space $On_xn_yn_z$, under the condition of maximum variation of the quantities Δn_x, Δn_y, Δn_z ($\Delta n_x = \Delta n_y = \Delta n_z = 1$). In this way we may obtain a criterion to decide whether the inequality in Eq. (7) is satisfied. The maximum value of $\Delta\rho$ in the graph shown in Fig. 1 can be calculated to be 1.73, so that we can write

$$\rho >> 2 \Rightarrow \nu >> \frac{c}{a} \approx 100 \text{ MHz}, \tag{8}$$

where we have taken the characteristic dimension a of the blackbody to be of about one meter. The assumption of continuity of the space $On_xn_yn_z$ becomes thus plausible in the limit of high frequencies and of wavelengths much smaller than a. Visible light and infrared radiation satisfy well this condition, so that we may conclude that no error has been made in considering the space $On_xn_yn_z$ as continuous in this limit.

The discrete character of the space $On_xn_yn_z$, however, has further consequences. We notice that the dimensionless quantity ρ^2 is expressed in terms of the sum of squares of three integers. From Lagrange theorem [4] in number theory it is known that every positive integer can be written as the sum of squares of four integers. It could therefore happen that a positive integer, as ρ^2 in our case, could not be given by the sum of three squares. From Gauss condition [4], indeed, it is known that the equation

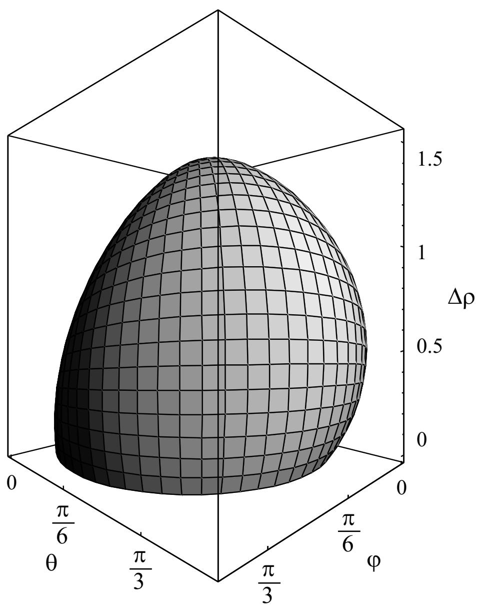

Figure 1 - The quantity $\Delta\rho$ as a function of the spherical coordinates associated to the discrete three-dimensional space $On_xn_yn_z$. It can be shown that the maximum value of $\Delta\rho$ in the spherical octant, where all frequencies are positive, is 1.73.

$$\rho^2 = i^2 + j^2 + k^2 \tag{9}$$

possesses a solution only if

$$\rho^2 \neq 4^\alpha \left(8\beta + 7\right), \tag{10}$$

where α and β are non-negative integers. Therefore, there might exist frequencies for which it is not possible to have standing waves inside the cavity. These frequencies are given by the following expression

$$\nu_{\alpha,\beta} = \frac{c}{a} 2^{\alpha-1} \sqrt{8\beta + 7}. \tag{11}$$

In order to represent these frequencies, we show a graph of the quantity $\rho_{\alpha,\beta} = \frac{2a}{c}\nu_{\alpha,\beta}$ in terms of non-negative integers α and β in Fig. 2. We notice that, by letting α and β vary in the intervals [0, 5] and [0, 100], respectively, we obtain a family of sequence of isolated points. The presence of frequencies not allowing standing waves inside the cavity, could therefore affect the validity of the hypothesis of uniform distribution of frequencies in the $On_xn_yn_z$ space. Before discussing this aspect and before giving a definitive solution to this logical question, we notice that, when we consider the directions in the $On_xn_yn_z$ space to which frequencies not allowing standing waves inside the cavity are associated, we find a non-trivial regular behavior. As already shown in Ref. [5], by showing the vectors $\frac{1}{\rho}(n_x, n_y, n_z)$ on the unitary sphere we may obtain the direction in the $On_xn_yn_z$ space along which the quantities $\rho_{\alpha,\beta}$ lie. By looking at the first octant of this unitary sphere, from Fig. 3 we notice the appearance of "stripes", given by the absence of points, showing a somewhat regular behavior. In these stripes, which have been denoted as *equatorial gaps* [5], the quantity $\rho = \rho_{\alpha,\beta}$ cannot be present for Gauss condition. Indeed, by extending this analysis to the whole unitary sphere, from Fig. 4 we notice that these stripes are located around maximum circumferences on the sphere, so that the above definition is justified. In Fig. 5, by looking from the top of the $n_x - n_y$ plane to the unitary sphere in Fig. 4, we notice two stripes of greater thickness along the n_x and n_y axes, while other two stripes, of smaller thickness with respect to the first two, lie along the $n_y = \pm n_x$ lines.

4. Remarks and conclusions

The consequences of discretization of points of a three-dimensional space are investigated. In particular, the assumption of continuous and uniform frequency distribution of standing waves inside a cubic cavity with perfectly reflecting inner walls is considered. By the same discrete character of the space in which the frequencies are defined, there could be frequencies for which standing waves are not allowed. This could affect the validity of the hypothesis of the uniformity of distribution of normal modes inside the cavity in the fictitious space $On_xn_yn_z$ used by Rayleigh to derive Eq. (1). In order to validate the hypothesis of a continuous frequency spectrum, a heuristic type of reasoning is in general adopted [3]. The appearance of not allowed frequencies, defined in a discrete space, has thus been investigated by means of number theory concepts.

In order to illustrate further these concepts, let us consider the analogous case of the light cone events in Minkowski space. By taking only events on the light cone, we might notice that the relation

$$c^2t^2 = x^2 + y^2 + z^2 \tag{12}$$

cannot be satisfied for some discrete values of (x, y, z). In this way, there could exist directions in the discrete Minkowski space for which light propagation is not possible. These directions, for what seen above, would correspond to the equatorial gaps on the unitary sphere (see Fig. 4, for example).

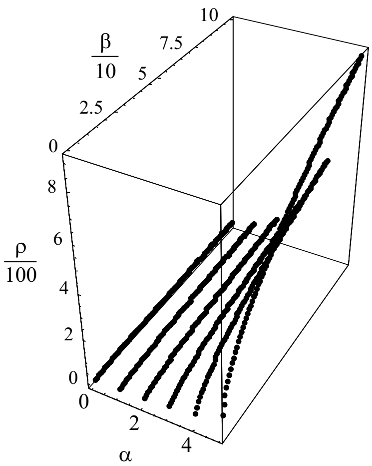

Figure 2 - A family of sequences of points obtained by letting the parameters α and β in Eq. (10) vary between 0 and 5 and between 0 and 100, respectively. These sequences represent the values of the quantity ρ which are not allowed in the space $On_xn_yn_z$.

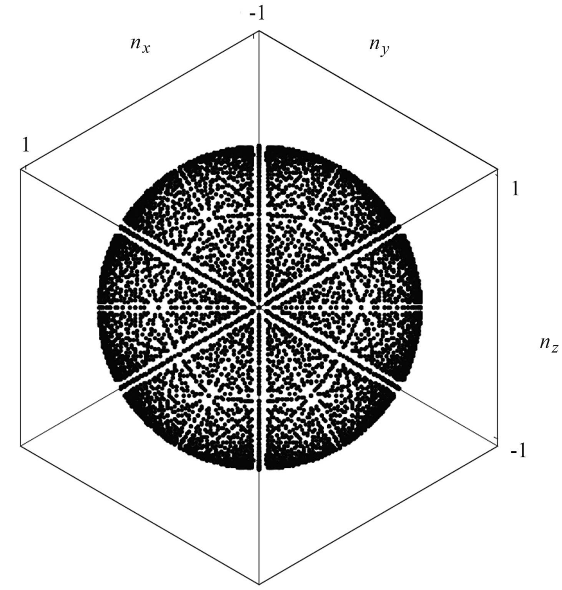

Figure 4 - Directions in the $On_xn_yn_z$ space for which the quantities ρ are allowed. These directions are calculated for indices n_x, n_y, n_z varying in an interval [-10, 10]. Notice the presence of equatorial gaps in which these quantities are not allowed.

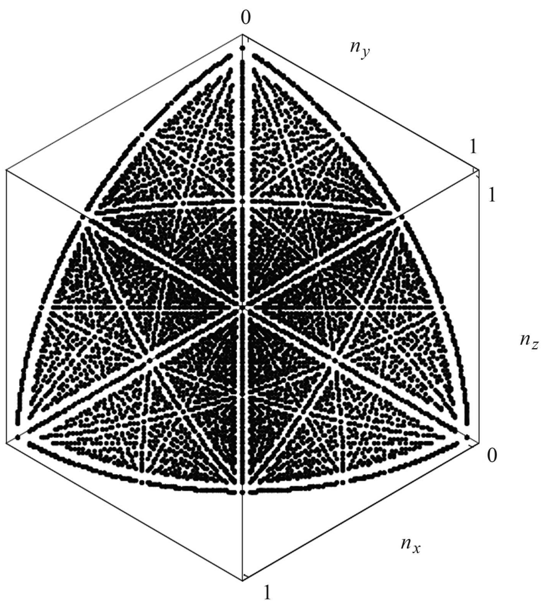

Figure 3 - The directions in the $On_xn_yn_z$space for which the dimensionless quantities ρ are allowed are indicated by points in the same space. These directions are calculated for indices n_x, n_y, n_z varying in an interval [0, 20].

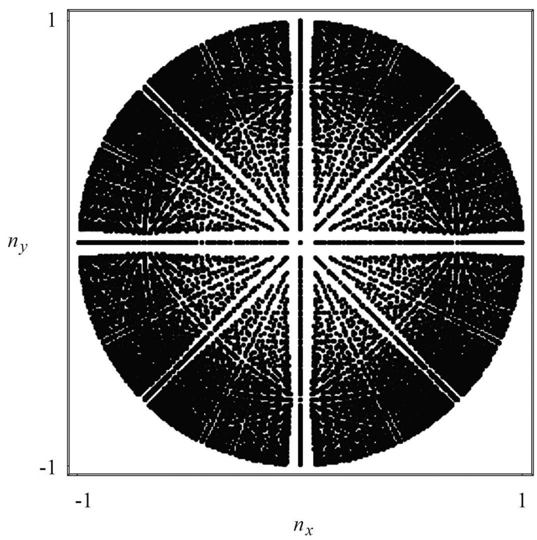

Figure 5 - Directions in the $On_xn_yn_z$ space for which the quantities ρ are allowed as seen by an observer placed on the top of the unitary sphere on the n_z axis. These directions are calculated for n_x, n_y, n_z varying in the interval [-15, 15].

However, as the discretized space increases in size, these directions would reduce to a subset with null measure on the unitary sphere (the equatorial gaps would reduce to lines, in practice). Therefore, as far as physically observable quantities are concerned, these points would acquire a vanishing statistical significance, which would make these effects not observable.

Going back to Rayleigh's assumption of continuity of the frequency spectrum, by this same argument, without recurring to the heuristic argument proposed in some textbooks, it could be once more concluded that Rayleigh correctly calculated the spectral density of the radiating wave inside the cubic cavity. Indeed, when one considers a wide range of modes in the cavity itself (*i.e.* one considers very many values of the indices n_x, n_y, n_z) the example above clarifies that the statistical significance of not allowed frequencies, in deriving the Rayleigh-Jeans formula, vanishes.

Acknowledgments

This work is dedicated to B. Savo, whose human warmth and scientific knowledge I've missed since his departure. I'll always remember with joy the productive time we spent studying and working in the room we shared at the Dipartimento di Fisica. I am very grateful to G. Gargiulo and F. Romeo for useful discussions.

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