Optimal All-pay Auction When Signals Are Correlated*

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Summary: 1. Introduction; 2. The model; 3. The all-pay auction; 4. Surplus extraction; 5. Conclusion.
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This paper proves the existence of the optimal all-pay auction when signals are correlated. In an all-pay auction every bidder pays his bid. The war of attrition is an auction in which every bidder but the winner pays his bid. The winner pays the second highest bid. Recently Krishna and Morgan showed that the war of attrition, if signals are correlated, dominates the all-pay auction. Examples in the paper show that the optimal all-pay auction may be optimal among all auctions and may dominate the war of attrition.

Neste artigo demonstro a existência do leilão *all-pay* ótimo quando os sinais são correlacionados. Um leilão *all-pay* é um leilão no qual cada licitante paga seu lance independentemente de ganhar o objeto. A guerra de atrito é um leilão no qual cada licitante paga seu lance, exceto o vencedor, que paga o segundo maior lance. Recentemente Krishna e Morgan demonstraram que a guerra de atrito, se os sinais são correlacionados, domina o leilão *all-pay.* Exemplos neste artigo mostram que o leilão ótimo *all-pay* pode ser ótimo entre todos os leilões e pode dominar a guerra de atrito.

1. Introduction

In an all-pay auction every bidder pays his bid and the highest bidder receives the object. Although not the most common auction, the all-pay auction also happens in the practice. An all-pay auction can be seen as a tournament where a participant's bid is equal to his effort. A raffle is an all-pay auction in which every participant buys one or more of a fixed number of tickets and the winner is chosen randomly among the sold tickets. Alternatively we can look at the all-pay auction as a generalized raffle.

The theoretical studies of all-pay auctions are not very numerous. For example, Baye et alii (1996) study the all-pay auction with complete information. Krishna and Morgan (1997) study the war of attrition and the all-pay

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auction. They calculate equilibrium strategies for both auctions and show that the war of attrition, in general, dominates the all-pay auction. The purpose of this paper is to find the optimal all-pay auction when signals are correlated.

The study of optimal auctions begun with Myerson (1981), who determined the optimal auction of one object when the bidders signals are stochastically independent. In the symmetric case the optimal auction is a second price sealed bid auction with a reserve price. However, not much is known about optimal auctions if signals are not stochastically independent. Page (1998) proves the existence of optimal auctions in a general setting. An explicit characterization of the optimal auction is, naturally, not possible in the very general model of Page. The optimal all-pay auction is an optimal auction in the independent private values case. An example will show that the optimal all-pay auction may dominate the war of attrition. Another example (from Bulow and Klemperer, 1996) shows that the all-pay auction can extract all the surplus.

2. The Model

The set of bidders is $\mathbb{I} = \{1, 2, \ldots, I\}$. There is one object to be sold at auction. Bidder's i has a signal $X_i : \Omega \to \mathbb{R}_+$, a random variable defined on the probability space $(\Omega, \mathcal{A}, P)$. The random vector of all signals is $X = (X_1, \ldots, X_I)$. It is possible to consider an unobserved signal $\mathbb{S} : \Omega \to \mathbb{R}^l_+$ as in Milgrom and Weber (1988). This would permit the inclusion of common value models. Though this generalization doesn't present special difficulties is notationally a bit uncomfortable. Therefore, I decided for the more concise model. Finally bidder's i evaluation of the object is the random variable $V_i = u_i(X)$, where $u_i : \mathbb{R}^I_+ \to \mathbb{R}$ is a continuously differentiable function. The distribution, F, of the random vector X has a continuously differentiable density $f : Z \to \mathbb{R}_{++}$, $\int_Z f(x)dx = 1$. For each vector $x = (x_1, \ldots, x_I)$ I define the vector $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_I)$ obtained by the removal of the i^{th} coordinate of x. I also write $x = (x_i, x_{-i})$. I suppose that $\sup\{u^i(z); z \in Z\} > 0$ and that the support Z of the distribution F is a cartesian product: $Z = \Pi_{i=1}^I [m_i, n_i]$. The bidder i signal X^i also has density $f^i, f^i(x_i) = \int_{Z_{-i}} f(x_i, x_{-i})dx_{-i}$. Here $Z_{-i} = \{z_{-i}; z \in Z\}$. The conditional density of bidder i given his signal is $f(x_{-i}|x_i) = \frac{f(x)}{f^i(x_i)}$. It is convenient to define $\widehat{u}_i(x) = u_i(x)f(x_{-i}|x_i)$. It is useful to include the seller as bidder 0. Thus, $\mathbb{B} = \mathbb{I} \cup \{0\}$ is the set of bidders. If $A \neq \emptyset$ is finite define $S(A) = \{(q_a)_{a\in A}; \sum_{a\in A} q_a = 1$ and for every $a \in A, q_a \geq 0\}$.

Lemma 1. *The densities $f(x)$ and $f(x_{-i}|x_i)$ are uniformly lipschitz in x.*

Proof. The density f is lipschitz since its support is compact and its derivative is continuous. From

$$\begin{aligned} |f^i(a) - f^i(b)| &= \left| \int f(a, x_{-i})dx_{-i} - \int f(b, x_{-i})dx_{-i} \right| \\ &\leq \int |f(a, x_{-i}) - f(b, x_{-i})|dx_{-i} \leq L'|a-b| \end{aligned}$$

it follows that f^i is uniformly lipschitiz too. Then, using that $inf_{z\in Z} f(z) > 0$ and the inequality:

$$\left| \frac{f(a, x_{-i})}{f^i(a)} - \frac{f(b, x_{-i})}{f^i(b)} \right| \leq \frac{|f(a, x_{-i}) - f(b, x_{-i})|}{f^i(a)} + f(b, x_{-i}) \frac{|f^i(b) - f^i(a)|}{f^i(a)f^i(b)}$$

the conditional density $f(x_{-i}|x_i) = \frac{f(x)}{f^i(x_i)}$ is proved lipschitiz. *QED*

The auction will proceed in the following way:

a) the seller publicly announces the functions[1] $q : \mathbb{R}^I_+ \to S(\mathbb{B})$ and $P = (P^1, \ldots, P^I), P^i : \mathbb{B} \times \mathbb{R}^{\mathbb{I}}_+ \to \mathbb{R}, 1 \leq i \leq I$;
b) each bidder, knowing his signal X_i, announces a number $x_i \geq 0$ privately to the seller; the seller forms the vector $x = (x_1, \ldots, x_I)$;
c) the objects are delivered accordingly to $u \in \mathbb{B}$ drawn with probability $q_u(x) = q(x)(u), u \in \mathbb{B}$;
d) bidder $i \in \mathbb{I}$ pays $P^i(u, x)$ if u happens.

As in Myerson (1981:62-3), the direct mechanisms (q, P) will be chosen among those that satisfy individual rationality and incentive compatibility constraints. Define the function $T_i(x_i)$ as the expected utility of bidder i if his signal is x_i. Thus, defining $P^i(x) = \sum_{u=0}^{I} q_u(x)P^i_u(x)$, we have that

$$T_i(x_i) = \int q_i(x)\widehat{u}_i(x)\, dx_{-i} - \int P^i(x)f(x_{-i}|x_i)dx_{-i} \tag{1}$$

[1] *Henceforward called direct mechanisms. The mechanism P^i satisfies $\int |P^i(x)|f(x_{-i}|x_i)$ $dx_{-i} < \infty$ for every x_i.*

If i has valuation x_i and declares valuation x_i', then we have the incentive compatibility constraints:

$$T_i(x_i) \geq \int q_i(x_i', x_{-i})\widehat{u}_i(x)\, dx_{-i} - \int P^i(x_i', x_{-i}) f(x_{-i}|x_i) dx_{-i} \tag{2}$$

The individual rationality constraints are:

$$T_i(x_i) \geq 0 \tag{3}$$

Lemma 2. *The following is true:*

(u) T_i *is a primitive:* $T_i(z) = r_i + \int_0^z T_i'(u)du$ *for every* $z \geq 0$*;*

(v) for almost every $x_i \in (m_i, n_i)$*,*

$$T_i'(x_i) = \int q_i(x) D_{x_i}(\widehat{u}_i(x)) dx_{-i} - \int P^i(x) D_{x_i} f(x_{-i}|x_i) dx_{-i} \tag{4}$$

Proof. Suppose $m_i < a < b < n_i$. From the incentive compatibility constraints we have

$$T_i(b) - T_i(a) \geq \int q_i(a, x_{-i}) \left[\widehat{u}_i(b, x_{-i}) - \widehat{u}_i(a, x_{-i})\right] dx_{-i} - \int P^i(a, x_{-i})(f(x_{-i}|b) - f(x_{-i}|a)) dx_{-i} \tag{5}$$

From lemma 1 there exists K such that $|\widehat{u}_i(b, x_{-i}) - \widehat{u}_i(a, x_{-i})| \leq K|b-a|$ and $|f(x_{-i}|b) - f(x_{-i}|a)| \leq K|b-a|$ for every x_{-i}. Since $\sum_{i=0}^{I} q_i(x) = 1$, we have that

$$T_i(b) - T_i(a) \geq -|b-a| \left(K + K \int |P^i(a, x_{-i})| dx_{-i}\right) \geq -K'|b-a|$$

Changing places between a and b, we obtain $T_i(a) - T_i(b) \geq -K'|b-a|$. So $|T_i(b) - T_i(a)| \leq K'|b-a|$. Therefore, T_i is absolutely continuous since it is lipschitz. Hence, it is almost everywhere differentiable and a primitive. This proves item (u). Suppose $a \in (m_i, n_i)$ is a point of differentiability of T_i. If $b > a$ then

$$\frac{T_i(b) - T_i(a)}{b - a} \geq \int q_i(a, x_{-i}) \frac{\widehat{u}_i(b, x_{-i}) - \widehat{u}_i(a, x_{-i})}{b - a} dx_{-i} - \int P^i(a, x_{-i}) \frac{f(x_{-i}|b) - f(x_{-i}|a)}{b - a} dx_{-i} \tag{6}$$

Hence, using the dominated convergence theorem,[2]

$$T_i'(a) \geq \int q_i(a, x_{-i}) D_a \widehat{u}_i(a, x_{-i}) dx_{-i} - \int P^i(a, x_{-i}) D_a f(x_{-i}|a) dx_{-i} \tag{7}$$

Analogously, if $b < a$ we obtain the inequality (7) reversed and therefore (4). *QED*

3. The All-pay Auction

3.1 All-pay auction definition

The all-pay auction is an auction in which every bidder makes a payment as a function only of his bid, independently of receiving the object. The payment function is therefore of the form $P^i(x) = \hat{P}^i(x_i)$. Formally the auction proceeds as follows:

a) the seller publicly announces the functions $q : \mathbb{R}_+^I \to S(\mathbb{B})$ and $P = (P^1, \ldots, P^I)$, $P^i : \mathbb{R}_+ \to \mathbb{R}$, $1 \leq i \leq I$;
b) each bidder, knowing his signal X_i, announces a number $x_i \geq 0$, privately to the seller; the seller forms the vector $x = (x_1, \ldots, x_I)$;
c) the object is delivered accordingly to $u \in \mathbb{B}$ drawn with probability $q(x)(u)$;
d) bidder i pays $P^i(x_i)$, $1 \leq i \leq I$.

Equation (4) above is crucial. In the independent private signals case the density $f(x_{-i}|x_i)$ does not depend on x_i. Therefore $D_{x_i} f(x_{-i}|x_i) = 0$ for every x_i and the payment function P^i disappears from equation (4). This property is preserved in the correlated signals case if the payment P^i

[2] *The individual rationality constraints imply that* $\int |P^i(a, x_{-i})| dx_{-i} < \infty$.

is a function of x_i only, that is, if $P^i(x) = \hat{P}^i(x_i)$. In this case, since $\int D_{x_i} f(x_{-i}|x_i)dx_{-i} = D_{x_i}\left(\int f(x_{-i}|x_i)dx_{-i}\right) = D_{x_{-i}}1 = 0$, we have that

$$\int \hat{P}^i(x_i) D_{x_i} f(x_{-i}|x_i)dx_{-i} = \hat{P}^i(x_i)\int D_{x_i} f(x_{-i}|x_i)dx_{-i} = 0$$

We have from lemma 2 that

$$T_i'(x_i) = \int q_i(x) D_{x_i}\widehat{u}_i(x)\,dx_{-i} \tag{8}$$

Equation (8) and lemma 4 show that an all-pay auction mechanism is determined by $(q_i)_{i\in\mathbb{I}}$. The next lemma gives a sufficient condition for such mechanisms to satisfy the incentive compatibility constraints.

Lemma 3. *Suppose $D_{x_i}(\widehat{u}_i(x)) \geq 0$ for every $x \in Z$ and $q_i(x)$ is increasing in x_i. If we define*

$$\begin{aligned}\widehat{P}^i(z) = &\int q_i(z, x_{-i})\widehat{u}_i(z, x_{-i})dx_{-i}\\ &- \int_0^z \left[\int q_i\left(\omega, x_{-i}\right) D_\omega\left(\widehat{u}_i(\omega, x_{-i})\right) dx_{-i}\right] d\omega\end{aligned}$$

then the mechanisms $(q_i, \widehat{P}^i)_{i\in\mathbb{I}}$ satisfy the individual rationality (3) and the incentive compatibility constraints (2).

Proof. If $z', z \geq 0$, we have to prove that

$$\begin{aligned}&\int_0^z \left[\int q_i(\omega, x_{-i}) D_\omega\left(\widehat{u}_i(\omega, x_{-i})\right) d_{x_{-i}}\right] d\omega =\\ &\int q_i(z, x_{-i})\widehat{u}_i(z, x_{-i})dx_{-i} - \widehat{P}^i(z) \geq\\ &\int q_i(z', x_{-i})\widehat{u}_i(z, x_{-i})dx_{-i} - \widehat{P}^i(z') =\\ &\int q_i(z', x_{-i})\left(\widehat{u}_i(z, x_{-i}) - \widehat{u}_i(z', x_{-i})\right) dx_{-i}+\\ &\int_0^{z'} \left[\int q_i(\omega, x_{-i}) D_\omega\left(\widehat{u}_i(\omega, x_{-i})\right) d_{x_{-i}}\right] d\omega\end{aligned}$$

Suppose first that $z \geq z'$. Then we have to prove that

$$\int_{z'}^{z} \left[\int q_i(\omega, x_{-i}) D_\omega(\widehat{u}_i(\omega, x_{-i})) d_{x_{-i}} \right] d\omega \geq$$
$$\int q_i(z', x_{-i})(\widehat{u}_i(z, x_{-i}) - \widehat{u}_i(z', x_{-i})) dx_{-i}$$

This is true since

$$\int_{z'}^{z} \left[\int q_i(\omega, x_{-i}) D_\omega(\widehat{u}_i(\omega, x_{-i})) d_{x_{-i}} \right] d\omega \geq$$
$$\int_{z'}^{z} \left[\int q_i(z', x_{-i}) D_\omega(\widehat{u}_i(\omega, x_{-i})) d_{x_{-i}} \right] d\omega =$$
$$\int q_i(z', x_{-i}) \left[\int_{z'}^{z} D_\omega(\widehat{u}_i(\omega, x_{-i})) d\omega \right] d_{x_{-i}} =$$
$$\int q_i(z', x_{-i})(\widehat{u}_i(z, x_{-i}) - \widehat{u}_i(z', x_{-i})) dx_{-i}$$

The case $z' < z$ is analogously proved. The individual rationality constraints follows from

$$\int q_i(z, x_{-i}) \widehat{u}_i(z, x_{-i}) dx_{-i} - \widehat{P}^i(z) =$$
$$\int_{0}^{z} \left[\int q_i(\omega, x_{-i}) D_\omega(\widehat{u}_i(\omega, x_{-i})) d_{x_{-i}} \right] d\omega \geq 0$$

3.2 The seller's revenue

From the definition of T_i we obtain the payment function, P^i:

$$P^i(z) = \int q_i(z, x_{-i}) \widehat{u}_i(z, x_{-i}) dx_{-i} - r_i - \int_0^z T_i'(\omega) d\omega$$

Since the individual rationality constraint requires that $T_i(0) = r_i \geq 0$, the seller will choose $r_i = 0$. Therefore, the seller's expected revenue from bidder i is

$$\int P^i(x) f(x) dx = \int P^i(x_i) f^i(x_i) dx_i =$$
$$\int q_i(x) \widehat{u}_i(x) f^i(x_i) dx - \int \chi_{(0, x_i)}(\omega) T_i'(\omega) f^i(x_i) d\omega \, dx_i =$$
$$\int q_i(x) u_i(x) f(x) dx - \int q_i(\omega, x_{-i})(1 - F^i(\omega)) D_\omega(\widehat{u}_i(\omega, x_{-i})) dx_{-i} \, d\omega$$

Above, equation (8) was used to eliminate T_i'. Define

$$h^i(x) = u_i(x) - \frac{1 - F^i(x_i)}{f^i(x_i)} \frac{D_{x_i}(\widehat{u}_i(x))}{f(x_{-i}|x_i)} \tag{9}$$

if $i \in \mathbb{I}$ and $h^0(x) = 0$. The seller revenue is

$$\int \sum_{i=1}^{I} P^i(x) f(x) dx = \int \sum_{i=1}^{I} q_i(x) h^i(x) f(x)\, dx = \int \sum_{i=0}^{I} q_i(x) h^i(x) f(x)\, dx \tag{10}$$

The following hypothesis permits to obtain the optimal auction in analogy with Myerson (1981):

$$h^i(x) = h^j(x) \text{ and } i \neq j \Rightarrow \frac{\partial}{\partial x_i}(h^i(x) - h^j(x)) > 0 \tag{11}$$

To understand condition (11) better, let us consider the particular case of private independent values and that $u_i(x) = x_i$. Then $h^i(x) = x_i - \frac{1-F^i(x_i)}{f^i(x_i)}$. Condition (11) is satisfied if $(x_i - \frac{1-F^i(x_i)}{f^i(x_i)})' > 0$. This is the regular case in Myerson's paper. Myerson also found the optimal auction in the non-regular case. His method, however, doesn't generalize, since the expected probability $\int q_i(x) f(x_{-i}|x_i) dx_{-i}$ may not be monotonic.

Theorem. *Suppose (11) and that $D_{x_i}(\widehat{u}^i(x)) \geq 0$ for every x_i. Then the optimal auction mechanisms $(q_i, P^i)_{i\in\mathbb{I}}$ are the following:*

a) $q_i(x) = \begin{cases} 0 & \text{if} \quad h^i(x) < \max_j h^j(x) \\ \frac{1}{\#\{i; h^i(x) = \max_j h^j(x)\}} & \text{if} \quad h^i(x) = \max_j h^j(x); \end{cases}$

b) $P^i(x_i) = \int q_i(x) \widehat{u}^i(x) d_{x_{-i}} - \int_0^{x_i} [q_i(\omega, x_{-i}) D_\omega \widehat{u}^i(\omega, x_{-i})] d\omega.$

Moreover, the seller expected revenue is $\int \max_i h^i(x) f(x) dx$.

Proof. Using equality (10) it is immediate that $q_i(x)$ defined above maximizes the seller revenue. In this case the seller's expected revenue is $\int \max_i h^i(x) f(x) dx$. It remains to be checked if the mechanisms $(q_i, P^i)_{i\in\mathbb{I}}$ satisfy the individual rationality and incentive compatibility contraints. However, this is clear from lemma (3).

Let us consider, for example, the two bidder symmetric case. I.e. $\mathbb{I} = \{1, 2\}$ and $f(x, y) = f(y, x)$. Then $h^1(x, y) = h^2(y, x)$. If (11) is true, then bidder 1 wins the object if $h^1(x, y) \geq 0$ and $x > y$.

In the next example the optimal all-pay auction is calculated with some detail.

Example. *There are two bidders with valuations $u_i(x_1, x_2) = x_i$, $i = 1, 2$. The signals vector $X = (X_1, X_2)$ has a distribution with density $f(x, y) = \frac{4(1+xy)}{5}$, $0 \leq x, y \leq 1$. The first thing to calculate is the marginal $f^1(x) = \int_0^1 f(x, y)dy = \frac{4+2x}{5}$. The distribution of $X_1$is, therefore, $F^1(x) = \int_0^x f^1(z)dz = \frac{4x+x^2}{5}$. We now find the conditional density $f(y|x) = \frac{f(x,y)}{f^1(x)} = \frac{2(1+xy)}{2+x}$.*

Thus, from (9) we have

$$\begin{aligned} h^1(x, y) &= x - \frac{(5 - 4x - x^2) D_x(x\frac{2(1+xy)}{2+x})}{4(1 + xy)} \\ &= x - \frac{(5 - 4x - x^2)(2 + 4xy + x^2 y)}{2(2 + x^2)(1 + xy)} \end{aligned}$$

We can see that (11) is true. The region in which bidder 1 never wins the object, i.e., $h^1(x, y) < 0$ is given by

$$2(x^3 + x^2 + 6x - 5) < yx(20 - 15x - 8x^2 - 3x^3)$$

If x is small, then $h^1(x, y) < 0$ for every y. If x is near 1, then $h^1(x, y) \geq 0$ for any y. But, for example, if $x = 3/4$, then $h^1(x, y) < 0$ if $y > 0.42$. Thus, in the region $0.42 < y < 3/4 = x$, the object is not delivered. The region in which bidder 2 predominates over bidder 2 is given by $h^1(x, y) - h^1(y, x) > 0$. Equivalently $x > y$. The seller's expected profit is $\int_0^1 \frac{\max\{h^1(x,y), h^1(y,x), 0\}(1+xy)}{5} dxdy \approx 0.457$.

To compare with the war of attrition, define $F(y|x) = \int_0^y f(z|x)dz$ and $\lambda(y|x) = \frac{f(x,y)}{1-F(y|x)}$. The expected payment of each bidder, if his signal is x, is (Krishna & Morgan, 1997:352, eq. 13):

$$e^W(x) = \int_0^x y f(y|x) \frac{\lambda(y|y|)}{\lambda(y|x)} \, dy$$

The seller's expected revenue is $2 \int_0^1 e^W(x) f^1(x) dx \approx 0.389$.

The following corollary will be used in the next section.

Corollary. *The seller's expected revenue is less than* $\int \max_{i\in\mathbb{I}} u^i(x)f(x)dx$ *if, for every* i, $D_{x_i}\widehat{u}^i(x) > 0$ *for every* $x \in Z$.

Proof. Immediate, since $h^i(x) < u^i(x)$ for every $i \in \mathbb{I}$.

4. Surplus Extraction

This section examines, briefly the issue of surplus extraction. It is easily seen from the individual rationality constraints that the seller's expected revenue is never greater than $\int(\max_{1\leq i\leq I} u_i(x))f(x)dx$. This supremum is never achieved if signals are independent. Moreover, the corollary in the previous section shows that the all-pay auction never extracts the full surplus when the distribution of signals has a density. The "full surplus extraction" is, therefore, not possible with independent signals or with the all-pay auction. However, full surplus extraction is possible if signals are correlated. McAfee and Reny's paper (1992) gives necessary and sufficient conditions for full surplus extraction. The condition is on the conditional density $f(x_{-i}|x_i)$. The optimal mechanism charges a bidder a fee that is a function of the *other bidders valuations.* Naturally, the optimal all-pay auction will not be, in general, optimal among all auctions. However, sometimes the all-pay auction extracts all the surplus and is, therefore, optimal among all auction mechanisms.The optimal all-pay auction doesn't extracts all the surplus when there are densities. In the following example[4] the all-pay auction extract all the surplus.

Example. *There are two bidders, whose values can be written as* $v_1 = A + B$ *and* $v_2 = C + B$, *where* A, B *and* C *are independent, uniformly distributed in* $[0, 1]$. *However, the bidders don't know* A, B *and* C. *The mechanism that charges* $P^i(v_i) = \frac{v_i^2}{2}$, $i = 1, 2$ *and gives the object to the highest bidder extracts all the surplus.*

[4] *This example is from Bulow and Klemperer (1984).*

5. Conclusion

This paper has studied optimal all-pay auctions. The signals can be correlated and bidders can be assymetric. The optimal all-pay auction is an optimal auction in the independent private values case. It is not in general optimal among all auctions. Although the all-pay auction is generally dominated by the war of attrition, the optimal all-pay auction is not so dominated. Definition (9) is the appropriate generalization of Myerson's marginal valuations to the correlated signals case.

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