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An algorithm for the automatic design of concrete shell reinforcement

Um algoritmo para o dimensionamento automático de cascas em concreto armado



A. B. COLOMBO [a]
abcolombo@gmail.com

J. C. DELLA BELLA [a]
dbella@usp.br

T. N. BITTENCOURT [a]
tbitten@gmail.com

Abstract

An algorithm developed for the design of reinforcement in concrete shells is presented in this text. The formulation and theory behind the development is shown, as well as results showing its robustness and capability of application on fairly large-scale structures. The design method is based on the three-layer model for reinforced concrete shell elements. A material model is also proposed in order to improve the numerical stability of the algorithm. Comparisons of single element design show that the modifications made to the material model don't effect significantly the final results while making for better numerical stability.

Keywords: reinforced concrete; shell elements; plate elements; structural design.

Resumo

Um algoritmo desenvolvido para o dimensionamento de armaduras para cascas de concreto armado é apresentado neste artigo. A formulação e aspectos teóricos que fundamentam o método são apresentados assim como, os resultados que mostram a robustez e capacidade de aplicação do algoritmo em estruturas de grande porte. O método de dimensionamento é baseada no modelo das três chapas para elementos de casca em concreto armado. Um modelo constitutivo é proposto para obter melhor estabilidade numérica no algoritmo. Comparações feitas do dimensionamento de um único elemento mostram que as modificações do modelo constitutivo não apresentam mudanças significantes nos resultados enquanto proporcionam melhor estabilidade numérica.

Palavras-chave: concreto armado, dimensionamento, cascas, chapas.

[a] *Departamento de Estruturas e Geotécnica, Escola Politécnica da Universidade de São Paulo, SP, Brasil.*

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1. Introduction

Shell type structural elements are used to model a great number of reinforced concrete structures, these elements can be found in structures such as nuclear power plants, offshore structures, and tunnel linings. This document focuses on finding the necessary reinforcement for a shell or plate element subjected to membrane forces N_x, N_y, N_{xy} and flexural forces M_x, M_y and M_{xy}. In the last decades many researchers (Baumann [1], Brodum-Nielsen [2] and Gupta [3]) have proposed solutions for this type of problem. The basic idea behind these solutions is that the forces and moments are resisted by the resultant tensile forces of the reinforcement and the resultant compressive forces of the concrete blocks.

More complex models, which focus on the behavioral analysis of reinforced shell elements, have also been developed (Scordelis [4], Hu [5], Cervera [6], Polak [7], Wang [8], Liu [9], Schulz [10] and Hara [11]). These formulations are fundamental for the development of new design techniques, on the other hand the complexity of the material models and analysis procedures involved in these, make them very difficult to apply in practical design situations.

For these situations a simpler procedure is more favorable. Gupta [3] presented a general solution procedure and an automatic solution algorithm based on it and on the CEB [12] formulation was presented by Lourenço [13], these authors proposed a general method of solution, including cases where there is no need for reinforcement. Tomás [14] used optimization techniques to design elements using this formulation. More recently Fall [15] suggested the same algorithm to reinforce tailor-made concrete structures. The algorithm presented here is proposed as an alternative to the one presented by Lourenço [13], it diverges on the algorithm structure and some modifications were made to the material model adopted.

2. Formulation

This section will discuss the basic theoretical concepts necessary in order to comprehend the proposed algorithm.

2.1 Three-layer model

Shells are two-dimensional elements that are subjected to combined membrane and plate load components. Generally the state of internal stresses in the shell can be described in function of eight resultant force components shown in Figure 1. An idealized shell composing of three layers is proposed by CEB [12], in this idealization the outer layers resist the bending moments and membrane forces acting on the shell and the inner layer resists the transverse shear.

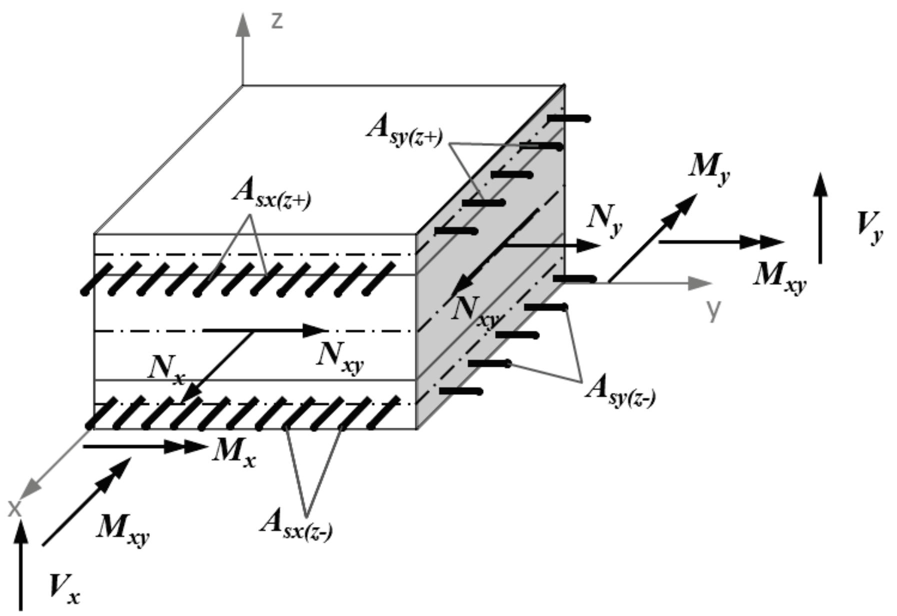
The membrane forces for the outer layers of the model can be found by applying equilibrium between the shell's membrane forces and bending moments and the membrane forces for the outer layers, as shown in Figure 2. Equations (1) - (6) are obtained through this equilibrium, in these equations, $h_{(z+)}$, $h_{(z-)}$, $h_{sx(z+)}$, $h_{sy(z+)}$, $h_{sx(z-)}$ and $h_{sy(z-)}$ are the indicated dimensions in Figure 3.

$$N_{x(z+)} = \frac{N_x h_{(z-)} + M_x}{(h_{(z+)} + h_{(z-)})} \quad (1)$$

$$N_{y(z+)} = \frac{N_y h_{(z-)} + M_y}{(h_{(z+)} + h_{(z-)})} \quad (2)$$

$$N_{xy(z+)} = \frac{N_{xy} h_{(z-)} + M_{xy}}{(h_{(z+)} + h_{(z-)})} \quad (3)$$

Figure 1 – Reinforced concrete shell element



$$N_{x(z-)} = \frac{N_x h_{(z+)} - M_x}{(h_{(z+)} + h_{(z-)})} \quad (4)$$

$$N_{y(z-)} = \frac{N_y h_{(z+)} - M_y}{(h_{(z+)} + h_{(z-)})} \quad (5)$$

$$N_{xy(z-)} = \frac{N_{xy} h_{(z+)} - M_{xy}}{(h_{(z+)} + h_{(z-)})} \quad (6)$$

As shown in Figure 2, the three-layer model can be thought of as being comprised of two membrane elements that resist the acting forces of the shell. A problem with this idealization is that, for this to be true, the reinforcement must always be in the mid-plane of the membranes, which is often not the case. This issue is overcome by correcting the reinforcement values

Figure 2 – Equilibrium between membrane forces in the outer layers and the shell active forces

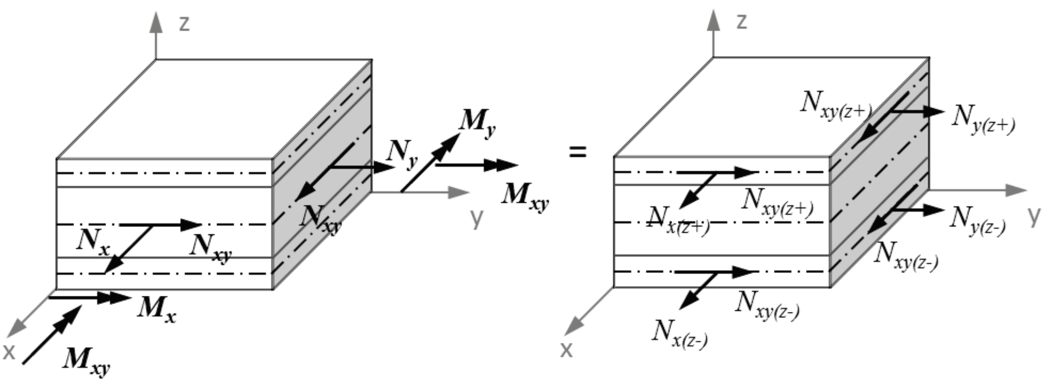


Figure 3 – Dimensions used in the three-layer model formulation

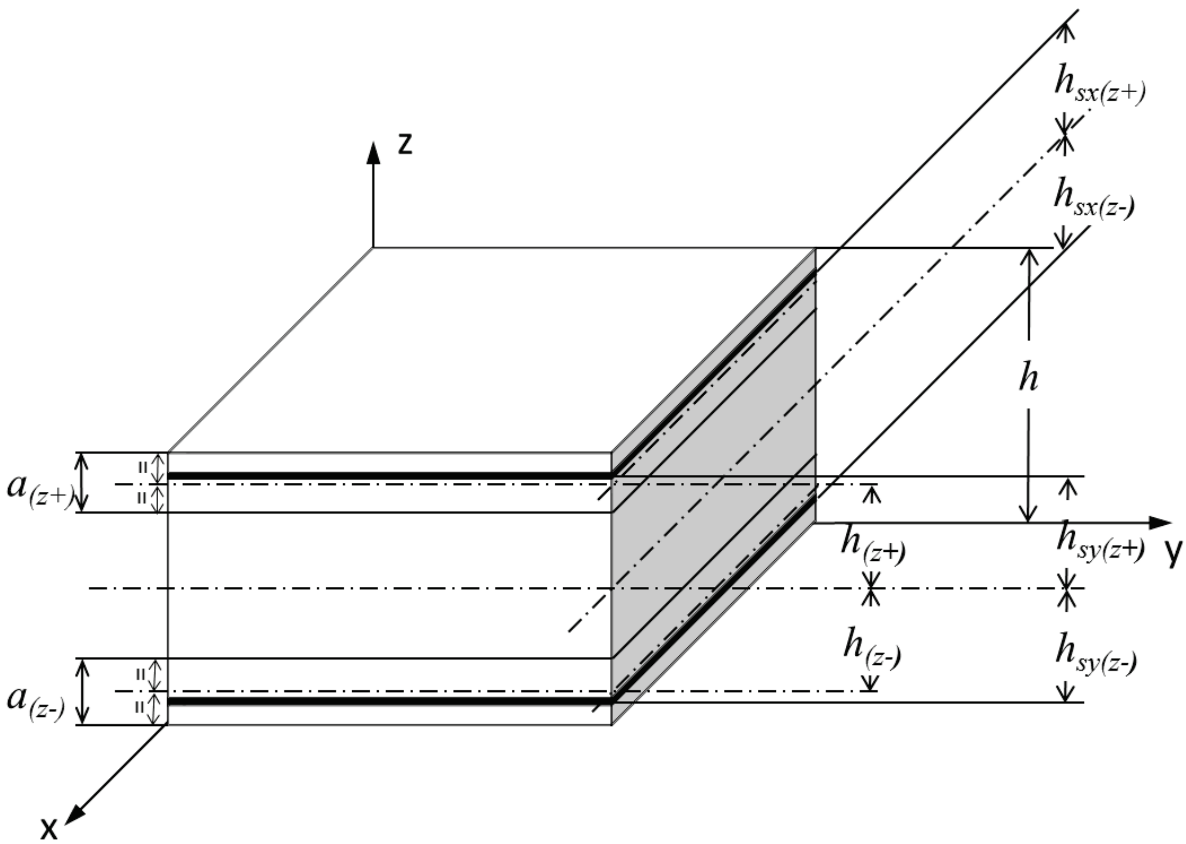
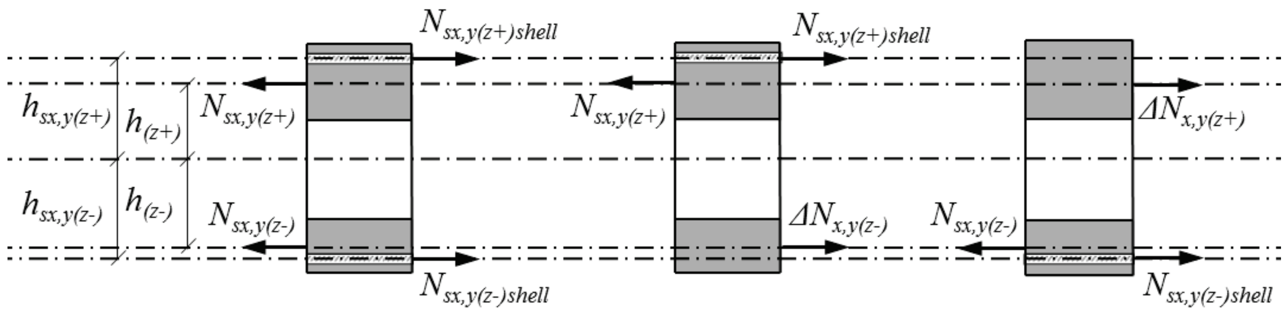


Figure 4 – Equilibrium conditions used to find the correct reinfocement force



and membrane forces in order to account for the real reinforcement depth.

The real forces acting on the reinforcement can be found by imposing equilibrium between the calculated values at the mid-plane of the membrane and the values at the real position of the reinforcement, as shown in Figure 4, this provides the following equations for the real reinforcement values.

$$N_{sx,y(z+)shell} = \frac{N_{sx,y(z+)} \cdot \left(h_{sx,y(z-)} + h_{(z+)}\right) + N_{sx,y(z-)} \cdot \left(h_{sx,y(z-)} - h_{(z-)}\right)}{\left(h_{sx,y(z-)} + h_{sx,y(z+)}\right)} \quad (7)$$

$$N_{sx,y(z-)\,shell} = N_{sx,y(z+)} + N_{sx,y(z-)} - N_{sx,y(z+)\,shell} \quad (8)$$

In the case where there is no membrane reinforcement in one of the layers, it's possible to find a correction value for the membrane forces as shown in Figure 4, based on this we find the following equations.

$$N_{sx,y(z+)shell} = \frac{N_{sx,y(z+)} \cdot \left(h_{(z+)} + h_{(z-)}\right)}{\left(h_{(z-)} + h_{sx,y(x-)}\right)} \quad (9)$$

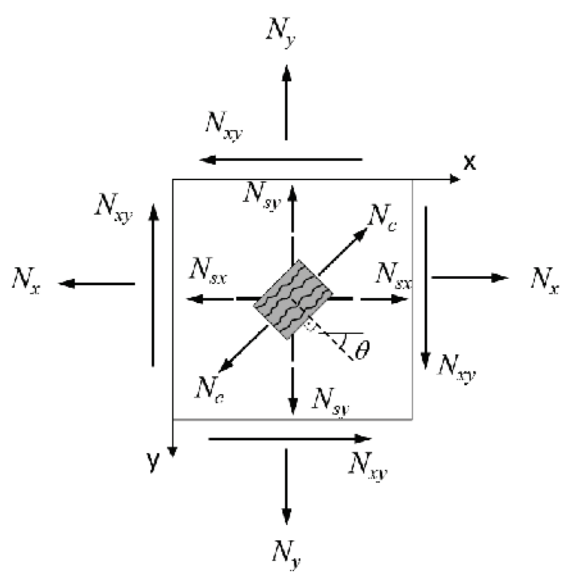
$$\Delta N_{x,y(z-)} = N_{sx,y(z+)} - N_{sx,y(z+)shell} \quad (10)$$

These equations are an important part of the proposed algorithm because the correction factor Δ*N* changes the membrane forces in a given layer, making it necessary to find new reinforcement values for this layer. These new values need to be corrected again to account for the real reinforcement position, creating in some cases an iterative process, this will be discussed in a latter part of this text.

2.2 Reinforcement for membrane elements

As shown in the previous item, determining the necessary reinforcement for membrane elements is an essential part of the solution for the three-layer model. Reinforcement in membrane elements has been studied by many authors including Baumann [1], Brodum-Nielsen [2] and Gupta [16]. The design method proposed by the structural code CEB [12] incorporates the main ideas proposed by these authors. The formulation shown below is derived for a generic membrane element shown in Figure 5, of thickness *a*. These equations can be applied to both the top (z+) and bottom (z-) membranes. Gupta [16] shows that by using the principal of minimum resistance it's possible obtain the following governing equations for the

Figure 5 – Reinfoced concrete membrane element



mechanical behavior of membrane elements with reinforcement in two orthogonal directions.

$$N_{sx} = N_x + N_{xy} \cdot \tan(\theta) \quad (13)$$

$$N_{sy} = N_y + N_{xy} \cdot \cot(\theta) \quad (14)$$

$$N_c = -\frac{N_{xy}}{\sin(\theta) \cdot \cos(\theta)} \quad (15)$$

The optimal design is obtained for θ = 45°, so long as $N_{sx} > 0$ and $N_{sy} > 0$, meaning that the reinforcement is needed in both x and y directions. This case will be referred to as Case I, applying the θ value above to eq. (13)-(15) yields the following equations.

$$N_{sx} = N_x + |N_{xy}| \quad (16)$$

$$N_{sy} = N_y + |N_{xy}| \quad (17)$$

$$N_c = -2 \cdot |N_{xy}| \quad (18)$$

Where N_{sx} and N_{sy} are the force by unit length in the reinforcement in the x and y directions, N_c is the force per unit length, acting on the concrete parallel to the crack. If eq.(16) gives a negative value for N_{sx}, setting $N_{sx} = 0$ to this equation gives the following equations which constitute Case II.

$$N_{sx} = 0 \quad (19)$$

$$N_{sy} = N_y - \frac{N_{xy}^2}{N_x} \quad (20)$$

$$N_c = N_x + \frac{N_{xy}^2}{N_x} \quad (21)$$

$$\tan(\theta) = -\frac{N_x}{N_{xy}} \quad (22)$$

In a similar form if eq. (14) yields a negative value for N_{sy}, its' possible to obtain the following equations for Case III.

$$N_{sx} = N_x - \frac{N_{xy}^2}{N_y} \quad (23)$$

$$N_{sy} = 0 \quad (24)$$

$$N_c = N_y + \frac{N_{xy}^2}{N_y} \quad (25)$$

$$\tan(\theta) = -\frac{N_{xy}}{N_y} \quad (26)$$

If eq. (20) or (23) result in a negative value no reinforcement is needed. This case will be referred to as Case IV. The concrete force N_c in this case is the minimum principal force N_{c2}, also the maximum principal stress N_{c1} has to be less than or equal to zero. The equations for the principal forces acting on the membrane are stated below.

$$N_{c1} = \frac{N_x + N_y}{2} + \sqrt{\left(\frac{N_x - N_y}{2}\right)^2 + N_{xy}^2} \quad (27)$$

$$N_{c2} = \frac{N_x + N_y}{2} - \sqrt{\left(\frac{N_x - N_y}{2}\right)^2 + N_{xy}^2} \quad (28)$$

The necessary reinforcement area per unit length can be found by dividing the reinforcement forces by the reinforcement tension.

$$A_{sx} = \frac{N_{sx}}{f_{yd}} \quad (29)$$

$$A_{sy} = \frac{N_{sy}}{f_{yd}} \quad (30)$$

When working with the three-layer model, the solution to the necessary reinforcement problem is finding the thickness of the outer layers. This thickness is found by dividing the compression force acting parallel to the crack direction of the concrete membrane N_c by a limit stress, the evaluation of this limit stress will be described in the next item. Eq. (31) may be used to find the thickness of the layer.

$$a = \frac{N_c}{f_c} \quad (31)$$

2.3 Material properties

Due to the high complexity of concrete a full description of the material behavior would imply in the use of a great amount of variables. For design purposes CEB [12] suggests values for average concrete strength based on the state of cracking of the structural element. In uncracked zones, the average concrete strength is given by f_{cd1} in eq.(32).

$$f_{cd1} = 0.85\left[1 - \frac{f_{ck}}{250}\right] f_{cd} \quad (32)$$

For concrete subjected to biaxial compression, f_{cd1} may be increased by the factor K given below,

$$K = \frac{1 + 3.80\alpha}{(1+\alpha)^2} \quad (33)$$

where $\alpha = \sigma_1/\sigma_2$ and σ_1 and σ_2 are the principal stresses at failure. For cracked zones the average strength is given by f_{cd2} in eq. (34).

$$f_{cd2} = 0.6\left[1 - \frac{f_{ck}}{250}\right] f_{cd} \quad (34)$$

The simplicity of the above model makes it ideal for practical use in structural design. A more complex model that represents the behavior of concrete in a better manner was presented by Vecchio [17]. This model was based on experimental results from reinforced concrete panels. In this model the maximum compressive strength decreases as the maximum tensile strain ε_1 increases, this compression softening equation is shown in eq. (35).

$$f_{c\max} = \frac{f_{ck}}{0.8 - 0.34 \cdot (\varepsilon_1/\varepsilon_{cp})} \leq 1.0 \quad (35)$$

The material model proposed in this paper adapts the above expression to interpolate between the values given by the CEB. In this model as ε_1 increases, the compressive strength is reduced from the value given by eq. (32) down to a minimum value given by expression (34). When the element is subjected to biaxial compression the increase in the concrete strength given by eq. (33) is considered. Equations (36)-(38) give a mathematical representation of the proposed model.

$$\beta = \frac{1.0}{0.8 - 0.34 \cdot (\varepsilon_1/\varepsilon_{cp})} \quad , \quad \left(0.6/0.85\right) \leq \beta \leq 1.0 \quad (36)$$

$$\text{Cases I - III} \quad f_c = \beta \cdot f_{cd1} \quad (37)$$

$$\text{Case IV} \quad f_c = K \; f_{cd1} \quad (38)$$

In order to use this model, it is necessary to be able to estimate ε_1 at failure for a membrane element. The study of the state of strain in membrane elements had great contributions by Gupta [16], the author presents equations (39) and (40) where the strains in the x and y reinforcements are related to the principal strains, ε_1 and ε_2, and the crack angle, θ. For more information on the development of these equations the reader may refer to Chen [18].

$$\varepsilon_x = \varepsilon_1 \cdot \cos^2(\theta) + \varepsilon_2 \cdot \sin^2(\theta) \quad (39)$$

$$\varepsilon_y = \varepsilon_1 \cdot \sin^2(\theta) + \varepsilon_2 \cdot \cos^2(\theta) \quad (40)$$

Using these equations it's possible to find ε_1 by estimating values for ε_2 and ε_{sx} or ε_{sy}. Expressions for ε_1 will be developed for the four cases of membrane reinforcement shown in item 2.2. For Case I where the reinforcement is needed in both the directions, setting $\varepsilon_2 = \varepsilon_{cp}$ (where ε_{cp} is the concrete peak compression strain), $\varepsilon_{sx} = \varepsilon_{yi}$ (where ε_{yi} is the steel yield strain), θ = 45°, and substituting these values in eq. (39) yields eq. (41). The same result is obtained using a similar approach with eq. (40).

$$\varepsilon_1 = 2 \cdot \left(\varepsilon_{yi} - 0.5 \cdot (\varepsilon_{cp})\right) \quad (41)$$

For Case II setting $\varepsilon_2 = \varepsilon_{cp}$ and $\varepsilon_{sx} = \varepsilon_{yi}$ in eq. (39) yields eq. (42). Working in a similar form with Case III we obtain eq.(43) from eq. (40). The ε_1 value in the equations below can be obtained from equations (22) and (26) for Case II and Case III, respectively.

$$\varepsilon_1 = \frac{\left(\varepsilon_{yi} - \varepsilon_{cp} \cdot \sin(\theta)^2\right)}{\cos(\theta)^2} \quad (42)$$

$$\varepsilon_1 = \frac{\left(\varepsilon_{yi} - \varepsilon_{cp} \cdot \cos(\theta)^2\right)}{\sin(\theta)^2} \quad (43)$$

Finally for Case IV ε_1 is set to zero, this is done in order to obtain concrete strength value equal to f_{cd1}.

3. Algorithms

The main objective of this procedure is finding the thickness of the outer layers of the three-layer model. In Appendix A three algorithms are presented, Algorithm 1 is the main algorithm and it calls the other two algorithms.

Algorithm 1 establishes an initial value for $a_{(z+)}$ and $a_{(z-)}$, calculates membrane forces acting on the outer layers and uses Algorithm 2 to evaluate the reinforcement forces for the outer membranes. These initial estimates for the reinforcement forces are inputted into Algorithm 3, this procedure reevaluates the membrane forces and reinforcement forces to take into account the difference between the position of the mid-plane of the outer layers and the actual reinforcement position. Using the values of $N_{c(z+)}$ and $N_{c(z-)}$, new thickness values $a_{(z+)}$ and $a_{(z-)}$ are obtained. This procedure is repeated until the thickness values converge.

To better illustrate the algorithms a numerical example is given in Appendix B. A complete iteration for element 3 from Table 2 is shown in the appendix.

4. Results

A computer routine that implements the algorithm shown in the previous item was developed using the Java programming language. Elements A and B, described below, were processed by Lourenço [13]. Comparisons between the results presented by these authors with the results obtained by the algorithm proposed here are shown in Table 1.

Element A

$$N_x = -20tf/m \quad N_y = 30tf/m \quad N_{xy} = 7.5tf/m$$
$$M_x = -6tf/m \quad M_y = 4tf/m \quad M_{xy} = -2tf/m$$
$$h = 0.2m \quad h_{sx(z+)} = h_{sy(z+)} = h_{sx(z-)} = h_{sy(z-)} = 0.08m$$
$$f_{cd} = 13.3MPa \quad f_{syd} = 348MPa$$

Element B

$$N_x = -20tf/m \quad N_y = 30tf/m \quad N_{xy} = 7.5tf/m$$
$$M_x = 6tf/m \quad M_y = 4tf/m \quad M_{xy} = -2tf/m$$
$$h = 0.2m \quad h_{sx(z+)} = h_{sy(z+)} = h_{sx(z-)} = h_{sy(z-)} = 0.08m$$
$$f_{cd} = 13.3MPa \quad f_{syd} = 348MPa$$

With the computer program developed it is also possible to find reinforcement values for processed finite element models. The program was used to design the reinforcement for a model of a subway station. The contour maps in Figure 6 show the reinforcement results for this model, in order to present numerical results for this model, some elements were chosen (see Figure 7) and the results for these elements are shown in Table 2.

5. Conclusions

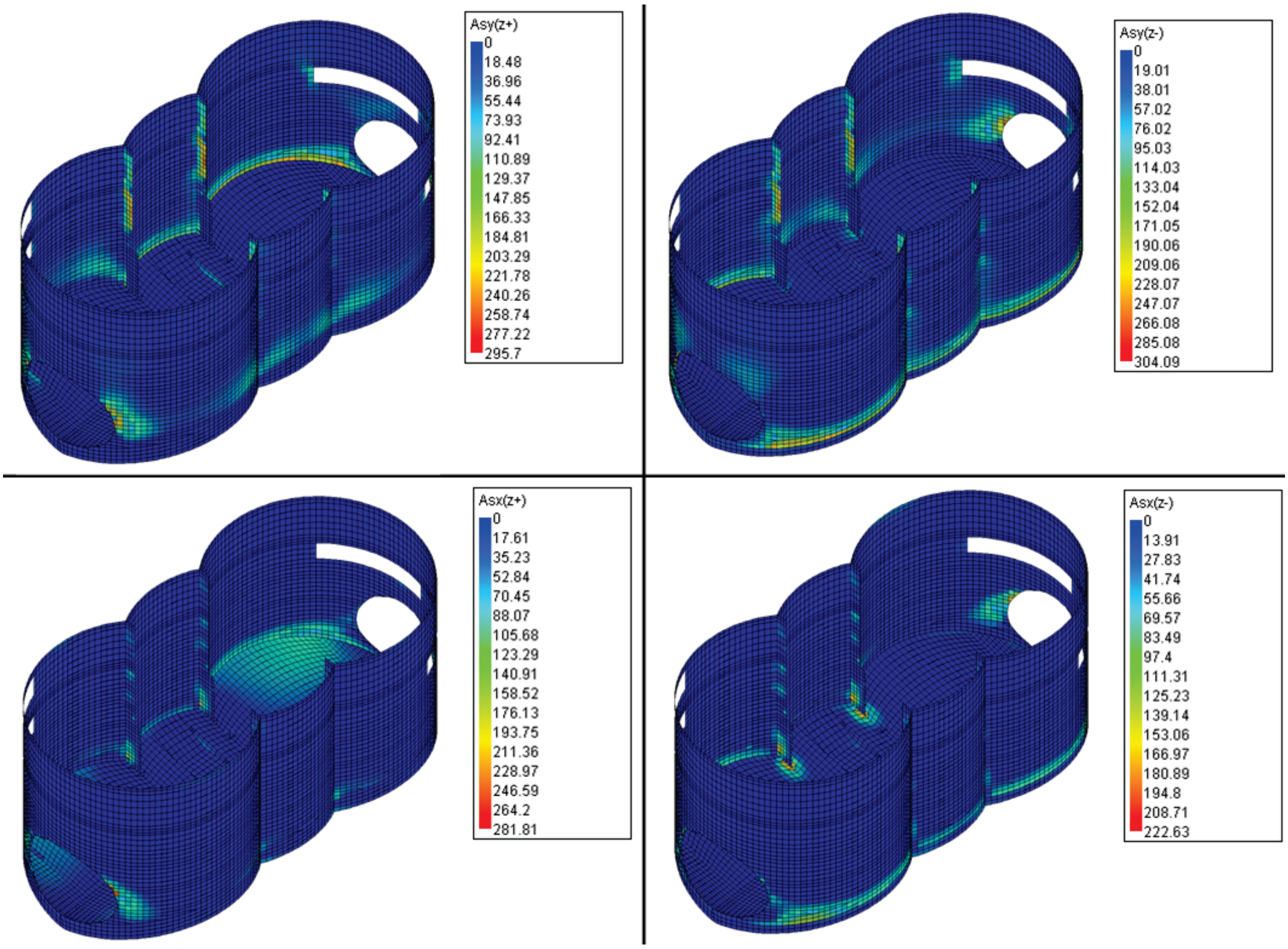
Shell-type elements can be used to model a large number of structures, for a designer, being able to determine the necessary reinforcement and check the concrete tension for these elements is fundamental. The three-layer model is a simplified model that has been adopted by structural codes such as the CEB Model Code 1990 [12]. The fundamental concept of this model is that internal forces of two outer membranes resist the shell's active forces. The presented procedure calculates trough an iterative method the thickness of these outer membranes, and therefore the necessary reinforcement.

The material model presented incorporates aspects of the CEB model and the compression softening model by Vecchio [17]. This was done in order to improve convergence of the algorithm, since the discontinuity introduced by the CEB model when the material goes from an uncracked state to a cracked state caused numerical difficulties. Using the compression softening equation it was possible to introduce some continuity to the material model which resulted in a much more stable behavior.

Results on Table 1 show a comparison between this algorithm and the one presented by Lourenço [13]. Both results are in equilibrium with the applied forces and the reinforcement values are consistent. For practical use in engineering the two methods yield basically the same results.

Overall this algorithm has proven to be reliable and to give consistent results. The material model developed reduced numerical

Figure 6 – Reinfocement area results for finite element model (Units: cm²/m)



problems significantly but there is still plenty of room for improvement. Another aspect that should be mentioned is the lack of displacement compatibility in the shell formulation. These issues should be studied in future works on the area.

6. References

[01] T. Baumann, "Zur Frage der Netzbewehrung von Flächentragwerken," Der Bauingenieur, vol. 47, no. 10, pp. 367-377, 1972.

[02] T. Brodum-Nielsen, "Optimum Design of Reinforced Concrete Shells and Slabs," Copenhagen, 1974.

[03] A. K. Gupta, "Combined Membrane and Flexural Reinforcement in Plates and Shells," Journal of Structural Engineering - ASCE, vol. 112, no. 3, pp. 550-557, March 1985.

[04] A. C. Scordelis and E. C. Chan, "Nonlinear Analysis of Reinforced Concrete Shells," Computer Applications in Concrete Technology, SP-98, pp. 25-57, 1987.

[05] H. T. Hu and W. C. Schnobrich, "Nonlinear Finite Element Analysis of Reinforced Concrete Plates and Shells under Monotonic Loading," Computers and Structures, vol. V. 38, no. No. 5/6, pp. 637-651, 1991.

[06] M. Cervera, E. Hinton and O. Hassan, "Nonlinear Analysis of Reinforced Concrete Plate and Shell Structures Using 20-Noded Isoparametric Brick Elements," Computers and Structures, vol. V. 25, no. No. 6, pp. 845-869, 1987.

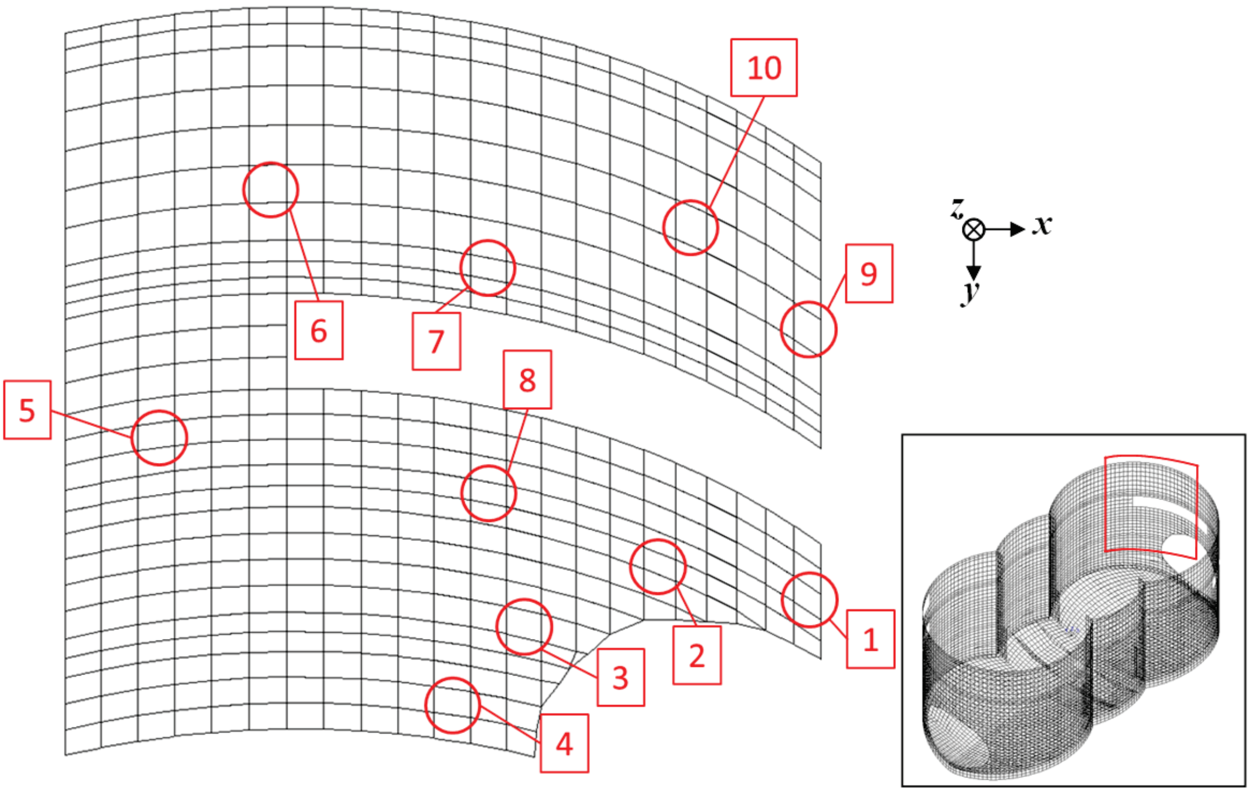
[07] M. A. Polak and F. J. Vecchio, "Reinforced Concrete Shell Elements Subjected to Bending and Membrane Loads," ACI Structural Journal, vol. V.91, no. No.3, pp. 261-268, May-June 1994.

[08] W. Wang and S. Teng, "Finite-Element Analysis of Reinforced Concrete Flat Plate Strucutres by Layered Shell Element," Journal of Structural Engineering - ASCE, vol. 134, no. 12, pp. 1862-1872, December 2008.

[09] Y. Liu and S. Teng, "Nonlinear Analysis of Reinforced Concrete Slabs Using Nonlayered Shell Element," Journal of Structural Engineering - ASCE, vol. 134, no. 7, pp. 1092-1100, July 2008.

[10] M. Schulz and M. P. S. d'Avila, "Analysis of Reinforced Concrete Shells with Transverse Shear Forces," Journal of

Figure 7 – Selected elements shown in Table 2



Structural Engineering - ASCE, pp. 837-848, July 2010.

[11] T. Hara, "Application of Computational Technologies to R/C Structural Analysis," Computers and Concrete, vol. 8, no. 1, pp. 97-110, 2011.

[12] Comité Euro-Internacional du Béton, Model Code 1990, London: Thomas Telford, 1990.

[13] P. B. Lourenço and J. A. Figueiras, "Automatic Design of Reinforcement in Concrete Plates and Shells," Engineering Computations, vol. 10, pp. 519-541, 1993.

[14] A. Tomás and P. Martí, "Design of reinforcement for concrete co-planar shell," MECCANICA, pp. 657-669, 2010.

[15] D. Fall, K. Lundgren, R. Rempling and K. Gylltoft, "Reinforcing tailor-made concrete structures: Alternatives and challenges," Engineering Structures, p. 372–378, 2012.

[16] A. K. Gupta, "Membrane Reinforcemnt in Shells," Journal of the Structural Division - ASCE, pp. 41-56, January 1981.

[17] F. J. Vecchio and M. P. Collins, "The Modified Compression-Field Theory for Reinforced Concrete Elements Subjected to

Table 1 – Comparison results with Lourenço & Figueiras (1993)

Element	Algorithm	$A_{sx(z+)}$ (cm²/m)	$A_{sy(z+)}$ (cm²/m)	$A_{sx(z-)}$ (cm²/m)	$A_{sy(z-)}$ (cm²/m)	$\theta_{(z+)}$ (deg)	$\theta_{(z-)}$ (deg)	$a_{(z+)}$ (m)	$a_{(z-)}$ (m)	$f_{c(z+)}$ (MPa)	$f_{c(z-)}$ (MPa)
A	Lourenço et. al.	1.00	12.14	15.14	2.27	-79.6°	45.0°	0.0816	0.0495	7.34	7.34
	Proposed algorithm	0.00	12.10	13.56	1.99	-79.1°	45.0°	0.0584	0.0464	8.88	7.39
	Δ (%)	N/A	-0.35	-10.45	-12.51	-0.65	0.00	-28.42	-6.18	21.0	0.62
B	Lourenço et. al.	10.85	14.19	0.00	0.00	-45.0°	N/A	0.0236	0.0474	57.3	10.40
	Proposed algorithm	11.23	14.34	0.00	0.00	-45.0°	N/A	0.0244	0.0536	47.3	10.46
	Δ (%)	3.51	1.09	N/A	N/A	0.00	N/A	3.42	13.13	90.6	0.61

Shear," ACI Journal, pp. 219-231, 1986.
[18] R. Chen and J. C. Della Bella, "Design of Reinforced Concrete Two-dimensional Structural Elements: Membranes, Plates and Shells," IBRACON Structural Journal, vol. 2, no. 3, pp. 320-344, September 2006.

7. Symbols

$a_{(z+)}$, $a_{(z-)}$ = thickness of the (z+) and (z-) layers (Fig. 3);
f_{ck} = characteristic strength for concrete;
f_{cd1} = average concrete strength for uncracked concrete;
f_{cd2} = average concrete strength for cracked concrete;
f_c = average concrete strength given by material model;
h = shell thickness;
$h_{(z+)}$ = distance from the shell mid-plane to the (z+) layer mid-plane;
$h_{(z-)}$ = distance from the shell mid-plane to the (z-) layer mid-plane;
$h_{sx(z+)}$ = distance from the shell mid-plane to the x direction (z+) reinforcement;
$h_{sy(z+)}$ = distance from the shell mid-plane to the y direction (z+) reinforcement;
$h_{sx(z-)}$ = distance from the shell mid-plane to the x direction (z-) reinforcement;
$h_{sy(z-)}$ = distance from the shell mid-plane to the y direction (z-) reinforcement;
M_x = bending moment per unit length in the x direction (Fig. 1);
M_y = bending moment per unit length in the y direction (Fig. 1);
M_{xy} = twisting moment per unit length (Fig. 1);
N_x = normal force per unit length in the x direction (Fig. 1);
N_y = normal force per unit length in the y direction (Fig. 1);
N_{xy} = shear force per unit length (Fig. 1);
N_{c1} = maximum principal tensile force per unit length;
N_{c2} = minimum principal tensile force per unit length;
$N_{x(z+)}$, $N_{y(z+)}$, $N_{xy\,(z+)}$ = membrane forces per unit length acting in (z+) layer (Fig. 2);
$N_{x(z-)}$, $N_{y(z-)}$, $N_{xy\,(z-)}$ = membrane forces per unit length acting in (z-) layer (Fig. 2);
$N_{sx,y\,(z+)}$ = resisting forces per unit length for the x and y reinforce-

Table 2 – Numerical results from the finite element model

Element	1	2	3	4	5	6	7	8	9	10
N_x (tf/m)	-1494.1	-1063.2	-263.6	-117.1	-648.7	-482.1	-682.2	-872.8	-541.7	-327.5
N_y (tf/m)	-116.2	101.1	276.4	-191.4	-68.6	23.6	-2.6	41.7	-5.3	0.6
N_{xy} (tf/m)	-29.5	57.5	83.6	-371.5	-114.2	30.1	30.4	190.2	10.4	13.1
M_x (tf.m/m)	291.3	32.2	-148.1	-107.8	-95.4	25.0	37.7	-61.5	-0.1	-0.7
M_y (tf.m/m)	36.4	72.4	-209.0	-295.9	-32.0	8.4	7.6	1.2	-1.7	-1.7
M_{xy} (tf.m/m)	-86.5	-64.0	113.3	31.8	56.1	-13.3	-11.5	59.2	2.2	0.6
h (m)	1.30	1.30	1.30	1.30	1.30	0.80	0.80	1.30	0.80	0.55
$h_{sx(z+)}$ (m)	0.400	0.400	0.400	0.400	0.400	0.250	0.250	0.400	0.250	0.125
$h_{sy(z+)}$ (m)	0.450	0.450	0.450	0.450	0.450	0.300	0.300	0.450	0.300	0.175
$h_{sx(z-)}$ (m)	0.400	0.400	0.400	0.400	0.400	0.250	0.250	0.400	0.250	0.125
$h_{sy(z-)}$ (m)	0.450	0.450	0.450	0.450	0.450	0.300	0.300	0.450	0.300	0.175
f_{cd} (MPa)	25	25	25	25	25	25	25	25	25	25
f_{syd} (MPa)	434.8	434.8	434.8	434.8	434.8	434.8	434.8	434.8	434.8	434.8
$A_{sx(z+)}$ (cm^2/m)	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$A_{sy(z+)}$ (cm_2/m)	22.85	30.46	0.00	0.00	0.00	5.97	2.74	16.47	0.00	0.00
$A_{sx(z-)}$ (cm^2/m)	0.00	0.00	16.75	67.95	0.00	0.00	0.00	0.00	0.00	0.00
$A_{sy(z-)}$ (cm^2/m)	0.00	0.00	98.35	99.86	11.61	0.55	0.00	5.01	0.00	1.09
$\theta_{(z+)}$	-53.9°	-85.8°	0.0°	0.0°	0.0°	-88.1°	-88.9°	73.1°	0.0°	0.0°
$\theta_{(z-)}$	0.0°	0.0°	-45.0°	-45.0°	-65.1°	82.9°	0.0°	85.3°	0.0°	88.1°
$f_{c(z+)}$ (MPa)	12.9	15.8	18.3	18.3	18.3	15.9	15.9	15.0	18.3	18.3
$f_{c(z-)}$ (MPa)	18.3	18.3	12.9	12.9	14.0	15.7	18.3	15.8	18.3	15.9
$a_{(z+)}$(m)	0.1837	0.3170	0.1943	0.2588	0.2278	0.1196	0.1630	0.3897	0.1485	0.0891
$a_{(z-)}$(m)	0.7360	0.3176	0.0898	0.3508	0.2014	0.1891	0.2336	0.2143	0.1481	0.1042

ment in (z+) layer (Fig. 2);

$N_{sx,y\ (z-)}$ = resisting forces per unit length for the x and y reinforcement in (z-) layer (Fig. 2);

$N_{sx,y\ (z+)shell}$ = resisting forces per unit length for the x and y reinforcement in the real reinforcement position in the (z+) layer (Fig. 4);

$N_{sx,y\ (z-)shell}$ = resisting forces per unit length for the x and y reinforcement in the real reinforcement position in the (z-) layer (Fig. 4);

$\Delta N_{x,y(z+)}$ = correction force per unit length for the (z+) layer (Fig. 4);

$\Delta N_{x,y(z-)}$ = correction force per unit length for the (z-) layer (Fig. 4);

ε_x , ε_y = strain in the x and y directions;

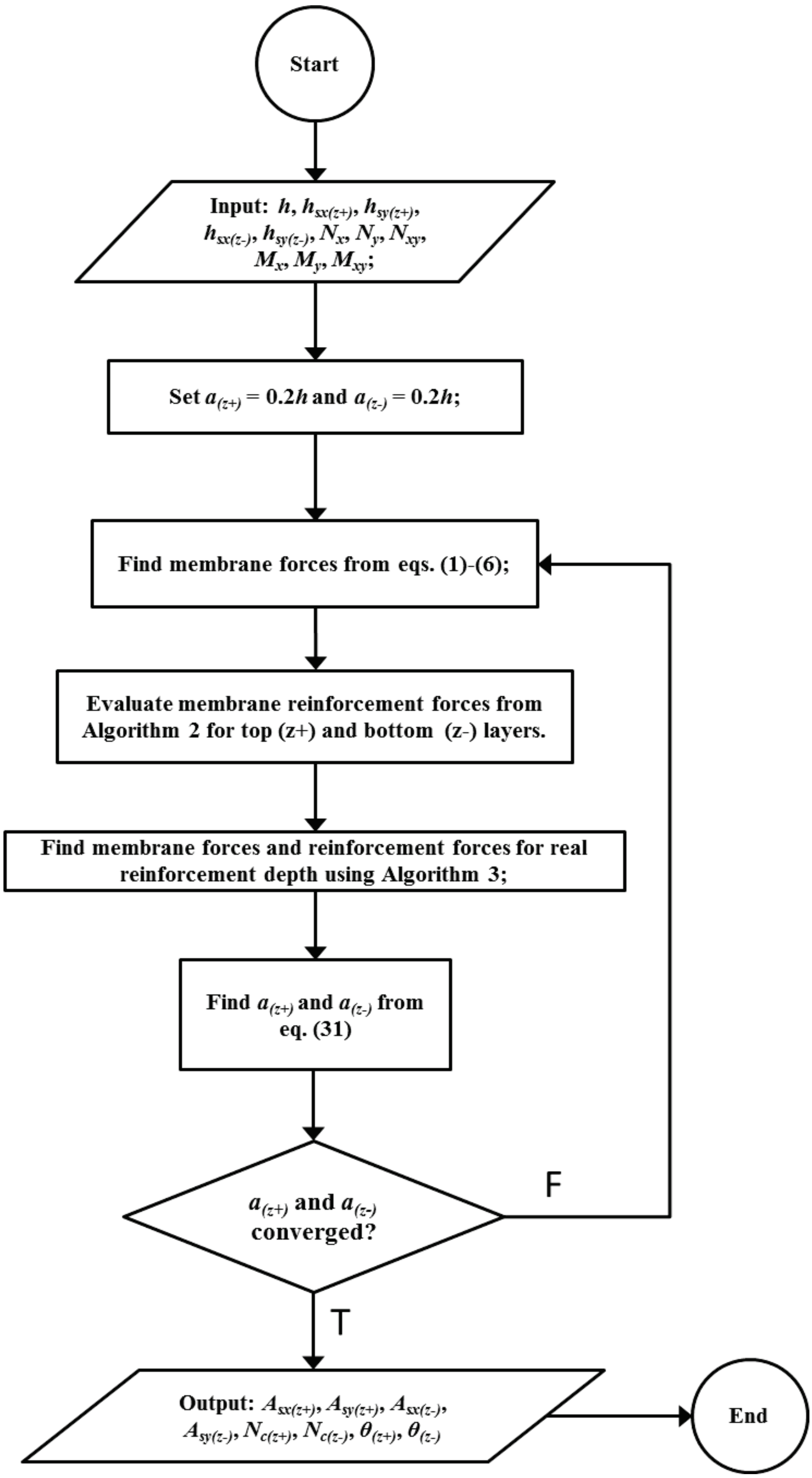
ε_1 = maximum principal strain;

ε_{cp} = concrete strain at peak strength;

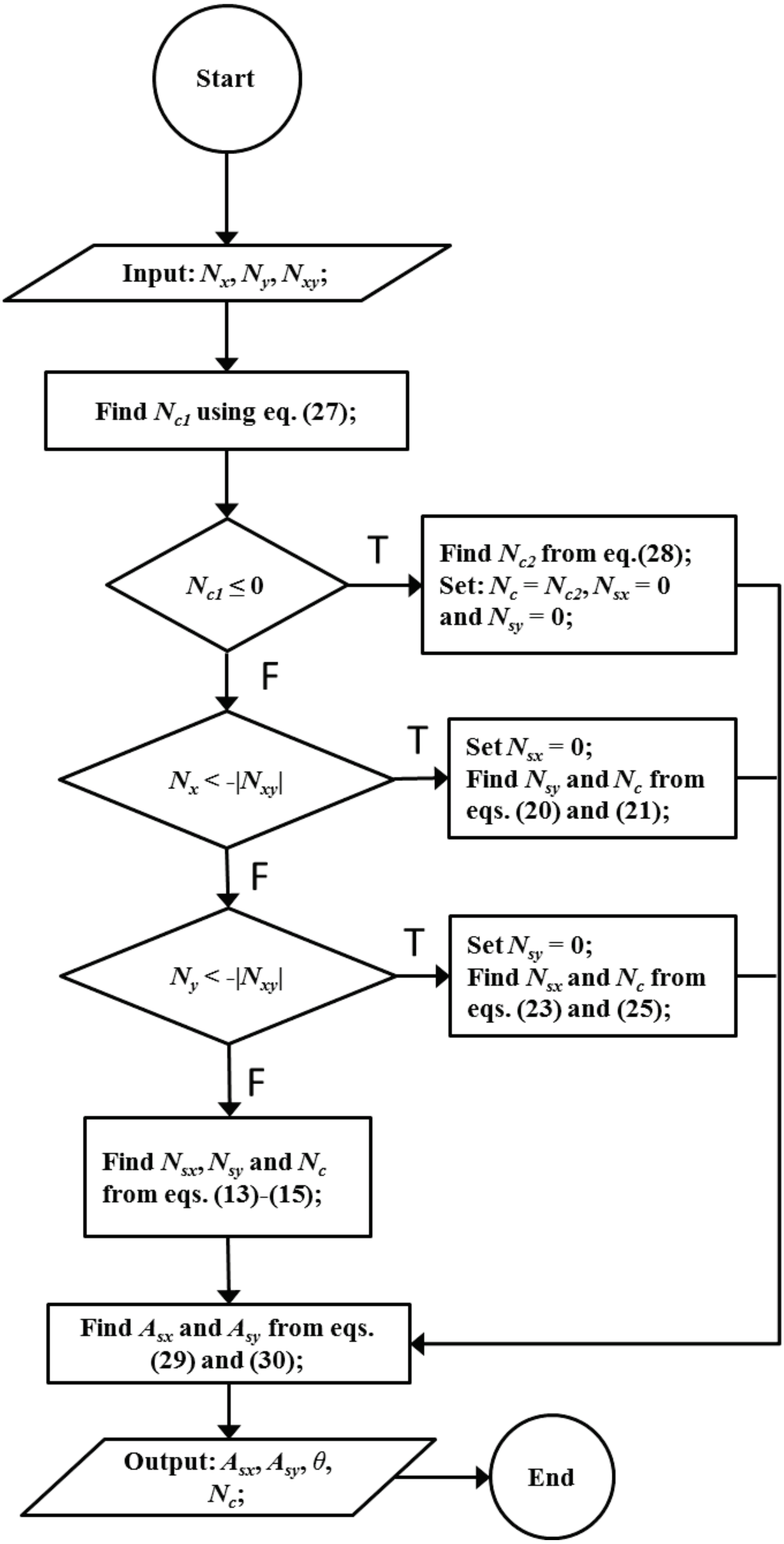
ε_{yi} = steel yield strain.

8. Appendix A – Flowcharts for the algorithms

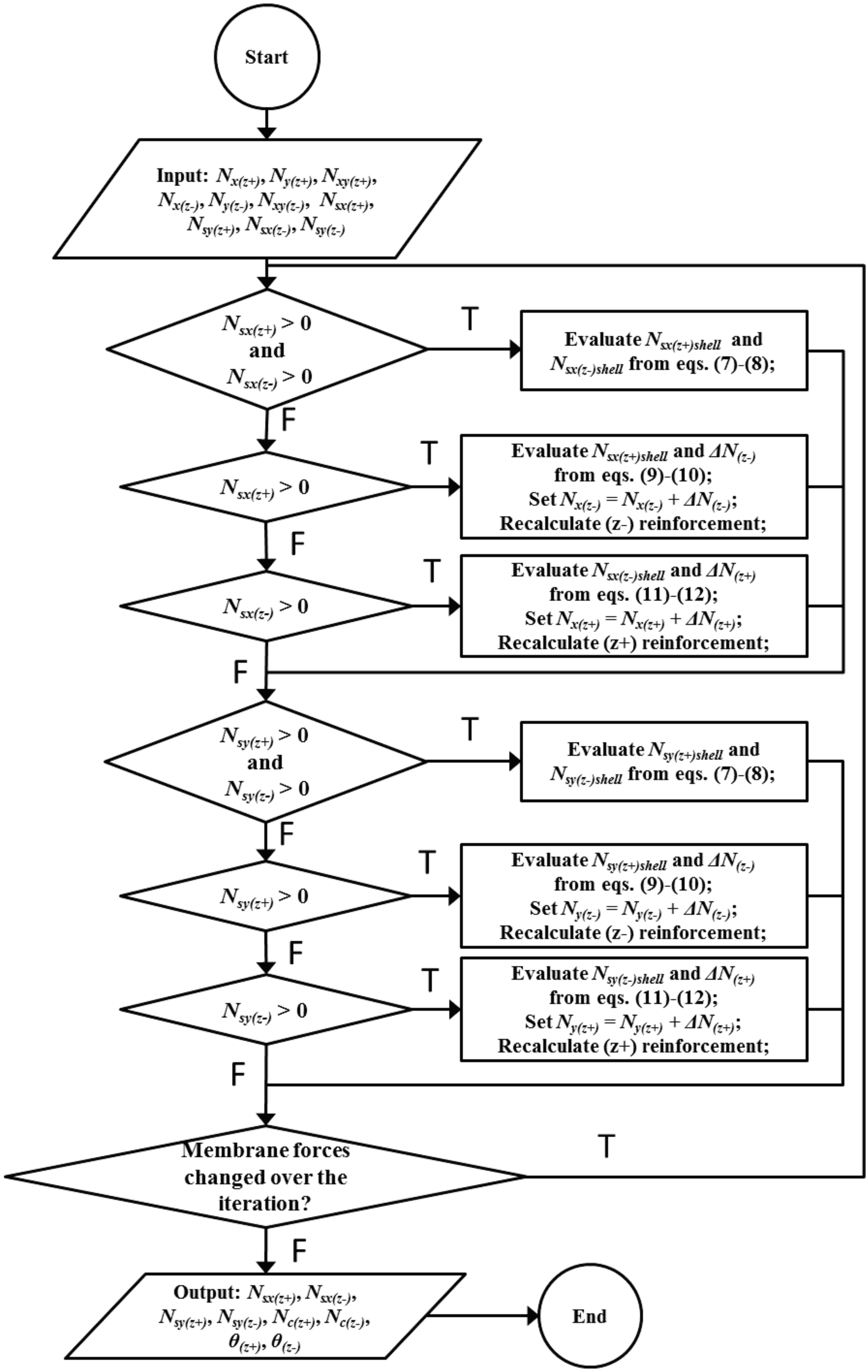
Algorithm 1 – Shell reinforcement design algorithm



Algorithm 2 – Membrane reinforcement design algorithm



Algorithm 3 – Algorithm for finding real reinforcement force



9. Appendix B – Numerical example of an iterative step

3rd iteration for Element 3 from Table 2:
The previous iteration yielded the following thickness values:

$$a_{(z+)} = 0.1949m \Rightarrow h_{(z+)} = h/2 - a_{(z+)}/2 = 0.5525m$$
$$a_{(z-)} = 0.0910m \Rightarrow h_{(z-)} = h/2 - a_{(z-)}/2 = 0.6045m$$

3rd iteration for Element 3 from Table 2:
The previous iteration yielded the following thickness values:

$$N_{x(z+)} = -265.71tf/m \quad N_{y(z+)} = -36.23tf/m \quad N_{xy(z+)} = 141.55tf/m$$
$$N_{x(z-)} = 2.09tf/m \quad N_{y(z-)} = 312.59tf/m \quad N_{xy(z+)} = -58.00tf/m$$

Calculating Reinforcement for the (z+) membrane:

From eq. (27) $N_{c1(z+)} = 31.24tf/m > 0 \Rightarrow$
Reinforcement is needed in at least one direction.

Since $N_{y(z+)} < -|N_{xy(z+)}|$
reinforcement is needed in the y direction only.

$$N_{sx(z+)} = 0tf/m \quad N_{sy(z+)} = 39.17tf/m \quad \theta_{(z+)} = 61.95° \quad N_{c(z+)} = -341.12tf/m$$

Calculating Reinforcement for the (z-) membrane:

$N_{c1(z-)} = 323.07tf/m > 0 \Rightarrow$
Reinforcement is needed in at least one direction.

Since $N_{x(z-)} > -|N_{xy(z-)}|$ and $N_{y(z-)} > -|N_{xy(z-)}|$
reinforcement is needed in both directions.

$$N_{sx(z-)} = 60.09tf/m \quad N_{sy(z-)} = 370.59tf/m \quad \theta_{(z-)} = -45° \quad N_{c(z-)} = -116tf/m$$

Calculate Reinforcement considering real depth:

X-Direction:
Since $N_{sx(z+)} = 0$ and $N_{sx(z-)} > 0$
a correct reinforcement force for the (z-) membrane will be calculated as well as a correction to the membrane force for the (z+) membrane. Using eq. (11)-(12):

$$N_{sx(z-)shell} = 72.98tf/m \quad \Delta N_{x(z+)} = -12.90tf/m$$

The new forces in the (z+) membranes are:

$$N_{x(z+)} = -278.61tf/m \quad N_{y(z+)} = -36.23tf/m \quad N_{xy(z+)} = 141.55tf/m$$

Recalculating the reinforcement for the (z+) membrane with these new values yield:

$$N_{sx(z+)} = 0tf/m \quad N_{sy(z+)} = 35.68tf/m \quad \theta_{(z+)} = 63.06° \quad N_{c(z+)} = -350.52tf/m$$

Y-Direction:
Since $N_{sy(z+)} > 0$ and $N_{sy(z-)} > 0$
it is necessary to find the correct reinforcement forces for using eq. (7)-(8).

$$N_{sy(z+)shell} = -23.87tf/m \quad N_{sy(z-)shell} = 430.14tf/m$$

A negative reinforcement force in this case indicates that using the real reinforcement depth leads to no need for reinforcement in the (z+) shell to correct the force acting in the (z+) membrane in the y-direction.
Since $\Delta N_{y(z+)}$ acts in the mid-plane of the membrane it's necessary to substitute $h_{sy(z+)} = h_{(z+)}$ in eq. (7).

$$\Delta N_{y(z+)} = \frac{N_{sy(z+)} \cdot (h_{y(z-)} + h_{(z+)}) + N_{sy(z-)memb} \cdot (h_{sy(z-)} - h_{(z-)})}{(h_{sy(z-)} + h_{(z+)})} = -57.11tf/m$$

$$N_{sy(z-)shell} = 427.70tf/m$$

The new membrane forces in the (z+) membrane are:

$$N_{x(z+)} = -278.61tf/m \quad N_{y(z+)} = -93.34tf/m \quad N_{xy(z+)} = 141.55tf/m$$

Recalculating the necessary reinforcement yields:
$N_{c1(z+)} = -21.11tf/m < 0 \Rightarrow$ No reinforcement is needed
$(N_{sx(z+)} = 0 \quad N_{sy(x+)} \ ?\ 0)$
Since the change in the (z+) membrane did not affect the forces in the x direction ($N_{sx(+z)}$ was already zero), we have a stable solution in the x and y directions and therefore it's possible to calculate new thickness values.

Calculate new thickness values:
For the (z+) membrane:
$N_{c(z+)} = N_{c2(z+)} = -355.14tf/m$
$f_{c(z+)} = 18.27MPa \Rightarrow a_{(z+)} = 0.1943m$
For the (z-) membrane:
$N_{c(z-)} = -116tf/m$
Using $\varepsilon_{yi} = 2.07 \times 10^{-3}$ and $\varepsilon_{cp} = -2 \times 10^{-3}$
in eq.(41) yields $\varepsilon_{1(z-)} = 6.14 \times 10^{-3}$.

Applying this to eq.(36) and (37):
$f_{c(z-)} = 12.90MPa \Rightarrow a_{(z-)} = 0.0810m$
Convergence was not achieved, further iterations are needed.