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Thouless theorem for one and two degrees of freedom

(Teorema de Thouless para um e dois graus de liberdade)

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We perform an accessible demonstration of Thouless theorem for systems of one and two degrees of freedom, using only elementary quantum mechanics. This theorem is specially useful to ¯nd the vacuum of a transformed set of bosonic operators. A further motivation comes from experiments involving few trapped ions or ¯elds in cavity quantum electrodynamics, usually modelled by few linearly interacting oscillators.
Keywords: quantum oscillators, variables transformations, bosonic systems.

Realizamos uma demonstração acessível do teorema de Thouless para um ou dois graus de liberdade, usando somente mecânica quântica elementar. Este teorema é especialmente útil para encontrar o vácuo relativo a um conjunto de operadores bosônicos transformados. Outra motivação é a existência de experimentos envolvendo íons capturados ou campos em eletrodinâmica quântica de cavidades, usualmente modelados por osciladores harmônicos interagindo linearmente.
Palavras-chave: osciladores quânticos, transformações de variáveis, sistemas bosônicos.

1. Introduction

Thouless theorem [1] is a crucial tool for the calculation of many boson observables. This theorem is often applied in traditional nuclear physics for the computation of low energy nuclear excitations. These excitations are of course due to the interaction between the fermions within the nucleus. The result of such interaction in the low energy domain is a collective excitation which for all practical purposes behaves as a bosonic degree of freedom. The language of Thouless theorem is very sophisticated and hard to understand for nonspecialists; however, it may turn out to be quite useful in systems which involve only few degrees of freedom, as we often encounter in quantum optics [2-17]. In these situations, it is usually convenient to rewrite the creation and annihilation bosonic operators in a new form, and it becomes necessary to find out the new eingenstates. This can be easily achieved once the new vacuum state is known, which solves the problem.

Just for illustration, let us consider a simple model for tunneling: assume one has two optical cavities connected by a waveguide. The Hamiltonian may be of the form [18, 19]

$$H = \hbar\omega_a a^\dagger a + \hbar\omega_b b^\dagger b + \hbar g \left(a^\dagger b + b^\dagger a\right). \tag{1}$$

The eigenvalues and eigenfunctions can be found by several techniques, and among those a geometrical one [4]. Diagonalizing Eq. (1) amounts to a rotation:

$$\begin{aligned} H &= \hbar \begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \begin{pmatrix} \omega_a & g \\ g & \omega_b \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \hbar \begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \mathbf{M}^T\mathbf{M} \begin{pmatrix} \omega_a & g \\ g & \omega_b \end{pmatrix} \mathbf{M}^T\mathbf{M} \begin{pmatrix} a \\ b \end{pmatrix} \\ &= \hbar \begin{pmatrix} \bar{a}^\dagger & \bar{b}^\dagger \end{pmatrix} \begin{pmatrix} \bar{\omega}_a & 0 \\ 0 & \bar{\omega}_b \end{pmatrix} \begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} \\ &= \hbar\bar{\omega}_a \bar{a}^\dagger \bar{a} + \hbar\bar{\omega}_b \bar{b}^\dagger \bar{b}, \end{aligned} \tag{2}$$

where

$$\begin{pmatrix} \bar{a} \\ \bar{b} \end{pmatrix} = \mathbf{M} \begin{pmatrix} a \\ b \end{pmatrix},$$
$$\begin{pmatrix} \bar{\omega}_a & 0 \\ 0 & \bar{\omega}_b \end{pmatrix} = \mathbf{M} \begin{pmatrix} \omega_a & g \\ g & \omega_b \end{pmatrix} \mathbf{M}^T, \tag{3}$$

and

$$\mathbf{M} = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \tag{4}$$

is a real orthogonal matrix ($\mathbf{M}\mathbf{M}^T = \mathbf{I}$, with $\mathbf{I}$ being an identity matrix). Since the explicit expressions for $\bar{\omega}_a$, $\bar{\omega}_b$ and $\mathbf{M}$ are not crucial for what follows, they are

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given as a function of the physical parameters in Appendix A. In the present case, the new vacuum is the same as the one before: if $a\left|0,0\right\rangle = b\left|0,0\right\rangle = 0$, then

$$\begin{aligned}\bar{a}\left|0,0\right\rangle &= M_{11}a\left|0,0\right\rangle + M_{12}b\left|0,0\right\rangle = 0, \\ \bar{b}\left|0,0\right\rangle &= M_{21}a\left|0,0\right\rangle + M_{22}b\left|0,0\right\rangle = 0.\end{aligned} \quad (5)$$

When one has a slightly more complicated coupling, *e.g.* bilinear terms like $a^\dagger b^\dagger$ and ab as in Hamiltonian

$$H = \hbar\omega_a\left(a^\dagger a + \frac{1}{2}\right) + \hbar\omega_b\left(b^\dagger b + \frac{1}{2}\right) + \hbar\frac{g}{2}\left(a^\dagger b^\dagger + a^\dagger b + b^\dagger a + ba\right), \quad (6)$$

the situation becomes more involved. In order to diagonalize this Hamiltonian, we may start by using the form

$$H = \frac{1}{2}\omega_a\left((q_a)^2 + (p_a)^2\right) + \frac{1}{2}\omega_b\left((q_b)^2 + (p_b)^2\right) + g q_a q_b, \quad (7)$$

where

$$\begin{aligned} q_a &= \sqrt{\frac{\hbar}{2}}\left(a^\dagger + a\right), \qquad p_a = i\sqrt{\frac{\hbar}{2}}\left(a^\dagger - a\right), \quad (8)\\ q_b &= \sqrt{\frac{\hbar}{2}}\left(b^\dagger + b\right), \qquad p_b = i\sqrt{\frac{\hbar}{2}}\left(b^\dagger - b\right).\end{aligned}$$

Using the results from Ref. [4], we get

$$H = \frac{1}{2}\bar{\omega}_a\left((\bar{q}_a)^2 + (\bar{p}_a)^2\right) + \frac{1}{2}\bar{\omega}_b\left((\bar{q}_b)^2 + (\bar{p}_b)^2\right), \quad (9)$$

where $\bar{\omega}_i$, $\bar{p}_i$ and $\bar{q}_i$ ($i = a$ or b) are given by

$$\begin{pmatrix}\bar{q}_a\\ \bar{q}_b\\ \bar{p}_a\\ \bar{p}_b\end{pmatrix} = \mathbf{M}\begin{pmatrix}q_a\\ q_b\\ p_a\\ p_b\end{pmatrix},$$

$$\begin{pmatrix}\bar{\omega}_a & 0 & 0 & 0\\ 0 & \bar{\omega}_b & 0 & 0\\ 0 & 0 & \bar{\omega}_a & 0\\ 0 & 0 & 0 & \bar{\omega}_b\end{pmatrix} = \left(\mathbf{M}^T\right)^{-1}\begin{pmatrix}\omega_a & g & 0 & 0\\ g & \omega_b & 0 & 0\\ 0 & 0 & \omega_a & 0\\ 0 & 0 & 0 & \omega_b\end{pmatrix}\mathbf{M}^{-1}. \quad (10)$$

The matrix $\mathbf{M}$ has the form

$$\begin{pmatrix}M_{11} & M_{12} & 0 & 0\\ M_{21} & M_{22} & 0 & 0\\ 0 & 0 & M_{33} & M_{34}\\ 0 & 0 & M_{43} & M_{44}\end{pmatrix}, \quad (11)$$

and corresponds to a squeezing followed by a rotation and another squeezing; $\bar{\omega}_a$, $\bar{\omega}_b$ and $\mathbf{M}$ are given in Appendix B. The new variables obey the usual commutation relations, and we can define related bosonic operators as

$$\bar{a} = \sqrt{\frac{1}{2\hbar}}\left(\bar{q}_a + i\bar{p}_a\right), \qquad \bar{b} = \sqrt{\frac{1}{2\hbar}}\left(\bar{q}_b + i\bar{p}_b\right). \quad (12)$$

Expressing $\bar{a}$ as

$$\bar{a} = \sqrt{\frac{1}{8\hbar}}\left\{\left(M_{11} + M_{33}\right)a + \left(M_{12} + M_{34}\right)b + \left(M_{11} - M_{33}\right)a^\dagger + \left(M_{12} - M_{34}\right)b^\dagger\right\}, \quad (13)$$

we note that the relations $a\left|0,0\right\rangle = b\left|0,0\right\rangle = 0$ do not guarantee that $\bar{a}\left|0,0\right\rangle = 0$, since in general $M_{11} \neq M_{33}$ and $M_{12} \neq M_{34}$. In this case, the relation between the old and the new vacuum states is not trivial.

It becomes, with these few examples, easy to imagine that if one needs the vacuum of a many-body operator the problem can become fastidiously cumbersome. In many-body physics, one usually needs to calculate matrix elements which involve these vacua. Thouless theorem is just the most powerful technical tool available. It teaches us how to, relate vacua of different pairs of quasiparticles. In full the theorem reads: starting with a general product wavefunction $|\phi_0\rangle$ which is the vacuum to quasi particle operators β, any other general product wavefunction $|\phi_1\rangle$ may be expressed in the form

$$|\phi_1\rangle = \mathcal{N}\exp\left\{\sum_{k<k'} Z_{kk'}\beta_k^\dagger\beta_{k'}^\dagger\right\}|\phi_0\rangle, \quad (14)$$

where $\mathcal{N} = \langle\phi_0\,|\phi_1\rangle$ is a normalization constant and Z a skew symmetric matrix. Thouless has given this theorem for pure Slater determinants ϕ_0, ϕ_1. Two quasiparticle states are, in this case, *particle-hole* states:

$$|\phi_1\rangle = \mathcal{N}\exp\left\{\sum_{m,i} Z_{mi}a_m^\dagger a_i^\dagger\right\}|\phi_0\rangle. \quad (15)$$

In the present contribution, we construct demonstrations of Thouless theorem for two special cases related to the description of the vacuum for a transformed set of bosonic operators. These demonstrations are sufficiently accessible and only elementary quantum mechanics is used. In section 2, we treat one degree of freedom systems, deducing the corresponding Thouless theorem and its relevant coefficients; the manner we perform the calculation allows us to understand why the theorem assume its form. Two degrees of freedom systems are treated in section 3.

2. One degree of freedom

Consider a canonical transformation that relates the bosonic operators a and $a^\dagger$ to the bosonic operators $\bar{a}$ and $\bar{a}^\dagger$ through:

$$\begin{pmatrix}\bar{a}\\ \bar{a}^\dagger\end{pmatrix} = \begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}\begin{pmatrix}a\\ a^\dagger\end{pmatrix}, \quad (16)$$

where $\left[a,a^\dagger\right] = \left[\bar{a},\bar{a}^\dagger\right] = 1$. We will deduce here an expression with the form of Eq. (15) connecting the vacuum state concerning $\bar{a}$ and $\bar{a}^\dagger$ to the vacuum state concerning a and $a^\dagger$. Since $\bar{a}^\dagger$ is the Hermitian conjugate of $\bar{a}$, we must have $M_{21} = M_{12}^*$ and $M_{22} = M_{11}^*$. In the following, the Fock states related to a and $a^\dagger$ will be indicated by $|n\rangle$:

$$a^\dagger a\,|n\rangle = n\,|n\rangle\,, \qquad |n\rangle = \frac{\left(a^\dagger\right)^n}{\sqrt{n!}}\,|0\rangle\,, \tag{17}$$

where $|0\rangle$ is the fundamental state for this basis. The vacuum related to $\bar{a}$ and $\bar{a}^\dagger$ will be indicated by $|\bar{0}\rangle$.

The vacuum state $|\bar{0}\rangle$ may be given by the superposition

$$|\bar{0}\rangle = \sum_{n=0}^{\infty} c_n\,|n\rangle\,, \tag{18}$$

and, since $\bar{a} = M_{11}a + M_{12}a^\dagger$ and $\bar{a}\,|\bar{0}\rangle = 0$,

$$\begin{aligned}
\bar{a}\,|\bar{0}\rangle &= \left(M_{11}a + M_{12}a^\dagger\right)\sum_{n=0}^{\infty} c_n\,|n\rangle \\
&= \sum_{n=1}^{\infty} M_{11}c_n\sqrt{n}\,|n-1\rangle + \sum_{n=0}^{\infty} M_{12}c_n\sqrt{n+1}\,|n+1\rangle \\
&= \sum_{n=0}^{\infty} M_{11}c_{n+1}\sqrt{n+1}\,|n\rangle + \sum_{n=1}^{\infty} M_{12}c_{n-1}\sqrt{n}\,|n\rangle \\
&= M_{11}c_1\,|0\rangle + \\
&\quad \sum_{n=1}^{\infty}\left(M_{11}c_{n+1}\sqrt{n+1} + M_{12}c_{n-1}\sqrt{n}\right)|n\rangle = 0
\end{aligned} \tag{19}$$

Due to the linear independency of the Fock states $|n\rangle$, the coefficients c_n must satisfy the relations: $c_1 = 0$; $M_{11}c_{n+1}\sqrt{n+1} + M_{12}c_{n-1}\sqrt{n} = 0$ for $n \geqslant 1$. Since this last equation may be written as

$$c_{n+2} = -\frac{M_{12}}{M_{11}}\frac{\sqrt{n+1}}{\sqrt{n+2}}c_n, \qquad \text{for} \qquad n \geqslant 0, \tag{20}$$

iterating and using $c_1 = 0$ the iteration process leads to

$$\begin{array}{lcl}
c_1 = 0 & , & c_2 = -\frac{M_{12}}{M_{11}}\frac{\sqrt{1}}{\sqrt{2}}c_0, \\
c_3 = 0 & , & c_4 = \left(-\frac{M_{12}}{M_{11}}\right)^2\frac{\sqrt{1\times 3}}{\sqrt{2\times 4}}c_0, \\
c_5 = 0 & , & c_6 = \left(-\frac{M_{12}}{M_{11}}\right)^3\frac{\sqrt{1\times 3\times 5}}{\sqrt{2\times 4\times 6}}c_0, \\
\vdots & \vdots & \vdots \\
c_{2n-1} = 0 & , & c_{2n} = \left(-\frac{M_{12}}{M_{11}}\right)^n\frac{\sqrt{(2n-1)!!}}{\sqrt{(2n)!!}}c_0, \\
 & & \text{for } n \geqslant 1.
\end{array} \tag{21}$$

We used the notation $n!! = n\,(n-2)\,(n-4)\,...$, where the last factor is 1 for odd n and 2 for even n. This gives the following expression for the vacuum state in the new basis:

$$\begin{aligned}
|\bar{0}\rangle &= c_0\,|0\rangle + c_0\sum_{n=1}^{\infty}\left(-\frac{M_{12}}{M_{11}}\right)^n\frac{\sqrt{(2n-1)!!}}{\sqrt{(2n)!!}}\,|2n\rangle \\
&= c_0\,|0\rangle + c_0\sum_{n=1}^{\infty}\left(-\frac{M_{12}}{M_{11}}\right)^n\frac{\sqrt{(2n-1)!!}}{\sqrt{(2n)!!}}\frac{\left(a^\dagger\right)^{2n}}{\sqrt{(2n)!}}\,|0\rangle\,.
\end{aligned} \tag{22}$$

Observing that

$$\frac{\sqrt{(2n-1)!!}}{\sqrt{(2n)!!}\sqrt{(2n)!}} = \frac{1}{\sqrt{(2n)!!}\sqrt{(2n)!!}} = \frac{1}{2^n}\frac{1}{n!}, \tag{23}$$

we get

$$\begin{aligned}
|\bar{0}\rangle &= c_0\,|0\rangle + c_0\sum_{n=1}^{\infty}\frac{1}{n!}\left(-\frac{1}{2}\frac{M_{12}}{M_{11}}\left(a^\dagger\right)^2\right)^n|0\rangle \\
&= c_0\exp\left(-\frac{1}{2}\frac{M_{12}}{M_{11}}\left(a^\dagger\right)^2\right)|0\rangle\,.
\end{aligned} \tag{24}$$

We may put the information about the canonical transformation in the parameter

$$Z = \frac{1}{2}\frac{M_{12}}{M_{11}}, \tag{25}$$

and write down the final expression for the new vacuum as:

$$|\bar{0}\rangle = c_0 e^{-Z\left(a^\dagger\right)^2}\,|0\rangle\,, \tag{26}$$

where c_0 is a normalization constant.

Using the identity [20]

$$\left[a, g(a^\dagger)\right] = \frac{\partial}{\partial a^\dagger}\left(g(a^\dagger)\right), \tag{27}$$

where $g(a^\dagger)$ is a function of the operator $a^\dagger$, and $\frac{\partial}{\partial a^\dagger}\left(g(a^\dagger)\right)$ is the derivative of this function,

$$g(a^\dagger) = \sum_{n=0}^{\infty}\left(a^\dagger\right)^n g_n$$

and

$$\frac{\partial}{\partial a^\dagger}\left(g(a^\dagger)\right) = \sum_{n=0}^{\infty}\left(a^\dagger\right)^{n-1} n g_n, \tag{28}$$

the g_n being constants or operators that commute with $a^\dagger$. It is easy to verify that $|\bar{0}\rangle$ is the new vacuum:

$$\begin{aligned}
\bar{a}\,|\bar{0}\rangle &= \left(M_{11}a + M_{12}a^\dagger\right)c_0 e^{-Z\left(a^\dagger\right)^2}\,|0\rangle \\
&= c_0\Big(M_{11}e^{-Z\left(a^\dagger\right)^2}a - 2M_{11}Za^\dagger e^{-Z\left(a^\dagger\right)^2} + \\
&\quad M_{12}a^\dagger e^{-Z\left(a^\dagger\right)^2}\Big)\,|0\rangle \\
&= c_0\left(-2M_{11}\left(\frac{1}{2}\frac{M_{12}}{M_{11}}\right) + M_{12}\right)a^\dagger e^{-Z\left(a^\dagger\right)^2}\,|0\rangle \\
&= 0.
\end{aligned} \tag{29}$$

3. Two degrees of freedom

Let us consider the transformation

$$\bar{\mathbf{A}} = \mathbf{M}\times\mathbf{A}, \tag{30}$$

where

$$\begin{aligned}
\bar{\mathbf{A}} &= \begin{pmatrix}\bar{a}\\ \bar{b}\\ \bar{a}^\dagger\\ \bar{b}^\dagger\end{pmatrix}, \qquad \mathbf{A} = \begin{pmatrix}a\\ b\\ a^\dagger\\ b^\dagger\end{pmatrix}, \\
\mathbf{M} &= \begin{pmatrix}\mathbf{D} & \mathbf{C}\\ \mathbf{C}^* & \mathbf{D}^*\end{pmatrix}, \qquad \mathbf{D} = \begin{pmatrix}M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}, \\
\mathbf{C} &= \begin{pmatrix}M_{13} & M_{14}\\ M_{23} & M_{24}\end{pmatrix}.
\end{aligned} \tag{31}$$

Here, a and b are annihilation bosonic operators, and $a^\dagger$ and $b^\dagger$ are creation bosonic operators. Notice that

$$\begin{aligned}\bar{a} &= M_{11}a + M_{12}b + M_{13}a^\dagger + M_{14}b^\dagger, \\ \bar{b} &= M_{21}a + M_{22}b + M_{23}a^\dagger + M_{24}b^\dagger. \end{aligned} \quad (32)$$

The fundamental state related to $\bar{a}$ and $\bar{b}$ will be expressed as $|\bar{0},\bar{0}\rangle$, and that one related to a and b as $|0,0\rangle$. We will obtain $|\bar{0},\bar{0}\rangle$ from $|0,0\rangle$ through an expression of the form of Eq. (15), *i.e.*,

$$|\bar{0},\bar{0}\rangle = \mathcal{N} f\left(a^\dagger, b^\dagger\right)|0,0\rangle, \quad (33)$$

where

$$f\left(a^\dagger,b^\dagger\right) = \exp\left\{\frac{1}{2}\left(Z_{aa}\left(a^\dagger\right)^2 + Z_{ab}a^\dagger b^\dagger + Z_{ba}b^\dagger a^\dagger + Z_{bb}\left(b^\dagger\right)^2\right)\right\}, \quad (34)$$

and $\mathcal{N}$ is a normalization constant.

Since $\left[a^\dagger, b^\dagger\right] = 0$, we may write

$$f\left(a^\dagger,b^\dagger\right) = \exp\left\{\frac{1}{2}\left(Z_{aa}\left(a^\dagger\right)^2 + \left(Z_{ab}+Z_{ba}\right)a^\dagger b^\dagger + Z_{bb}\left(b^\dagger\right)^2\right)\right\}. \quad (35)$$

The states $|\bar{0},\bar{0}\rangle$ must satisfy

$$\bar{a}\,|\bar{0},\bar{0}\rangle = 0, \quad (36)$$

$$\bar{b}\,|\bar{0},\bar{0}\rangle = 0, \quad (37)$$

and we will find coefficients Z_{ij} (i and $j = a$ or b) that make Eq. (36) and Eq. (37) true. Equation (36) leads to

$$\left(M_{11}b + M_{12}b + M_{13}a^\dagger + M_{14}b^\dagger\right) \mathcal{N}_t f\left(a^\dagger,b^\dagger\right)|0,0\rangle = 0. \quad (38)$$

Using identity (27) we obtain

$$\left[a, f\left(a^\dagger,b^\dagger\right)\right] = a^\dagger f\left(a^\dagger,b^\dagger\right) Z_{aa} + b^\dagger f\left(a^\dagger,b^\dagger\right)\left(\frac{Z_{ab}+Z_{ba}}{2}\right), \quad (39)$$

$$\left[b, f\left(a^\dagger,b^\dagger\right)\right] = a^\dagger f\left(a^\dagger,b^\dagger\right)\left(\frac{Z_{ab}+Z_{ba}}{2}\right) + b^\dagger f\left(a^\dagger,b^\dagger\right) Z_{bb}. \quad (40)$$

Thus Eq. (38) may be written in the form

$$\begin{aligned}0 = &\Big\{M_{11}\left(\left[a, f\left(a^\dagger,b^\dagger\right)\right] + f\left(a^\dagger,b^\dagger\right)a\right) + \\ &M_{12}\left(\left[b, f\left(a^\dagger,b^\dagger\right)\right] + f\left(a^\dagger,b^\dagger\right)b\right)\Big\}|0,0\rangle \\ &+\left\{M_{13}a^\dagger f\left(a^\dagger,b^\dagger\right) + M_{14}b^\dagger f\left(a^\dagger,b^\dagger\right)\right\}|0,0\rangle . \end{aligned} \quad (41)$$

Since $a\,|0,0\rangle = b\,|0,0\rangle = 0$, we have

$$\begin{aligned}0 = &\Big\{M_{11}\left[a, f\left(a^\dagger,b^\dagger\right)\right] + M_{12}\left[b, f\left(a^\dagger,b^\dagger\right)\right] + \\ &M_{13}a^\dagger f\left(a^\dagger,b^\dagger\right) + M_{14}b^\dagger f\left(a^\dagger,b^\dagger\right)\Big\}|0,0\rangle, \end{aligned} \quad (42)$$

and, using Eqs. (39) and (40),

$$\begin{aligned}0 = &\left\{M_{11}Z_{aa} + M_{12}\left(\frac{Z_{ab}+Z_{ba}}{2}\right) + M_{13}\right\} \\ &a^\dagger f\left(a^\dagger,b^\dagger\right)|0,0\rangle + \Big\{M_{11}\left(\frac{Z_{ab}+Z_{ba}}{2}\right) + \\ &M_{12}Z_{bb} + M_{14}\Big\}\, b^\dagger f\left(a^\dagger,b^\dagger\right)|0,0\rangle . \end{aligned} \quad (43)$$

Thus, we may choose

$$\begin{aligned}M_{11}Z_{aa} + M_{12}\left(\frac{Z_{ab}+Z_{ba}}{2}\right) + M_{13} &= 0, \\ M_{11}\left(\frac{Z_{ab}+Z_{ba}}{2}\right) + M_{12}Z_{bb} + M_{14} &= 0. \end{aligned} \quad (44)$$

Now, using Eq. (37) and performing an analogous calculation, just changing M_{1i} by M_{2i} ($i = 1$ to 4), we get

$$\begin{aligned}M_{21}Z_{aa} + M_{22}\left(\frac{Z_{ab}+Z_{ba}}{2}\right) + M_{23} &= 0, \\ M_{21}\left(\frac{Z_{ab}+Z_{ba}}{2}\right) + M_{22}Z_{bb} + M_{24} &= 0. \end{aligned} \quad (45)$$

Defining

$$Z_m = \left(\frac{Z_{ab}+Z_{ba}}{2}\right), \quad (46)$$

we can write the set of equations above in the form

$$M_{11}Z_{aa} + M_{12}Z_m + M_{13} = 0, \quad (47)$$

$$M_{21}Z_{aa} + M_{22}Z_m + M_{23} = 0, \quad (48)$$

$$M_{11}Z_m + M_{12}Z_{bb} + M_{14} = 0, \quad (49)$$

$$M_{21}Z_m + M_{22}Z_{bb} + M_{24} = 0. \quad (50)$$

Using Eqs. (47) and (48) we find

$$\begin{aligned}Z_{aa} &= \frac{M_{22}M_{13} - M_{12}M_{23}}{M_{12}M_{21} - M_{22}M_{11}}, \\ Z_m &= \frac{M_{11}M_{23} - M_{13}M_{21}}{M_{12}M_{21} - M_{22}M_{11}}. \end{aligned} \quad (51)$$

Using Eqs. (49) and (50) we find

$$\begin{aligned}Z_{bb} &= \frac{M_{21}M_{14} - M_{11}M_{24}}{M_{11}M_{22} - M_{21}M_{12}}, \\ Z_m &= \frac{M_{14}M_{22} - M_{12}M_{24}}{M_{12}M_{21} - M_{22}M_{11}}. \end{aligned} \quad (52)$$

For the system of Eqs. (47-50) to have a solution, the two Z_m values encountered must be equal, *i.e.*, the condition

$$M_{14}M_{22} - M_{12}M_{24} = M_{11}M_{23} - M_{13}M_{21} \quad (53)$$

must be satisfied. This occurs if the transformation matrix $\mathbf{T}$ obeys the sympletic condition,

$$\mathbf{M} \times \mathbf{J} \times \mathbf{M}^T = \mathbf{J}, \quad (54)$$

where

$$\mathbf{J} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{I} & \mathbf{0} \end{pmatrix}, \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{0} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad (55)$$

as may be seen by performing the matrices multiplication in Eq. (54) and comparing the element in the first line and second column of $\mathbf{M}\times\mathbf{J}\times\mathbf{M}^T$ with the one of $\mathbf{J}$. If $\mathbf{M}$ obeys the simpletic condition, it may be associated to a classical canonical transformation [21].

Thus we can express $|\bar{0},\bar{0}\rangle$ through

$$|\bar{0},\bar{0}\rangle = \mathcal{N}_t \exp\left\{\frac{1}{2}\left(Z_{aa}\left(a^\dagger\right)^2 + Z_{ab}a^\dagger b^\dagger + Z_{ba}b^\dagger a^\dagger + Z_{bb}\left(b^\dagger\right)^2\right)\right\}|0,0\rangle\,, \tag{56}$$

with

$$\begin{aligned} Z_{aa} &= \frac{M_{22}M_{13} - M_{12}M_{23}}{M_{12}M_{21} - M_{22}M_{11}}, \\ Z_{ab} &= \frac{M_{11}M_{23} - M_{13}M_{21}}{M_{12}M_{21} - M_{22}M_{11}}, \\ Z_{ba} &= \frac{M_{14}M_{22} - M_{12}M_{24}}{M_{12}M_{21} - M_{22}M_{11}}, \\ Z_{bb} &= \frac{M_{21}M_{14} - M_{11}M_{24}}{M_{11}M_{22} - MM_{12}}, \end{aligned} \tag{57}$$

since the values chosen for Z_{ab} and Z_{ba} lead to the value of Z_m found by solving Eqs. (47), (48), (49) and (50). It is easy to see that

$$\begin{pmatrix} Z_{aa} & Z_{ab} \\ Z_{ba} & Z_{bb} \end{pmatrix} = -\left(\mathbf{C}^\dagger\left(\mathbf{D}^\dagger\right)^{-1}\right)^*, \tag{58}$$

which is the Thouless theorem for the present situation.

4. Conclusion

We demonstrated the Thouless theorem for the particular case where the states connected are vacuum states of two sets of bosonic operators, for one and two degrees of freedom. We also found the coefficients necessary for the application of the theorem, which are useful to effectively find the vacuum states in the new basis. Thus, knowing the transformation for the operators, we can calculate the transformation for the vacuum states and obtain the new Fock states, the new coherent states and so on.

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Appendix A

The rotation matrix $\mathbf{M}$ in Eq. (4) may be written as [4]

$$\mathbf{M} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \tag{59}$$

where

$$\begin{aligned} \sin\theta &= \sqrt{\frac{1}{2}\left(1 - \frac{\omega_a - \omega_b}{\sqrt{\left(\omega_a - \omega_b\right)^2 + 4g^2}}\right)}, \\ \cos\theta &= \sqrt{\frac{1}{2}\left(1 + \frac{\omega_a - \omega_b}{\sqrt{\left(\omega_a - \omega_b\right)^2 + 4g^2}}\right)}, \end{aligned} \tag{60}$$

what leads to the following expression for the frequencies $\bar{\omega}_a$ and $\bar{\omega}_b$ in Eqs. (3):

$$\begin{aligned} \bar{\omega}_a &= \frac{1}{2}\left(\omega_a + \omega_b + \sqrt{\left(\omega_a - \omega_b\right)^2 + 4g^2}\right), \\ \bar{\omega}_b &= \frac{1}{2}\left(\omega_a + \omega_b - \sqrt{\left(\omega_a - \omega_b\right)^2 + 4g^2}\right). \end{aligned} \tag{61}$$

Appendix B

The transformation matrix $\mathbf{M}$ in Eq. (11) may be written in the form [4]

$$\mathbf{M} = \begin{pmatrix} \eta\mu\cos\theta & \eta^{-1}\mu\sin\theta & 0 & 0 \\ -\eta\nu\sin\theta & \eta^{-1}\nu\cos\theta & 0 & 0 \\ 0 & 0 & \eta^{-1}\mu^{-1}\cos\theta & \eta\mu^{-1}\sin\theta \\ 0 & 0 & -\eta^{-1}\nu^{-1}\sin\theta & \eta\nu^{-1}\cos\theta \end{pmatrix}, \tag{62}$$

where

$$\eta = \left(\frac{\omega_a}{\omega_b}\right)^{\frac{1}{4}},$$

$$\mu = \left\{\frac{1}{2}\left(\frac{\omega_a}{\omega_b} + \frac{\omega_b}{\omega_a} + \sqrt{\left(\frac{\omega_a}{\omega_b} - \frac{\omega_b}{\omega_a}\right)^2 + \frac{4g^2}{\omega_a\omega_b}}\right)\right\}^{\frac{1}{4}},$$

$$\nu = \left\{\frac{1}{2}\left(\frac{\omega_a}{\omega_b} + \frac{\omega_b}{\omega_a} - \sqrt{\left(\frac{\omega_a}{\omega_b} - \frac{\omega_b}{\omega_a}\right)^2 + \frac{4g^2}{\omega_a\omega_b}}\right)\right\}^{\frac{1}{4}},$$

$$\sin\theta = \sqrt{\frac{1}{2}\left(1 - \frac{\frac{\omega_a}{\omega_b} - \frac{\omega_b}{\omega_a}}{\sqrt{\left(\frac{\omega_a}{\omega_b} - \frac{\omega_b}{\omega_a}\right)^2 + \frac{4g^2}{\omega_a\omega_b}}}\right)},$$

$$\cos\theta = \sqrt{\frac{1}{2}\left(1 + \frac{\frac{\omega_a}{\omega_b} - \frac{\omega_b}{\omega_a}}{\sqrt{\left(\frac{\omega_a}{\omega_b} - \frac{\omega_b}{\omega_a}\right)^2 + \frac{4g^2}{\omega_a\omega_b}}}\right)}. \tag{63}$$

The frequencies $\bar{\omega}_a$ and $\bar{\omega}_b$ in Eqs. (10)) are given by

$$\bar{\omega}_a = \sqrt{\frac{1}{2}\left(\omega_a^2+\omega_b^2+\sqrt{\left(\omega_a^2-\omega_b^2\right)^2+4g^2\omega_a\omega_b}\right)},$$

$$\bar{\omega}_b = \sqrt{\frac{1}{2}\left(\omega_a^2+\omega_b^2-\sqrt{\left(\omega_a^2-\omega_b^2\right)^2+4g^2\omega_a\omega_b}\right)}. \quad (64)$$

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