Revista Brasileira de Ensino de Física, v. 36, n. 3, 3301 (2014)
www.sbfisica.org.br

Artigos Gerais

On the electrostatic energy of two point charges

(A energia eletrostática de duas cargas pontuais)

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Recebido em 26/11/13; Aceito em 24/2/14; Publicado em 31/7/2014

The electrostatic field energy due to two fixed point-like charges shows some peculiar features concerning the distribution in space of the field energy density of the system. Here we discuss the evaluation of the field energy and the mathematical details that lead to those peculiar and non-intuitive physical features.
Keywords: electrostatic energy, self-energy, classical renormalization.

A energia eletrostática de duas cargas puntiformes fixas exibe algumas peculiaridades que dizem respeito à distribuição da densidade de energia associada ao campo elétrico total do sistema. Discutimos aqui o cálculo da energia do ponto de vista do campo e os detalhes que levam a essas características físicas peculiares e não intuitivas.
Palavras-chave: eletrostática, auto-energia, renormalização clássica.

1. Introduction

The electrostatic energy of two point charges is given by the simple expression

$$U_{12} = \frac{1}{4\pi\epsilon_0}\,\frac{q_1\,q_2}{R}, \tag{1}$$

where R is distance between the charges the values of which are q_1 and q_2. If the charges have the same algebraic sign the electrostatic energy is positive but if the algebraic signs are not equal then the electrostatic energy is negative. Eq. (1) is interpreted as a potential or energy due to the spatial configuration of the charges. If constraints forces are removed this energy will be transformed into kinetic energy of the charges. On the other hand, from the field point of view the total energy of the system is given by the expression

$$U = \frac{\epsilon_0}{2}\iiint \|\mathbf{E}(P)\|^2\,dV \geq 0, \tag{2}$$

where $\mathbf{E}(P)$ is the total field of the system at a point P of the space, that is

$$\mathbf{E}(P) = \mathbf{E}_1(P) + \mathbf{E}_2(P). \tag{3}$$

In order to extract the interaction or potential energy of the configuration we must subtract the self-energies of the point charges and next we show how this can be accomplished.

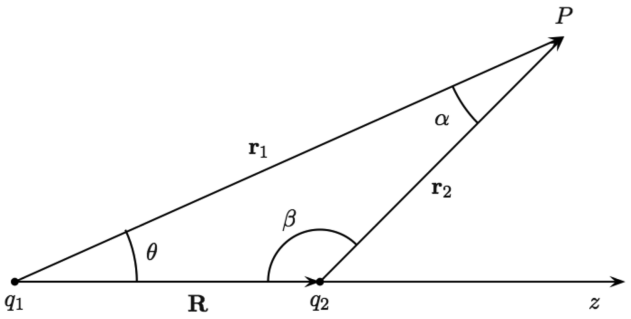

Figure 1 - Geometry for the evaluation of the interaction energy of two point charges.

The field energy can be split into three separate contributions

$$\frac{\epsilon_0}{2}\iiint \mathbf{E}^2(P)\,dV = \frac{\epsilon_0}{2}\iiint \mathbf{E}_1^2(P)\,dV + \frac{\epsilon_0}{2}\iiint \mathbf{E}_2^2(P)\,dV + \epsilon_0\iiint \mathbf{E}_1(P)\cdot\mathbf{E_2}(P)\,dV. \tag{4}$$

The first two terms on RHS of Eq. (4) can be interpreted as the classical self-energies of the point charges. From a classical point of view both terms lead to divergent contributions. To deal with this problem we must introduce a regularization and renormalization scheme. A simple one is to compare two configurations, say the configuration shown in Fig. 1 and the configuration where both point charges are at a different distance, say D, from each other and then subtract one con-

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figuration from the other. This procedure leads to a subtraction of infinities and may cause discomfort even among the not so mathematically-minded. A simple way out of this situation is to replace the point charges by two small identical spherical distributions, for example, a spherical shell of radius δ uniformly charged with a charge density σ. This is convenient because inside the shells considered one at a time the electric field is zero. This will yield a finite self-energy contribution given by $2 \times e^2/(2 \cdot 4\pi\epsilon_0\delta)^2$. Upon subtraction the self-energy terms will cancel out and we will be left with two crossed terms. Then we let the distance between the two shells in one of the configurations approach infinity ($D \to \infty$). This procedure will leave us with one relevant finite crossed term to be calculated. Now we let $\delta \to 0$ keeping the charge constant. The surviving crossed term then represents the finite variation of the interaction energy of the two point charges with respect to the reference configuration and must reproduce Eq. (1), that is

$$\epsilon_0 \iiint \mathbf{E}_1(P) \cdot \mathbf{E_2}(P)\, dV = \frac{1}{4\pi\epsilon_0}\,\frac{q_1\, q_2}{R}, \qquad (5)$$

but this must be proved by explicit evaluation of integral.

The first time the author heard of this problem was when he was reading the first edition of Ref. [1] where it was proposed as an advanced problem in a special chapter at the end of the book. The challenge was to prove that the potential energy point of view and the field energy one were not mutually incompatible by arguing, not necessarily by explicit calculations. Recently, in a paper on the role of field energy in introductory physics courses Hilborn [2] commented on this and some peculiar features concerning the distribution in space of the field energy of the system. To appreciate these features consider for simplicity two equal positive charges. Then

1. There is a spherical region centered at one of the charges that does not contribute to the total interaction energy.
2. This region can be divided into two subregions that contribute with algebraically opposite energies and the amount of negative energy is very small when compared with the total energy.
3. And, finally, 90% of the field energy lies in a limited part of the space.

Here the present author will try to show to the interested reader the details of those peculiar and non-intuitive aspects by performing explicitly the calculations.

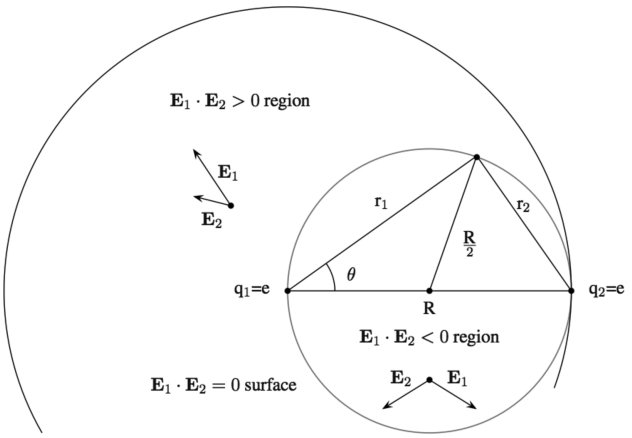

Figure 2 - The distribution of the electrostatic energy density for two similar positive point charges.

2. Evaluation of the electrostatic energy

It is convenient to use spherical coordinates with the origin at one of the charges and the polar z-axis along the line that passes through both, see Fig. 1. Let us set $q_1 = q_2 = e$. The crossed term reads

$$\mathbf{E}_1 \cdot \mathbf{E_2} = \frac{e^2}{(4\pi\epsilon_0)^2 r_1^2 r_2^2}\, \mathbf{e}_{r_1} \cdot \mathbf{e}_{r_2}. \qquad (6)$$

Now we introduce, see Fig. 1

$$\mathbf{r}_1 = \mathbf{R} + \mathbf{r}_2 \qquad (7)$$

hence

$$r_2^2 = r_1^2 + R^2 - 2r_1 R\, \cos\,\theta, \qquad (8)$$

and, see Fig. 1,

$$\mathbf{e}_{r_1} \cdot \mathbf{e}_{r_2} = \cos\,\alpha. \qquad (9)$$

Therefore

$$\mathbf{E}_1 \cdot \mathbf{E_2} = \frac{e^2}{(4\pi\epsilon_0)^2}\, \frac{\cos\,\alpha}{r_1^2\,(r_1^2 + R^2 - 2r_1 R\, \cos\,\theta)}. \qquad (10)$$

To relate α and θ we apply the sine law to the triangle in Fig. 1

$$\frac{R}{\sin\,\alpha} = \frac{r_2}{\sin\,\theta}. \qquad (11)$$

Combining this relation with the fundamental trigonometric identity $\cos^2\,\alpha + \sin^2\,\alpha = 1$, we find after some simple manipulations

$$\cos\,\alpha = \sqrt{1 - \frac{R^2}{r_2^2}\,(1 - \cos^2\,\theta)}. \qquad (12)$$

[2] Recall that the electrostatic energy associated with a uniformly charged spherical shell whose radius is a and total charge is q is given by

$$\frac{1}{2}\,\frac{q^2}{4\pi\epsilon_0 a}.$$

Now we take Eq. (8) into the equation above and after some simplifications we get

$$1-\frac{R^2}{r_2^2}\left(1-\cos^2\,\theta\right)=\frac{\left(r_1-R\cos\,\theta\right)^2}{r_1^2+R^2-2r_1R\,\cos\,\theta}. \quad (13)$$

It follows that

$$\cos\,\alpha=\frac{r_1-R\cos\,\theta}{\sqrt{r_1^2+R^2-2r_1R\,\cos\,\theta}}, \quad (14)$$

and

$$\mathbf{E}_1\cdot\mathbf{E_2}=\frac{e^2}{(4\pi\epsilon_0)^2}\,\frac{r_1-R\cos\,\theta}{r_1^2\left(r_1^2+R^2-2r_1R\,\cos\,\theta\right)^{3/2}}. \quad (15)$$

Therefore we must now compute

$$U_{12}=\epsilon_0\,\times\,\frac{e^2}{(4\pi\epsilon_0)^2}\int_0^\infty r_1^2\,dr_1\,\times \int_\Omega\frac{r_1-R\cos\,\theta}{r_1^2\left(r_1^2+R^2-2r_1R\,\cos\,\theta\right)^{3/2}}\,d\Omega, \quad (16)$$

where $d\Omega=\sin\,\theta\,d\theta\,d\phi$. The integration over the azimuthal angle is trivial and yields a factor 2π, and upon introducing the variable $\xi=\cos\,\theta$ we have

$$U_{12}=\frac{e^2}{8\pi\epsilon_0}\int_0^\infty f(r_1)\,dr_1, \quad (17)$$

where we have defined

$$f(r_1)=\int_{-1}^{+1}\frac{r_1-R\,\xi}{\left(r_1^2+R^2-2r_1R\,\xi\right)^{3/2}}\,d\xi. \quad (18)$$

This integral can evaluated straightforwardly and because $R>0$, $r_1>0$, we can write the result as

$$f(r_1)=\frac{|R-r_1|\,(R+r_1)-(R+r_1)\,(R-r_1)}{|R-r_1|\,(R+r_1)\,r_1^2}. \quad (19)$$

To proceeed we must consider two cases.

Case $r_1<R$. In this case it is easily seen that

$$f(r_1)=0. \quad (20)$$

This means that inside of an imaginary sphere of radius equal to R, the interaction energy of the two charges is zero.

Case $r_1>R$. In this case, Eq. (19) yields

$$f(r_1)=\frac{2}{r_1^2}. \quad (21)$$

Taking this result into Eq. (17) we obtain the expected result

$$U_{12}=\frac{e^2}{4\pi\epsilon_0}\int_R^\infty\frac{dr_1}{r_1^2}=\frac{e^2}{4\pi\epsilon_0\,R}. \quad (22)$$

Notice that if we set the upper limit equal to $10R$, then a simple calculation shows that

$$U'_{12}=\frac{e^2}{4\pi\epsilon_0}\int_R^{10R}\frac{dr_1}{r_1^2}=\frac{9}{10}\frac{e^2}{4\pi\epsilon_0\,R}, \quad (23)$$

that is, as stated in Ref. [2], 90% of energy is contained between two spheres, one of radius R and the other one of radius $10R$. If the charges are not identical, all we have to do is replace e^2 by q_1q_2.

3. The energy density distribution

Though the final result is the one we expected the way it was obtained reveals some details that are somewhat surprising, to wit, the part of the field energy that corresponds to the interaction energy of the two point charges comes from the region $r>R$. The region $r<R$ makes no contribution at all. This conclusion agrees with Ref. [2]. How can this physically be?

To answer this question we must first realize that the electrostatic interaction energy density of the system is essentially given by the dot product of the fields. In the case of two positive identical charges it is not difficult to see that the angle between the fields, let us denote it by α as before, is obtuse near the charges and acute far away from them, see Fig. 2. In fact, the negative contributions comes from a spherical region of radius equal to $R/2$ centered at the midpoint between the two charges, see Fig. 2. The positive contribution comes from the rest. The volume of the spherical region is smaller than the volume of the rest, but the fields are more intense near the charges than far away from them. Therefore, we conclude that in the end there is a cancellation between the corresponding contributions and this is reason why the entire region $r<R$ makes no contribution at all to the final result. The electrostatic field energy of this system comes from the region $R<r<\infty$.

Inside the region $r<R$ there is a surface that separates the negative energy density region from the positve one. On this surface the fields are perpendicular to each other and the energy density is null. This can be seen from Eq. (15). The dot product between the fields is zero if and only if

$$r_1=R\,\cos\,\theta, \quad (24)$$

but if we inspect Fig. 2 this relation is a consequence of Thales' theorem that states that any triangle inscribed in a semicircle is a right triangle which is the case of the triangle formed by the three segments of line whose lengths are r_1, r_2, and R. It follows easily that on the

spherical surface of radius equal to $R/2$ the fields are perpendicular to each other and consequently the interaction energy is zero.

4. The ratio between negative and positive energy

Let us now evaluate the ratio between the negative and the positive field interaction energy. The hard part is the evaluation of the contribution of the negative energy. In order to perform this calculation some geometrical transformations will have to be made. Consider Fig. 3. In order to shift the origin of the coordinate system to the midpoint between the charges we introduce the position vectors of the charges with respect to the midpoint, $\mathbf{x}_1$ and $\mathbf{x}_2$, such that $\mathbf{x}_1 + \mathbf{x}_2 = 0$ and $\|\mathbf{x}_1\| = \|\mathbf{x}_2\| = R/2$. We introduce also the position vector $\mathbf{X}$ of an arbitrary point P with respect to the midpoint. Notice that the magnitude $\|\mathbf{X}\| = X$ of this vector lies in the interval $0 \leq \mathbf{X} \leq R/2$. The new polar angle is θ' and the following vector relations are easily seen to hold

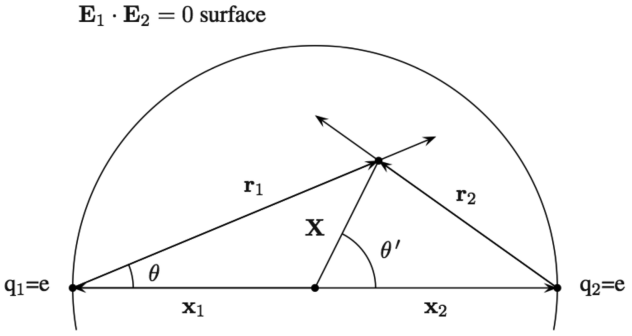

Figure 3 - Geometry for the evaluation of the negative energy density contribution.

$$\mathbf{r}_1 = \mathbf{x}_2 + \mathbf{X}; \qquad \mathbf{r}_2 = -\mathbf{x}_2 + \mathbf{X}. \tag{25}$$

The interaction energy as before depends on the dot product

$$\mathbf{E}_1 \cdot \mathbf{E_2} = \frac{e^2}{(4\pi\epsilon_0)^2 r_1^2 r_2^2}\, \mathbf{e}_{r_1} \cdot \mathbf{e}_{r_2} = \frac{e^2}{(4\pi\epsilon_0)^2 r_1^2 r_2^2}\, \frac{\mathbf{r}_1}{r_1} \cdot \frac{\mathbf{r}_2}{r_2}. \tag{26}$$

From the vector relations above it follows that

$$r_1^2 = \frac{R^2}{4} + X^2 + RX\, \cos\theta', \tag{27}$$

and

$$r_2^2 = \frac{R^2}{4} + X^2 - RX\, \cos\theta'. \tag{28}$$

We also have

$$\frac{\mathbf{r}_1}{r_1} \cdot \frac{\mathbf{r}_2}{r_2} = \frac{X^2 - \dfrac{R^2}{4}}{r_1 r_2}. \tag{29}$$

Defining the dimensionless variable

$$u := \frac{X}{R}, \qquad 0 \leq u \leq 1, \tag{30}$$

the field energy content of this region A_{12} can be written as

$$A_{12} = U_{12} \int_0^{1/2} du\, u^2 \left(u^2 - \frac{1}{4}\right) \times \int_0^1 \frac{d\xi}{\left(\frac{1}{4} + u^2 + u\,\xi\right)^{3/2} \left(\frac{1}{4} + u^2 - u\,\xi\right)^{3/2}}, \tag{31}$$

where $\xi := \cos\theta'$. The integral in ξ is

$$\int_0^1 \frac{d\xi}{\left(\frac{1}{4} + u^2 + u\,\xi\right)^{3/2} \left(\frac{1}{4} + u^2 - u\,\xi\right)^{3/2}} = \frac{64}{\|2u-1\|\,(2u+1)\,(16u^4 + 8u^2 + 1)}\,, \tag{32}$$

and after performing the integral in u we obtain

$$A_{12} = -U_{12}\, \frac{\pi - 2}{4}. \tag{33}$$

In order to get zero energy inside the spherical region of radius R centered at one of the charges we must have an equal amount of a positive contribution B_{12}

$$B_{12} = +U_{12}\, \frac{\pi - 2}{4}. \tag{34}$$

Therefore the ratio of the negative energy to the positive energy is

$$\frac{A_{12}}{U_{12} + B_{12}} = -\left(\frac{\pi - 2}{\pi + 2}\right) \approx -0.2220. \tag{35}$$

This means that the negative energy content is considerable less than the positive energy one.

5. Final remarks

To conclude let us call the reader's attention to two points. The first one is that if the point charges have opposite algebraic signs then as before there will still be a sphere centered at one of the charges of radius equal to their separation inside of which the total content of energy is zero, but the energy contained in the smaller sphere of radius $R/2$ will be positive and the rest of the energy will be negative. This can be easily seen by sketching the dot product $\mathbf{E}_1 \cdot \mathbf{E}_2$.

The second one is that all calculations done for the configuration considered here – *mutatis muntandis* – apply to the corresponding gravitational case. The content of energy stored in the gravitational field is given by

$$U = -\frac{1}{8\pi G} \iiint \mathbf{g}^2(P)\, dV, \tag{36}$$

where $\mathbf{g}(P)$ is the resultant field at a point P. For a gravitational configuration similar to the electrostatic one considered here $\mathbf{g}(P) = \mathbf{g}_1(P) + \mathbf{g}_2(P)$, and depending on the model we choose for the mass distribution we will not need to deal with infinities due to self-energies.The same can be said about extended charge distributions.

As a final remark, we would like to emphasize that the regularization and renormalization of the divergent self-energy of a point charge is an important problem in classical and quantum electrodynamics, and quantum fields in general. For a pedagogical introduction to renormalization in a classical context see Refs. [3, 4]. The reader interested in the quantum aspects of the problem will find plenty of references in the literature. An introduction to its general aspects can be found in Ref. [5]. More technical approaches can be found in Ref. [6]. See also the pioneering work of M. Schönberg in Ref. [7].

Acknowledgments

The author is grateful to the referee and to C. Farina for their helpful comments.

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