

Noncommutative Configuration Space. Classical and Quantum Mechanical Aspects

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In this work we examine noncommutativity of position coordinates in classical symplectic mechanics and its quantisation. In coordinates $\{q^i, p_k\}$ the canonical symplectic two-form is $\omega_0 = dq^i \wedge dp_i$. It is well known in symplectic mechanics [5, 6, 9] that the interaction of a charged particle with a magnetic field can be described in a Hamiltonian formalism without a choice of a potential. This is done by means of a modified symplectic two-form $\omega = \omega_0 - e\mathbf{F}$, where e is the charge and the (time-independent) magnetic field \mathbf{F} is closed: $d\mathbf{F} = 0$. With this symplectic structure, the canonical momentum variables acquire non-vanishing Poisson brackets: $\{p_k, p_l\} = eF_{kl}(q)$. Similarly a closed two-form in p -space \mathbf{G} may be introduced. Such a *dual magnetic field* \mathbf{G} interacts with the particle's *dual charge* r . A new modified symplectic two-form $\omega = \omega_0 - e\mathbf{F} + r\mathbf{G}$ is then defined. Now, both p - and q -variables will cease to Poisson commute and upon quantisation they become noncommuting operators. In the particular case of a linear phase space \mathbf{R}^{2N} , it makes sense to consider constant \mathbf{F} and \mathbf{G} fields. It is then possible to define, by a linear transformation, global Darboux coordinates: $\{\xi^i, \pi_k\} = \delta^i_k$. These can then be quantised in the usual way $[\widehat{\xi}^i, \widehat{\pi}_k] = i\hbar\delta^i_k$. The case of a quadratic potential is examined with some detail when N equals 2 and 3.

Keywords: Noncommutativity; Symplectic mechanics; Quantization

I. INTRODUCTION

The idea to consider non vanishing commutation relations between position operators $[\mathbf{x}, \mathbf{y}] = i\ell^2$, analogous to the canonical commutation relations between position and conjugate momentum $[\mathbf{x}, \mathbf{p}_x] = i\hbar$, is ascribed to Heisenberg, who saw there a possibility to introduce a fundamental length ℓ which might control the short distance singularities of quantum field theory. However, noncommutativity of coordinates appeared first nonrelativistically in the work of Peierls [2] on the diamagnetism of conduction electrons. In the limit of a strong magnetic field in the z -direction, the gap between Landau levels becomes large and, to leading order, one obtains $[\mathbf{x}, \mathbf{y}] = i\hbar c/eB$. In relativistic quantum mechanics, noncommutativity was first examined in 1947 by Snyder [3] and, in the last five years, inspired by string and brane-theory, many papers on field theory in noncommutative spaces appeared in the physics literature. The apparent unitarity problem related to time-space noncommutativity in field theory was studied and solved in [10]. Also (nonrelativistic) quantum mechanics on noncommutative twodimensional spaces has been examined more thoroughly in the recent years: [11–16]. The above mentioned unitarity problem in quantum physics is also examined in Balachandran et al. [17].

In this work we discuss noncommutativity of configuration space Q in classical mechanics on the cotangent bundle $T^*(Q)$ and its canonical quantisation in the most simple case. In section II we review the classical theory of a non relativistic particle interacting with a time-independent magnetic field $\mathbf{F} = 1/2F_{ij}(q)dq^i \wedge dq^j$; $d\mathbf{F} = 0$. This is done in every textbook introducing a potential in a Lagrangian formalism. The Legendre transformation defines then the Hamiltonian and the

canonical symplectic two-form $dq^i \wedge dp_i$ implements the corresponding Hamiltonian vector field. We also recall the less well known procedure of avoiding the introduction of a potential using a modified symplectic structure: $\omega = dq^i \wedge dp_i - e\mathbf{F}$. The coupling with the charge e is hidden in the symplectic structure and does not show up in the Hamiltonian: $H_0(q, p) = \delta^{kl} p_k p_l / 2m + \mathcal{V}(q)$. In section III, a closed two-form in p -space, the *dual field*: $\mathbf{G} = 1/2G^{kl}(p)dp_k \wedge dp_l$, is added to the symplectic structure $\omega = dq^i \wedge dp_i - e\mathbf{F} + r\mathbf{G}$, where r is a *dual charge*.

Such an approach with a modified symplectic structure has been previously considered by Duval and Horvathy [11, 14] emphasizing the $N = 2$ -dimensional case in connection with the quantum Hall effect. We should also mention Plyushchay's interpretation [18] of such a dual charge r when $N = 2$ as the anyon spin. Considering here an arbitrary number of dimensions N , no such interpretation of r is assumed. The crucial point is that, now, both p - and q -variables cease to Poisson commute and upon quantisation they should become noncommuting operators. In the particular case of a linear phase space \mathbf{R}^{2N} , it makes sense to consider constant \mathbf{F} and \mathbf{G} fields. It is then possible to define global Darboux coordinates with Poisson brackets $\{\xi^i, \pi_k\} = \delta^i_k$. These can then be quantised uniquely [1] in the usual way: $[\widehat{\xi}^i, \widehat{\pi}_k] = i\hbar\delta^i_k$. However, in general, the dynamics become non-linear and there is no guarantee that the Hamiltonian vector field is complete. It is then not trivial to quantise the Hamiltonian, which becomes nonlocal. However, for a linear or quadratic Hamiltonian, this is possible and it is seen that the noncommutativity generates a magnetic moment type interaction. The cases $N = 2$ and $N = 3$ are discussed in detail in section IV. In section V we examine the problem of symmetries in the modified symplectic manifold. Finally, in section VI general comments are made and further developments are suggested. In appendix A we recall

basic notions in symplectic geometry and in appendix B we give a brief account of the Gotay-Nester-Hinds algorithm [7] for constrained Hamiltonian systems.

II. NON RELATIVISTIC PARTICLE INTERACTING WITH A TIME-INDEPENDENT MAGNETIC FIELD

A particle of mass m and charge e , with potential energy \mathcal{V} , moving in a Euclidean configuration space Q , with cartesian coordinates q^i , interacts with a (time-independent) magnetic field given by a closed two-form $\mathbf{F}(q) = \frac{1}{2} F_{ij}(q) \mathbf{d}q^i \wedge \mathbf{d}q^j$. The dynamics is given by the Laplace equation:

$$m \frac{\mathbf{d}^2 q^i}{\mathbf{d}t^2} = \delta^{ij} \left(e F_{jk}(q) \frac{\mathbf{d}q^k}{\mathbf{d}t} - \frac{\partial \mathcal{V}(q)}{\partial q^j} \right). \quad (\text{II.1})$$

Assuming Q to be Euclidean avoids topological subtleties, so that there exists a global potential one-form $\mathbf{A}(q) = A_i(q) \mathbf{d}q^i$ such that $\mathbf{F} = \mathbf{d}\mathbf{A}$. A global Lagrangian formalism can then be established with a Lagrangian function on the tangent bundle $\{\tau: T(Q) \rightarrow Q\}$:

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} m \delta_{ij} \dot{q}^i \dot{q}^j + e \dot{q}^i A_i(q) - \mathcal{V}(q).$$

The Euler-Lagrange equation is obtained as:

$$\begin{aligned} 0 &= \frac{\partial \mathcal{L}}{\partial q^i} - \frac{\mathbf{d}}{\mathbf{d}t} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = -\frac{\partial \mathcal{V}}{\partial q^i} + e \dot{q}^k \frac{\partial A_k(q)}{\partial q^i} - \frac{\mathbf{d}}{\mathbf{d}t} (m \delta_{ij} \dot{q}^j + e A_i(q)) \\ &= -\frac{\partial \mathcal{V}}{\partial q^i} + e \dot{q}^k \left(\frac{\partial A_k(q)}{\partial q^i} - \frac{\partial A_i(q)}{\partial q^k} \right) - m \frac{\mathbf{d}}{\mathbf{d}t} \delta_{ij} \dot{q}^j \\ &= -\frac{\partial \mathcal{V}}{\partial q^i} + e \mathbf{F}_{ik}(q) \dot{q}^k - m \delta_{ij} \ddot{q}^j, \end{aligned} \quad (\text{II.2})$$

and coincides with the Laplace equation (II.1).

The Legendre transform

$$(q^i, \dot{q}^j) \rightarrow \left(q^i, p_k = \frac{\partial \mathcal{L}}{\partial \dot{q}^k} = m \delta_{kl} \dot{q}^l + e A_k(q) \right),$$

defines the Hamiltonian on the cotangent bundle $\{T^*(Q) \xrightarrow{\kappa} Q\}$:

$$\mathcal{H}_{\mathbf{A}}(q, p) = -\mathcal{L}(q, \dot{q}) + p_i \dot{q}^i =$$

$$\frac{1}{2m} \delta^{kl} (p_k - e A_k(q)) (p_l - e A_l(q)) + \mathcal{V}(q).$$

With the canonical symplectic two-form

$$\omega_0 = \mathbf{d}q^i \wedge \mathbf{d}p_i, \quad (\text{II.3})$$

the Hamiltonian vector field of $\mathcal{H}_{\mathbf{A}}$ is:

$$\mathbf{X}_{\mathcal{H}} = \frac{\delta^{ij}}{m} (p_j - e A_j) \frac{\partial}{\partial q^i} + \left(\frac{e}{m} \delta^{kl} \frac{\partial A_k}{\partial q^i} (p_l - e A_l) - \frac{\partial \mathcal{V}}{\partial q^i} \right) \frac{\partial}{\partial p_i}.$$

Its integral curves are solutions of:

$$\frac{\mathbf{d}q^i}{\mathbf{d}t} = \frac{\delta^{ij}}{m} (p_j - e A_j), \quad \frac{\mathbf{d}p_i}{\mathbf{d}t} = \frac{e}{m} \delta^{kl} \frac{\partial A_k}{\partial q^i} (p_l - e A_l) - \frac{\partial \mathcal{V}}{\partial q^i}, \quad (\text{II.4})$$

which is again equivalent to (II.1).

If the second de Rham cohomology were not trivial, $H_{dR}^2(Q) \neq 0$, there is no global potential \mathbf{A} and a local Lagrangian formalism is needed. This can be done enlarging the configuration space Q to the total space \mathcal{P} of a principal $U(1)$ bundle over Q with a connection, given locally by \mathbf{A} [19]. This can be avoided using a global Hamiltonian formalism[20] in the cotangent bundle $T^*(Q)$ using a modified symplectic two-form:

$$\omega = \omega_0 - e \mathbf{F} = \mathbf{d}q^i \wedge \mathbf{d}p_i - \frac{1}{2} e F_{ij}(q) \mathbf{d}q^i \wedge \mathbf{d}q^j, \quad (\text{II.5})$$

and a "charge-free" Hamiltonian:

$$\mathcal{H}_0(p, q) = \frac{1}{2m} \delta^{kl} p_k p_l + \mathcal{V}(q).$$

The Hamiltonian vector fields corresponding to an observable $f(q, p)$ are now defined relative to ω as $\iota_{\mathbf{X}_f} \omega = \mathbf{d}f$ and given by:

$$\mathbf{X}_f = \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \left(\frac{\partial f}{\partial q^i} + \frac{\partial f}{\partial p_k} e F_{ki}(q) \right) \frac{\partial}{\partial p_i}.$$

With the Hamiltonian \mathcal{H}_0 , the dynamics are again given by the Laplace equation (II.1) in the form:

$$\frac{\mathbf{d}q^i}{\mathbf{d}t} = \frac{\delta^{ij}}{m} p_j; \quad \frac{\mathbf{d}p_l}{\mathbf{d}t} = -\delta^{ki} \left(\frac{\partial \mathcal{V}}{\partial q^i} + \frac{e}{m} p_i F_{kl}(q) \right). \quad (\text{II.6})$$

The Poisson brackets, relative to the symplectic structure II.5, are:

$$\{f, g\} = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} + \frac{\partial f}{\partial p_k} e F_{kl}(q) \frac{\partial g}{\partial p_l}. \quad (\text{II.7})$$

In particular, the coordinates themselves have Poisson brackets:

$$\begin{aligned} \{q^i, q^j\} &= 0, \quad \{q^i, p_l\} = \delta^i_l, \\ \{p_k, q^j\} &= -\delta_k^j, \quad \{p_k, p_l\} = e F_{kl}(q). \end{aligned} \quad (\text{II.8})$$

Obviously, the meaning of the $\{q, p\}$ variables in (II.3) and (II.5) are different. However both formalisms $(\omega_0, \mathcal{H}_A)$ and (ω, \mathcal{H}_0) lead to the same equations of motion and thus, they must be equivalent. Indeed, in each open set U homeomorphic to \mathbf{R}^6 , the vanishing $\mathbf{dF} = 0$ implies the existence of \mathbf{A} such that $\mathbf{F} = \mathbf{dA}$ in U and, locally:

$$\omega = \mathbf{d}q^i \wedge \mathbf{d}p_i - \frac{1}{2} e F_{ij} \mathbf{d}q^i \wedge \mathbf{d}q^j = -\mathbf{d}[(p_i + e A_i) \mathbf{d}q^i].$$

Thus there exist local Darboux coordinates:

$$\xi^i = q^i, \quad \pi_k = p_k + e A_k(q), \quad (\text{II.9})$$

such that $\omega = \mathbf{d}\xi^i \wedge \mathbf{d}\pi_i$, which is the form (II.3).

The dynamics defined by the Hamiltonian $\mathcal{H}_0(q, p) = p^2/2m + \mathcal{V}(q)$, with symplectic two-form ω , is equivalent to the dynamics defined by the Hamiltonian $\mathcal{H}_A(\xi, \pi) = (\pi - e A(\xi))^2/2m + \mathcal{V}(\xi)$ and canonical symplectic structure $\omega = \mathbf{d}\xi^i \wedge \mathbf{d}\pi_i$. Equivalence is trivial since both symplectic two-forms are equal, but expressed in different coordinates $\{q, p\}$ and $\{\xi, \pi\}$, related by (II.9). It seems worthwhile to note that a gauge transformation $\mathbf{A} \rightarrow \mathbf{A}' = \mathbf{A} + \mathbf{grad}\phi$ corresponds to a change of Darboux coordinates

$$\{\xi^i, \pi_k\} \Rightarrow \{\xi^{i'}, \pi'_k\} = \{\xi^i, \pi_k + e \partial_k \phi\},$$

i.e. a symplectic transformation.

III. NONCOMMUTATIVE COORDINATES

Let us consider an affine configuration space $Q = \mathbf{A}^N$ so that points of phase space, identified with $\mathcal{M} \doteq \mathbf{R}^{2N} = \mathbf{R}_q^N \times \mathbf{R}_p^N$, may be given by linear coordinates (q, p) . Together with the (usual) magnetic field \mathbf{F} , we may introduce a (dual) magnetic field $\mathbf{G} = 1/2 G^{kl}(p) \mathbf{d}p_k \wedge \mathbf{d}p_l$, a closed two-form, $\mathbf{dG} = 0$, in \mathbf{R}_p^N space. Let e be the usual electric charge and r , a dual charge, which couples the particle with \mathbf{F} and \mathbf{G} . Consider the closed two-form:

$$\begin{aligned} \omega &= \omega_0 - e\mathbf{F} + r\mathbf{G} \\ &= \mathbf{d}q^i \wedge \mathbf{d}p_i - \frac{1}{2} e F_{ij}(q) \mathbf{d}q^i \wedge \mathbf{d}q^j + \frac{1}{2} r G^{kl}(p) \mathbf{d}p_k \wedge \mathbf{d}p_l. \end{aligned} \quad (\text{III.1})$$

In matrix notation this two-form (III.1) is represented as:

$$\begin{aligned} (\Omega) &= \begin{pmatrix} -e\mathbf{F} & \mathbf{1} \\ -\mathbf{1} & +r\mathbf{G} \end{pmatrix} \\ &= \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & +r\mathbf{G} \end{pmatrix} \begin{pmatrix} -\Psi & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ -e\mathbf{F} & \mathbf{1} \end{pmatrix} \\ &= \begin{pmatrix} e\mathbf{F} & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \Phi \end{pmatrix} \begin{pmatrix} \mathbf{1} & -r\mathbf{G} \\ 0 & \mathbf{1} \end{pmatrix}. \end{aligned} \quad (\text{III.2})$$

where [21] $\Phi = (\mathbf{1} - e\mathbf{F}r\mathbf{G})$; $\Psi = (\mathbf{1} - r\mathbf{G}e\mathbf{F})$.

The fundamental Hamiltonian equation $\iota_{\mathbf{X}}\omega = \mathbf{d}f$, in (A.1), reads:

$$(X^i - rG^{ij}X_j) \mathbf{d}p_i - (X_k - eF_{kl}X^l) \mathbf{d}q^k = \frac{\partial f}{\partial q^k} \mathbf{d}q^k + \frac{\partial f}{\partial p_i} \mathbf{d}p_i. \quad (\text{III.3})$$

This can be rewritten as

$$\left(\frac{\partial f}{\partial p_i} - rG^{ij} \frac{\partial f}{\partial q^j} \right) = \Psi^i_j X^j; \quad \left(\frac{\partial f}{\partial q^k} - eF_{kl} \frac{\partial f}{\partial p^l} \right) = -\Phi_k^l X_l. \quad (\text{III.4})$$

Obviously, from (III.2) or (III.4), the closed two-form ω will be non degenerate, and hence symplectic, if $\mathbf{det}(\Omega) = \mathbf{det}(\Psi) = \mathbf{det}(\Phi) \neq 0$, so that (Ω) has an inverse:

$$\begin{aligned} (\Omega)^{-1} &= \begin{pmatrix} \mathbf{1} & 0 \\ +e\mathbf{F} & \mathbf{1} \end{pmatrix} \begin{pmatrix} -\Psi^{-1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} -r\mathbf{G} & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix} \\ &= \begin{pmatrix} +\Psi^{-1}r\mathbf{G} & -\Psi^{-1} \\ +e\mathbf{F}\Psi^{-1}r\mathbf{G} + \mathbf{1} & -e\mathbf{F}\Psi^{-1} \end{pmatrix}; \end{aligned} \quad (\text{III.5})$$

$$\begin{aligned} &= \begin{pmatrix} \mathbf{1} + r\mathbf{G} & \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \Phi^{-1} \end{pmatrix} \\ 0 & \mathbf{1} \end{pmatrix} \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & -e\mathbf{F} \end{pmatrix} \\ &= \begin{pmatrix} +r\mathbf{G}\Phi^{-1} & -r\mathbf{G}\Phi^{-1}e\mathbf{F} - \mathbf{1} \\ \Phi^{-1} & -\Phi^{-1}e\mathbf{F} \end{pmatrix}. \end{aligned} \quad (\text{III.6})$$

Explicitly:

$$\omega^\flat : \mathbf{d}f \rightarrow \begin{cases} (X_f)^i = (\Psi^{-1})^i_j (\partial f / \partial p_j - rG^{jk} \partial f / \partial q^k) \\ (X_f)_k = -(\Phi^{-1})_k^l (\partial f / \partial q^l - eF_{lj} \partial f / \partial p_j) \end{cases} \quad (\text{III.7})$$

The corresponding Poisson brackets are given by:

$$\{f, g\} = \omega(\mathbf{X}_f, \mathbf{X}_g) = (\partial_q f \quad \partial_p f) (\Lambda) \begin{pmatrix} \partial_q g \\ \partial_p g \end{pmatrix} \quad (\text{III.8})$$

with the matrix

$$(\Lambda) = -(\Omega)^{-1} = \begin{pmatrix} -(\Psi^{-1} r\mathbf{G} = r\mathbf{G}\Phi^{-1}) & +\Psi^{-1} \\ -\Phi^{-1} & +(\Phi^{-1} e\mathbf{F} = e\mathbf{F}\Psi^{-1}) \end{pmatrix}. \quad (\text{III.9})$$

Explicitly:

$$\{f, g\} = -\frac{\partial f}{\partial q}(\Psi^{-1} r\mathbf{G})\frac{\partial g}{\partial q} - \frac{\partial f}{\partial p}(\Phi^{-1})\frac{\partial g}{\partial q} + \frac{\partial f}{\partial q}(\Psi^{-1})\frac{\partial g}{\partial p} + \frac{\partial f}{\partial p}(\Phi^{-1} e\mathbf{F})\frac{\partial g}{\partial p}. \quad (\text{III.10})$$

In particular, for the coordinates (q^i, p_k) , we have:

$$\begin{aligned} \{q^i, q^j\} &= -(\Psi^{-1})^i_k r\mathbf{G}^{kj} = -r\mathbf{G}^{ik}(\Phi^{-1})_k^j, \\ \{q^i, p_l\} &= (\Psi^{-1})^i_l, \\ \{p_k, q^j\} &= -(\Phi^{-1})_k^j, \\ \{p_k, p_l\} &= (\Phi^{-1})_k^j eF_{jl} = eF_{kj}(\Psi^{-1})^j_l. \end{aligned} \quad (\text{III.11})$$

With $\mathcal{H}(q, p) = (\delta^{kl} p_k p_l / 2m) + \mathcal{V}(q)$, the equations of motion read:

$$\begin{aligned} \frac{dq^i}{dt} &= \{q^i, \mathcal{H}\} = (\Psi^{-1})^i_j \left(-r\mathbf{G}^{jk} \frac{\partial \mathcal{H}}{\partial q^k} + \frac{\partial \mathcal{H}}{\partial p_j} \right), \\ &= (\Psi^{-1})^i_j \left(-r\mathbf{G}^{jk} \frac{\partial \mathcal{V}}{\partial q^k} + \frac{p^j}{m} \right), \\ \frac{dp_k}{dt} &= \{p_k, \mathcal{H}\} = (\Phi^{-1})_k^l \left(-\frac{\partial \mathcal{H}}{\partial q^l} + eF_{lj} \frac{\partial \mathcal{H}}{\partial p_j} \right) \\ &= (\Phi^{-1})_k^l \left(-\frac{\partial \mathcal{V}}{\partial q^l} + eF_{lj} \frac{p^j}{m} \right). \end{aligned} \quad (\text{III.12})$$

The celebrated Darboux theorem guarantees the existence of local coordinates (ξ^i, π_k) , such that $\omega = \mathbf{d}\xi^i \wedge \mathbf{d}\pi_i$. When one of the charges (e, r) vanishes, such Darboux coordinates are easily obtained using the potential one-forms $\mathbf{A} = A_i(q)\mathbf{d}q^i$ and $\tilde{\mathbf{A}} = \tilde{A}^k(p)\mathbf{d}p_k$, such that $\mathbf{F} = \mathbf{d}\mathbf{A}$ and $\mathbf{G} = \mathbf{d}\tilde{\mathbf{A}}$.

Indeed, if $r = 0$, as in section II, Darboux coordinates are provided by $\xi^i = q^i; \pi_k = p_k + eA_k(q)$. A modified symplectic potential and two-form are defined by:

$$\theta = (p_k + eA_k)\mathbf{d}q^k; \omega = -\mathbf{d}\theta. \quad (\text{III.13})$$

The Hamiltonian and corresponding equations of motion are:

$$\mathcal{H}(\xi, \pi) = \frac{1}{2}\delta^{kl}(\pi_k - eA_k(\xi))(\pi_l - eA_l(\xi)) + \mathcal{V}(\xi), \quad (\text{III.14})$$

$$\frac{\mathbf{d}\xi^i}{\mathbf{d}t} = \delta^{ij}(\pi_j - eA_j(\xi)), \quad \frac{\mathbf{d}\pi_i}{\mathbf{d}t} = e\delta^{kl}(\pi_k - eA_k)\frac{\partial A_l}{\partial \xi^i} - \frac{\partial \mathcal{V}}{\partial \xi^i}, \quad (\text{III.15})$$

which yields the second order equation in ξ , as in (II.1):

$$\frac{\mathbf{d}^2 \xi^i}{\mathbf{d}t^2} = \delta^{ij} \left(-\frac{\partial \mathcal{V}(\xi)}{\partial \xi^j} + eF_{jl}(\xi) \frac{\mathbf{d}\xi^l}{\mathbf{d}t} \right). \quad (\text{III.16})$$

When $e = 0$, Darboux variables are

$$\xi^i = q^i + r\tilde{A}^i(p); \pi_k = p_k, \quad (\text{III.17})$$

and we define

$$\theta = p_k \mathbf{d}(q^k + r\tilde{A}^k); \omega = -\mathbf{d}\theta. \quad (\text{III.18})$$

The Hamiltonian and equations of motion are now given by:

$$\mathcal{H}(\xi, \pi) = \frac{1}{2}\delta^{kl} \pi_k \pi_l + \mathcal{V}(\xi - r\tilde{A}(\pi)), \quad (\text{III.19})$$

$$\frac{\mathbf{d}\xi^i}{\mathbf{d}t} = \delta^{ij} \pi_j - r\partial_k \mathcal{V}(q) \frac{\partial \tilde{A}^k}{\partial \pi_i}, \quad \frac{\mathbf{d}\pi_i}{\mathbf{d}t} = -\frac{\partial \mathcal{V}}{\partial q^i}(q). \quad (\text{III.20})$$

The second order equation, obeyed by π (!), is given by

$$\frac{\mathbf{d}^2 \pi_i}{\mathbf{d}t^2} = \partial_{ij}^2 \mathcal{V}(q) \left(-\delta^{jk} \pi_k + r\mathbf{G}^{jk}(\pi) \frac{\mathbf{d}\pi_l}{\mathbf{d}t} \right). \quad (\text{III.21})$$

Here the q -variable is assumed to be solved in terms of $\tilde{\pi}$ from equation $\tilde{\pi}_k = -\partial \mathcal{V}(q)/\partial q^k$ and this is possible if $\mathbf{det}(\partial_{ij}^2 \mathcal{V}(q)) \neq 0$!

In the case of nonzero charges (e, r) and non-constant \mathbf{F} and \mathbf{G} fields, there is no generic formula to define global Darboux coordinates (ξ^i, π_k) . However, if the fields \mathbf{F} and \mathbf{G} are constant, the Poisson matrix (III.2) is brought in canonical Darboux form by a linear symplectic orthogonalization procedure, à la Hilbert-Schmidt. In the next section this is done explicitly for $N = 2$ and $N = 3$. Obviously such a linear transformation: $(q^i, p_k) \Rightarrow (\xi^i, \pi_k)$ is defined up to a linear symplectic map of $\mathbf{Sp}(2n)$. These variables $(\xi^i, \pi_k) \in \mathbf{R}^{2n}$ can be canonically quantised as operators obeying the commutation relations

$$[\hat{\xi}^i, \hat{\xi}^j] = 0; [\hat{\xi}^i, \hat{\pi}_l] = i\hbar \delta^i_l; [\hat{\pi}_k, \hat{\pi}_l] = 0. \quad (\text{III.22})$$

As von Neumann taught us in [1], they are realised on the Hilbert space of square integrable functions of the variable ξ as

$$(\hat{\xi}^i \Psi)(\xi) = \xi^i \Psi(\xi); (\hat{\pi}_k \Psi)(\xi) = \frac{\hbar}{i} \frac{\partial \Psi(\xi)}{\partial \xi^k}. \quad (\text{III.23})$$

The original variables (q^i, p_k) being linear functions of the (ξ^i, π_k) are then also quantised.

When $\mathbf{det}(\Psi) = \mathbf{det}(\Phi) = 0$, the closed two-form ω is singular. When its rank is constant, ω defines a presymplectic structure on phase space which we call the primary constraint manifold denoted by \mathcal{M}_1 . The consistency of the resulting constrained Hamiltonian system will be examined in the $N = 2$ and $N = 3$ cases.

IV. EXAMPLES: $N = 2$ AND 3

In the two examples below, we consider a classical Hamiltonian of the form

$$\mathcal{H} = \frac{1}{2m} \delta^{kl} p_k p_l + \mathcal{V}(q). \quad (\text{IV.1})$$

A complete resolution will be given for a harmonic oscillator potential:

$$\mathcal{V}(q) \doteq \frac{\kappa}{2} \delta_{ij} q^i q^j. \quad (\text{IV.2})$$

Also of interest is the case of a constant "electric field": $\mathcal{V}(q) = -\mathbf{E}_k q^k$, which is exactly solvable and left to the reader.

A. Dynamics in the noncommutative plane

The magnetic fields in two dimensions, are written as:

$$e F_{ij} = B \varepsilon_{ij}; r G^{kl} = C \varepsilon^{kl}, \quad (\text{IV.3})$$

where B and C are pseudoscalars. The closed two-form (III.1) becomes

$$\omega = \mathbf{d}q^i \wedge \mathbf{d}p_i - B \mathbf{d}q^1 \wedge \mathbf{d}q^2 + C \mathbf{d}p_1 \wedge \mathbf{d}p_2. \quad (\text{IV.4})$$

The equation $\iota_X \omega = \mathbf{d}f$ reads

$$X^i - C \varepsilon^{ij} X_j = \frac{\partial f}{\partial p_i}; X_k - B \varepsilon_{kl} X^l = -\frac{\partial f}{\partial q^k}. \quad (\text{IV.5})$$

Denoting $\chi \doteq (1 + CB)$, the matrices Φ and Ψ are written as $\Phi_i^j = \chi \delta_i^j$ and $\Psi^k_l = \chi \delta^k_l$. The matrix (III.2) is then invertible if χ does not vanish.

1. The non degenerate case

Here, we will assume χ to be strictly positive. The above equation (IV.5) can then be inverted with Hamiltonian vector fields given by:

$$X^i = \chi^{-1} \left(\frac{\partial f}{\partial p_i} - C \varepsilon^{ij} \frac{\partial f}{\partial q^j} \right), X_k = -\chi^{-1} \left(\frac{\partial f}{\partial q^k} - B \varepsilon_{kl} \frac{\partial f}{\partial p^l} \right). \quad (\text{IV.6})$$

The Poisson brackets (III.11) become:

$$\{q^i, q^j\} = -C \chi^{-1} \varepsilon^{ij}; \{q^i, p_l\} = \chi^{-1} \delta^i_l, \\ \{p_k, q^j\} = -\chi^{-1} \delta_k^j; \{p_k, p_l\} = B \chi^{-1} \varepsilon_{kl}. \quad (\text{IV.7})$$

Substitution of the Ansatz

$$\xi^i = \alpha q^i + \beta \frac{C}{2} p_k \varepsilon^{ki}; \pi_k = \gamma \frac{B}{2} q^j \varepsilon_{jk} + \delta p_k, \quad (\text{IV.8})$$

in the canonical Poisson brackets, leads to the equations

$$\alpha^2 - \alpha\beta - \frac{CB}{4} \beta^2 = 0, \delta^2 - \delta\gamma - \frac{CB}{4} \gamma^2 = 0, \\ \alpha\delta + \frac{CB}{2} (\alpha\gamma + \delta\beta) - \frac{CB}{4} \beta\gamma = \chi. \quad (\text{IV.9})$$

We choose the solution:

$$\alpha = \delta = \sqrt{u}; \beta = \gamma = \frac{1}{\sqrt{u}}; u = \frac{1}{2}(1 + \sqrt{\chi}), \quad (\text{IV.10})$$

such that (IV.8) reduces to (II.9) when $C = 0$ or to (III.17) in case $B = 0$. The 2-form (III.1) has the canonical Darboux form $\omega = d\xi^i \wedge d\pi_i$ in the variables

$$\xi^i = \sqrt{u} \left(q^i - \frac{C}{2u} \varepsilon^{ik} p_k \right); \pi_k = \sqrt{u} \left(p_k - \frac{B}{2u} \varepsilon_{ki} q^i \right). \quad (\text{IV.11})$$

These have an inverse if, and only if $\chi \neq 0$:

$$\sqrt{\chi} q^i = \sqrt{u} \left(\xi^i + \frac{C}{2u} \varepsilon^{ik} \pi_k \right); \sqrt{\chi} p_k = \sqrt{u} \left(\pi_k + \frac{B}{2u} \varepsilon_{ki} \xi^i \right). \quad (\text{IV.12})$$

With the complex variables

$$q = q^1 + \mathbf{i}q^2, p = p_1 + \mathbf{i}p_2; \xi = \xi^1 + \mathbf{i}\xi^2, \pi = \pi_1 + \mathbf{i}\pi_2, \quad (\text{IV.13})$$

the above changes of variables are written as:

$$\xi = \sqrt{u} \left(q + \mathbf{i} \frac{C}{2u} p \right); \pi = \sqrt{u} \left(p + \mathbf{i} \frac{B}{2u} q \right). \quad (\text{IV.14})$$

The inverse transformations are:

$$q = \sqrt{u/\chi} \left(\xi - \mathbf{i} \frac{C}{2u} \pi \right); p = \sqrt{u/\chi} \left(\pi - \mathbf{i} \frac{B}{2u} \xi \right). \quad (\text{IV.15})$$

The Hamiltonian (IV.2) becomes

$$\mathcal{H} = \frac{1}{2m'} \delta^{kl} \pi_k \pi_l + \frac{\kappa'}{2} \delta_{ij} \xi^i \xi^j - \omega'_L \Lambda \\ = \frac{1}{2m'} \frac{\pi^\dagger \pi + \pi \pi^\dagger}{2} + \frac{\kappa'}{2} \frac{\xi^\dagger \xi + \xi \xi^\dagger}{2} - \omega'_L \Lambda \quad (\text{IV.16})$$

where Λ is angular momentum

$$\Lambda = \frac{1}{2} \left(\varepsilon_{ij} \xi^i \delta^{jk} \pi_k - \varepsilon^{kl} \pi_k \delta_{lj} \xi^j \right) \\ = \frac{1}{2} \left((\xi^1 \pi_2 - \xi^2 \pi_1) - (\pi_1 \xi^2 + \pi_2 \xi^1) \right) \\ = \frac{1}{4\mathbf{i}} \left((\xi^\dagger \pi - \xi \pi^\dagger) - (\pi \xi^\dagger + \pi^\dagger \xi) \right). \quad (\text{IV.17})$$

The "renormalised" mass and elasticity constant are given by:

$$\frac{1}{m'} = \frac{1}{m} \frac{u}{\chi} \left(1 + \frac{c^2}{4u^2} \right); \kappa' = \kappa \frac{u}{\chi} \left(1 + \frac{b^2}{4u^2} \right). \quad (\text{IV.18})$$

where

$$b = \frac{B}{\sqrt{m\kappa}}; c = C\sqrt{m\kappa}. \quad (\text{IV.19})$$

The corresponding frequency $\omega'_0 = \sqrt{\kappa'/m'}$ is given in terms of the "bare" frequency $\omega_0 = \sqrt{\kappa/m}$ by:

$$\omega'_0 = \frac{\omega_0}{2\chi} \left((b-c)^2 + 4\chi \right)^{1/2}. \quad (\text{IV.20})$$

and ω'_L , the induced Larmor frequency, by:

$$\omega'_L = \frac{\omega_0}{2\chi} (b - c). \quad (\text{IV.21})$$

The solution of Hamiltonian's equations with (IV.16) is standard. With[22]

$$m'\omega'_0 = \sqrt{m'\kappa'} = \sqrt{m\kappa} \left(\left(1 + \frac{b^2}{4u^2}\right) \left(1 + \frac{c^2}{4u^2}\right)^{-1} \right)^{1/2} \quad (\text{IV.22})$$

reduced variables are introduced by:

$$Q \doteq (m'\omega'_0)^{1/2} \xi; P \doteq (m'\omega'_0)^{-1/2} \pi. \quad (\text{IV.23})$$

The original (q, p) are expressed as:

$$\begin{aligned} q &= \sqrt{u/\chi} (m'\omega'_0)^{-1/2} \left(Q - \mathbf{i} \frac{c'}{2u} P \right), \\ p &= \sqrt{u/\chi} (m'\omega'_0)^{+1/2} \left(P - \mathbf{i} \frac{b'}{2u} Q \right), \end{aligned} \quad (\text{IV.24})$$

where

$$c' = C(m'\omega'_0) = C\sqrt{m'\kappa'}, \quad b' = B/(m'\omega'_0) = B/\sqrt{m'\kappa'}. \quad (\text{IV.25})$$

The symplectic structure and the Poisson brackets are:

$$\begin{aligned} \omega &= \frac{1}{2} \left(\mathbf{d}Q^\dagger \wedge \mathbf{d}P + \mathbf{d}Q \wedge \mathbf{d}P^\dagger \right) \\ \{f, g\} &= 2 \left(\frac{\partial f}{\partial Q} \frac{\partial g}{\partial P^\dagger} + \frac{\partial f}{\partial Q^\dagger} \frac{\partial g}{\partial P} - \frac{\partial f}{\partial P} \frac{\partial g}{\partial Q^\dagger} - \frac{\partial f}{\partial P^\dagger} \frac{\partial g}{\partial Q} \right). \end{aligned} \quad (\text{IV.26})$$

The fundamental nonzero Poisson bracket is

$$\{Q, P^\dagger\} = 2. \quad (\text{IV.27})$$

In these variables, the Hamiltonian (IV.16) reads:

$$\mathcal{H} = \frac{\omega'_0}{4} \left((P^\dagger P + PP^\dagger) + (Q^\dagger Q + QQ^\dagger) \right) - \omega'_L \Lambda, \quad (\text{IV.28})$$

where

$$\Lambda = \frac{1}{4\mathbf{i}} \left((Q^\dagger P - PQ^\dagger) - (PQ^\dagger + P^\dagger Q) \right). \quad (\text{IV.29})$$

The corresponding equations of motion are:

$$\begin{aligned} \frac{dQ}{dt} = \{Q, \mathcal{H}\} &= 2 \frac{\partial \mathcal{H}}{\partial P^\dagger} = \omega'_0 P - \mathbf{i} \omega'_L Q \\ \frac{dP}{dt} = \{P, \mathcal{H}\} &= -2 \frac{\partial \mathcal{H}}{\partial Q^\dagger} = -\omega'_0 Q - \mathbf{i} \omega'_L P \end{aligned} \quad (\text{IV.30})$$

With the shift variables

$$A_{(+)} = \frac{1}{2} (Q + \mathbf{i}P); A_{(-)} = \frac{1}{2} (Q^\dagger + \mathbf{i}P^\dagger), \quad (\text{IV.31})$$

the symplectic structure and the Poisson brackets are given by:

$$\omega = -\mathbf{i} \left(\mathbf{d}A_{(+)}^\dagger \wedge \mathbf{d}A_{(+)} + \mathbf{d}A_{(-)}^\dagger \wedge \mathbf{d}A_{(-)} \right), \quad (\text{IV.32})$$

$$\begin{aligned} \{f, g\} &= -\mathbf{i} \left(\frac{\partial f}{\partial A_{(+)}^\dagger} \frac{\partial g}{\partial A_{(+)}^\dagger} + \frac{\partial f}{\partial A_{(-)}^\dagger} \frac{\partial g}{\partial A_{(-)}^\dagger} \right. \\ &\quad \left. - \frac{\partial f}{\partial A_{(+)}^\dagger} \frac{\partial g}{\partial A_{(+)}^\dagger} - \frac{\partial f}{\partial A_{(-)}^\dagger} \frac{\partial g}{\partial A_{(-)}^\dagger} \right) \end{aligned} \quad (\text{IV.33})$$

with fundamental nonzero brackets:

$$\{A_{(\pm)}, A_{(\pm)}^\dagger\} = -\mathbf{i}. \quad (\text{IV.34})$$

The Hamiltonian, with the (positive !) frequencies

$$\omega_{(\pm)} = (\omega'_0 \pm \omega'_L), \quad (\text{IV.35})$$

reads now:

$$\mathcal{H} = \frac{\omega_{(+)}}{2} \left(A_{(+)}^\dagger A_{(+)} + A_{(+)} A_{(+)}^\dagger \right) + \frac{\omega_{(-)}}{2} \left(A_{(-)}^\dagger A_{(-)} + A_{(-)} A_{(-)}^\dagger \right). \quad (\text{IV.36})$$

The corresponding equations of motion and their solutions are given by:

$$\frac{dA_{(\pm)}}{dt} = \{A_{(\pm)}, \mathcal{H}\} = -\mathbf{i} \frac{\partial \mathcal{H}}{\partial A_{(\pm)}^\dagger} = -\mathbf{i} \omega_{(\pm)} A_{(\pm)}; \quad (\text{IV.37})$$

$$A_{(\pm)}(t) = \exp \{ -\mathbf{i} \omega_{(\pm)} t \} A_{(\pm)}(0). \quad (\text{IV.38})$$

The relations between variables are given by:

$$\begin{aligned}
 A_{(+)} &= \frac{1}{2} (Q + \mathbf{i}P) \\
 &= \frac{\sqrt{u}}{2} \left((m'\omega'_0)^{+1/2} \left(1 - \frac{b'}{2u}\right) q + \mathbf{i} (m'\omega'_0)^{-1/2} \left(1 + \frac{c'}{2u}\right) p \right) \\
 A_{(-)}^\dagger &= \frac{1}{2} (Q - \mathbf{i}P) \\
 &= \frac{\sqrt{u}}{2} \left((m'\omega'_0)^{+1/2} \left(1 + \frac{b'}{2u}\right) q - \mathbf{i} (m'\omega'_0)^{-1/2} \left(1 - \frac{c'}{2u}\right) p \right).
 \end{aligned} \tag{IV.39}$$

The inverse transformations are:

$$\begin{aligned}
 q &= (m'\omega'_0)^{-1/2} \sqrt{u/\chi} \left(Q - \mathbf{i} \frac{c'}{2u} P \right), \\
 &= (m'\omega'_0)^{-1/2} \sqrt{u/\chi} \left(\left(1 - \frac{c'}{2u}\right) A_{(+)} + \left(1 + \frac{c'}{2u}\right) A_{(-)}^\dagger \right), \\
 p &= (m'\omega'_0)^{+1/2} \sqrt{u/\chi} \left(P - \mathbf{i} \frac{b'}{2u} Q \right) \\
 &= \mathbf{i} (m'\omega'_0)^{+1/2} \sqrt{u/\chi} \left(\left(1 - \frac{b'}{2u}\right) A_{(-)}^\dagger - \left(1 + \frac{b'}{2u}\right) A_{(+)} \right).
 \end{aligned} \tag{IV.40}$$

Quantisation is trivial though the substitution of the fundamental Poisson brackets (IV.27),(IV.34) by operator commutators

$$\left[\mathbf{Q}, \mathbf{P}^\dagger \right] = 2\mathbf{i}\hbar; \quad \left[\mathbf{A}_{(\pm)}, \mathbf{A}_{(\pm)}^\dagger \right] = \hbar. \tag{IV.41}$$

Having kept the initial ordering, the quantum Hamiltonian has eigenvalues:

$$E(n_{(+)}, n_{(-)}) = \hbar\omega_{(+)} (n_{(+)} + 1/2) + \hbar\omega_{(-)} (n_{(-)} + 1/2), \tag{IV.42}$$

where $n_{(\pm)}$ are nonnegative integers. The corresponding eigenvectors are denoted by $|n_{(+)}, n_{(-)}\rangle$.

2. The degenerate or constraint case

The condition $\chi \doteq (1 + BC) = 0$ determines ω as a presymplectic structure on \mathcal{M} and shall be called the primary constraint. Again, the notation is simplified using complex variables[23]. The presymplectic two-form reads

$$\begin{aligned}
 \omega &= \frac{1}{2} \left(dq^\dagger \wedge dp + dq \wedge dp^\dagger \right) \\
 &\quad - \frac{B}{4\mathbf{i}} \left(dq^\dagger \wedge dq - dq \wedge dq^\dagger \right) + \frac{C}{4\mathbf{i}} \left(dp^\dagger \wedge dp - dp \wedge dp^\dagger \right).
 \end{aligned} \tag{IV.43}$$

The Hamiltonian (IV.2) becomes

$$\mathcal{H} = \frac{1}{2m} \frac{p^\dagger p + p p^\dagger}{2} + \frac{\kappa}{2} \frac{q^\dagger q + q q^\dagger}{2}, \tag{IV.44}$$

Writing a vector field as

$$\mathbf{X} = X^i \partial/\partial q^i + X_k \partial/\partial p_k = U \partial/\partial q + U^\dagger \partial/\partial q^\dagger + V \partial/\partial p + V^\dagger \partial/\partial p^\dagger,$$

$$\begin{aligned}
 \iota_{\mathbf{X}} \omega &= \frac{1}{2} \left((U + \mathbf{i}CV) dq^\dagger + (U^\dagger - \mathbf{i}CV^\dagger) dq \right. \\
 &\quad \left. - (V + \mathbf{i}BU) dp^\dagger - (V^\dagger - \mathbf{i}BU^\dagger) dp \right).
 \end{aligned} \tag{IV.45}$$

The homogeneous equation, $\iota_{\mathbf{X}} \omega = 0$ has nontrivial solutions. Indeed, with $U_0 = Z^1 + \mathbf{i}Z^2$ and $V_0 = Z_1 + \mathbf{i}Z_2$, equation

(IV.45) yields the system:

$$U_0 + \mathbf{i}CV_0 = 0; \quad \text{or} \quad V_0 + \mathbf{i}BU_0 = 0, \tag{IV.46}$$

of which the determinant is $\chi = 1 + BC = 0$.

The inhomogeneous equation $\iota_{\mathbf{X}}\omega = \mathbf{d}\mathcal{H}$, i.e. the Hamiltonian dynamics, reads

$$U + \mathbf{i}CV = 2 \frac{\partial \mathcal{H}}{\partial p^\dagger} = \frac{p}{m}; V + \mathbf{i}BU = -2 \frac{\partial \mathcal{H}}{\partial q^\dagger} = \kappa q. \quad (\text{IV.47})$$

It will have a solution if

$$\langle \mathbf{d}\mathcal{H} | \mathbf{Z} \rangle = 0. \quad (\text{IV.48})$$

This condition, termed secondary constraint, is explicitly given by:

$$\frac{\partial \mathcal{H}}{\partial p} - \mathbf{i}C \frac{\partial \mathcal{H}}{\partial q} = 0; \text{ or } \frac{\partial \mathcal{H}}{\partial q} - \mathbf{i}B \frac{\partial \mathcal{H}}{\partial p} = 0. \quad (\text{IV.49})$$

For the Hamiltonian (IV.44) this condition (IV.49) is linear:

$$\frac{1}{m} p + \mathbf{i}C \kappa q = 0; \text{ or } \kappa q + \mathbf{i}B \frac{1}{m} p = 0. \quad (\text{IV.50})$$

and defines the secondary constraint manifold \mathcal{M}_2 .

On \mathcal{M}_2 , a particular solution of $\iota_{\mathbf{X}}\omega = \mathbf{d}\mathcal{H}$ is given by:

$$U_P = \frac{p}{m}; V_P = 0. \quad (\text{IV.51})$$

The general solution is given by:

$$U = \frac{p}{m} + U_0; V = V_0. \quad (\text{IV.52})$$

where (U_0, V_0) is restricted to obey (IV.46). This vector field, restricted to \mathcal{M}_2 , should conserve the constraints i.e. must be tangent to \mathcal{M}_2 :

$$0 = \left\langle \frac{1}{m} \mathbf{d}p + \mathbf{i}C \kappa \mathbf{d}q | X \right\rangle, \quad (\text{IV.53})$$

The vector fields U and V are completely defined on \mathcal{M}_2 , with ensuing equations of motion:

$$\begin{aligned} \frac{dq}{dt} &= U = -\mathbf{i} \frac{\sqrt{m\kappa C}}{1+m\kappa C^2} \omega_0 q = \frac{1}{1+m\kappa C^2} \frac{p}{m}, \\ \frac{dp}{dt} &= V = -\mathbf{i} \frac{\sqrt{m\kappa C}}{1+m\kappa C^2} \omega_0 p = -\frac{m\kappa C^2}{1+m\kappa C^2} \kappa q. \end{aligned} \quad (\text{IV.54})$$

In terms of the frequency:

$$\omega_r = -\frac{\sqrt{m\kappa C}}{1+m\kappa C^2} \omega_0 = \frac{B/\sqrt{m\kappa}}{1+B^2/m\kappa} \omega_0, \quad (\text{IV.55})$$

the solution is given by

$$q(t) = \exp\{\mathbf{i}\omega_r t\} q_0; p(t) = \exp\{\mathbf{i}\omega_r t\} p_0. \quad (\text{IV.56})$$

Obviously, if q_0 and p_0 obey the secondary constraints (IV.50), $q(t)$ and $p(t)$ obey them at all times.

The same result can be obtained by symplectic reduction, restricting the pre-symplectic two-form (IV.43) to \mathcal{M}_2 :

$$\omega|_{\mathcal{M}_2} = -\mathbf{i} \frac{(1+m\kappa C^2)^2}{2C} dq^\dagger \wedge dq. \quad (\text{IV.57})$$

$$\{f, g\}_{\mathcal{M}_2} = \frac{2\mathbf{i}C}{(1+m\kappa C^2)^2} \left(\frac{\partial f}{\partial q^\dagger} \frac{\partial g}{\partial q} - \frac{\partial f}{\partial q} \frac{\partial g}{\partial q^\dagger} \right). \quad (\text{IV.58})$$

The fundamental Poisson bracket is

$$\{q, q^\dagger\}_{\mathcal{M}_2} = \frac{-2\mathbf{i}C}{(1+m\kappa C^2)^2} \quad (\text{IV.59})$$

The dynamics are given by:

$$\frac{dq}{dt} = -\frac{2\mathbf{i}C}{(1+m\kappa C^2)^2} \frac{\partial \mathcal{H}_r}{\partial q^\dagger}. \quad (\text{IV.60})$$

And, with the reduced Hamiltonian \mathcal{H}_r given by

$$\mathcal{H}_r = (1+m\kappa C^2) \frac{\kappa}{2} q^\dagger q, \quad (\text{IV.61})$$

this yields equation (IV.56). When $B > 0$, hence $C < 0$, we define

$$a = \frac{(1+m\kappa C^2)}{|2C|} q^\dagger, \quad (\text{IV.62})$$

such that

$$\{a, a^\dagger\} = -\mathbf{i}; \mathcal{H}_r = \frac{\omega_r}{2} (a^\dagger a + a a^\dagger). \quad (\text{IV.63})$$

Quantisation is again trivial introducing operators \mathbf{a} and \mathbf{a}^\dagger , obeying

$$[\mathbf{a}, \mathbf{a}^\dagger] = \hbar \quad (\text{IV.64})$$

such that the quantum Hamiltonian

$$\mathbf{H}_r = \frac{\omega_r}{2} (\mathbf{a}^\dagger \mathbf{a} + \mathbf{a} \mathbf{a}^\dagger). \quad (\text{IV.65})$$

has eigenvalues:

$$E(n) = \hbar \omega_r (n + 1/2). \quad (\text{IV.66})$$

3. The $\chi \rightarrow 0$ limit of (IV A 1).

We need the expansion of

$$(m'\omega'_0) = (m\omega_0) \times \left(\left(1 + \frac{b^2}{4u^2}\right) \left(1 + \frac{c^2}{4u^2}\right)^{-1} \right)^{1/2}, \tag{IV.67}$$

in powers of $\varepsilon = \sqrt{\chi}$, where $1 + bc = \varepsilon^2$ and $2u = 1 + \varepsilon$.

$$\begin{aligned} (m'\omega'_0) &= \frac{m\omega_0}{|c|} \left(1 + \frac{c^2 - 1}{c^2 + 1} \varepsilon + \dots\right) \\ &= \frac{1}{|C|} \left(1 + \frac{c^2 - 1}{c^2 + 1} \varepsilon + \dots\right) \\ (m'\omega'_0)^{-1} &= \frac{(m\omega_0)^{-1}}{|b|} \left(1 + \frac{b^2 - 1}{b^2 + 1} \varepsilon + \dots\right) \\ &= \frac{1}{|B|} \left(1 + \frac{b^2 - 1}{b^2 + 1} \varepsilon + \dots\right). \end{aligned} \tag{IV.68}$$

Also, from (IV.25), we obtain

$$\begin{aligned} \frac{c'}{2u} &= \frac{(m'\omega'_0)C}{2u} = \frac{C}{|C|} \left(1 - \frac{2}{c^2 + 1} \varepsilon + \dots\right) \\ \frac{b'}{2u} &= \frac{B}{(m'\omega'_0)2u} = \frac{B}{|B|} \left(1 - \frac{2}{b^2 + 1} \varepsilon + \dots\right) \end{aligned} \tag{IV.69}$$

For definiteness, we assume in the following $B > 0$ and so $C < 0$ in the limit $\varepsilon \rightarrow 0$. We obtain

$$\begin{aligned} 1 - \frac{b'}{2u} &= \frac{2}{1 + b^2} \varepsilon + \dots ; \\ 1 + \frac{b'}{2u} &= 2 - \frac{2}{1 + b^2} \varepsilon + \dots \\ 1 + \frac{c'}{2u} &= \frac{2}{1 + c^2} \varepsilon + \dots ; \\ 1 - \frac{c'}{2u} &= 2 - \frac{2}{1 + b^2} \varepsilon + \dots \end{aligned} \tag{IV.70}$$

Also:

$$\omega'_0 = \frac{\omega_0}{2\varepsilon^2} (b - c) \left(1 + \frac{2\varepsilon^2}{(b - c)^2}\right), \quad \omega'_L = \frac{\omega_0}{2\varepsilon^2} (b - c), \tag{IV.71}$$

$$\begin{aligned} \omega_{(+)} &= \omega'_0 + \omega'_L = -\omega_0 \frac{1 + (m\omega_0)^2 C^2}{(m\omega_0)C} \frac{1}{\varepsilon^2}, \\ \omega_{(-)} &= \omega'_0 - \omega'_L = -\omega_0 \frac{(m\omega_0)C}{1 + (m\omega_0)^2 C^2}. \end{aligned} \tag{IV.72}$$

One of the frequencies $\omega_{(+)}$ diverges, while the other $\omega_{(-)}$ tends to ω_r defined in (IV.55). The relations in (IV.39) yield the initial conditions:

$$\begin{aligned} A_{(+)}(0) &\approx \sqrt{\frac{|B|}{2}} (1 + b^2)^{-1} \left(q_0 + \mathbf{i} \frac{B}{(m\omega_0)^2} p_0\right) (\varepsilon + O(\varepsilon^2)) \\ A_{(-)}^\dagger(0) &\approx \sqrt{\frac{|B|}{2}} \left(q_0 - \mathbf{i} \frac{1}{|B|} p_0\right) (1 + O(\varepsilon^2)). \end{aligned} \tag{IV.73}$$

The solutions (IV.40), in the $\varepsilon \rightarrow 0$ limit are then written as

$$\begin{aligned} q(t) &\approx \sqrt{\frac{2}{|B|}} \left(\frac{1}{\varepsilon} A_{(+)}(0) \mathbf{exp}\{-\mathbf{i}\omega_{(+)}t\} + \frac{1}{1 + c^2} A_{(-)}^\dagger(0) \mathbf{exp}\{\mathbf{i}\omega_r t\} \right) \\ &\approx (1 + b^2)^{-1} \left(q_0 + \mathbf{i} \frac{|B|}{(m\omega_0)^2} p_0 \right) \mathbf{exp}\{-\mathbf{i}\omega_{(+)}t\} \\ &\quad + (1 + c^2)^{-1} (q_0 - \mathbf{i}|B|^{-1} p_0) \mathbf{exp}\{\mathbf{i}\omega_r t\}; \end{aligned} \tag{IV.74}$$

The first term is a fast oscillating function with diverging frequency and so averages to zero. Furthermore, if the initial conditions are on \mathcal{M}_2 , i.e. if $(q_0 + \mathbf{i}|B|p_0/(m\omega_0)^2) = 0$, this first term behaves as $O(\varepsilon) \mathbf{exp}\{\mathbf{i}vt/\varepsilon^2\}$ converging to zero. The second term is then reduced to the expression (IV.56) of

$q(t)$. Similar considerations hold for $p(t)$ in such a way that the solution stays on M_2 .

B. Noncommutative \mathbf{R}^3

In \mathbf{R}^3 , the magnetic fields \mathbf{F} and \mathbf{G} are written in terms of pseudovectors $\underline{\mathbf{B}} = \{B^k\}$ and $\underline{\mathbf{C}} = \{C_k\}$ as:

$$eF_{ij} = \varepsilon_{ijk} B^k; rG^{ij} = \varepsilon^{ijk} C_k. \quad (\text{IV.75})$$

The closed two-form (III.1) is written as:

$$\omega = \mathbf{d}q^i \wedge \mathbf{d}p_i - \frac{1}{2} \varepsilon_{ijk} B^k \mathbf{d}q^i \wedge \mathbf{d}q^j + \frac{1}{2} \varepsilon^{klm} C_m \mathbf{d}p_k \wedge \mathbf{d}p_l. \quad (\text{IV.76})$$

The fundamental equation $\iota_X \omega = \mathbf{d}f$ reads

$$X^i - C_k \varepsilon^{ijk} X_j = \frac{\partial f}{\partial p_i}; X_k - B^i \varepsilon_{kli} X^l = -\frac{\partial f}{\partial q^k}. \quad (\text{IV.77})$$

Defining $\vartheta = \underline{\mathbf{C}} \cdot \underline{\mathbf{B}} = C_k B^k$ and $\chi = 1 + \vartheta$, this is also written as

$$\begin{aligned} \chi X^i &= (\delta^i_j + B^i C_j) \frac{\partial f}{\partial p_j} - C_k \varepsilon^{ijk} \frac{\partial f}{\partial q^k} \\ \chi X_k &= -\left((\delta_k^l + C_k B^l) \frac{\partial f}{\partial q^l} - B^i \varepsilon_{kli} \frac{\partial f}{\partial p_i} \right). \end{aligned} \quad (\text{IV.78})$$

The 3×3 matrices Φ and Ψ read:

$$\Phi_i^j = \chi \delta_i^j - C_i B^j; \Psi^k_l = \chi \delta_k^l - B^k C_l,$$

with $\det \Phi = \det \Psi = \chi^2$. Assuming again $\chi \neq 0$ [24], these matrices have inverses:

$$(\Phi^{-1})_i^j = \frac{1}{\chi} (\delta_i^j + C_i B^j), \quad (\Psi^{-1})^k_l = \frac{1}{\chi} (\delta_k^l + B^k C_l).$$

The Hamiltonian vector fields are obtained from (IV.78):

$$\begin{aligned} X^i &= \chi^{-1} \left((\delta^i_j + B^i C_j) \frac{\partial f}{\partial p_j} - C_k \varepsilon^{ijk} \frac{\partial f}{\partial q^k} \right), \\ X_k &= -\chi^{-1} \left((\delta_k^l + C_k B^l) \frac{\partial f}{\partial q^l} - B^i \varepsilon_{kli} \frac{\partial f}{\partial p_i} \right). \end{aligned} \quad (\text{IV.79})$$

The Poisson brackets are given by:

$$\begin{aligned} \{q^i, q^j\} &= -\chi^{-1} \varepsilon^{ijk} C_k, \quad \{q^i, p_l\} = \chi^{-1} (\delta_l^i + B^i C_l), \\ \{p_k, q^j\} &= -\chi^{-1} (\delta_k^j + C_k B^j), \quad \{p_k, p_l\} = \chi^{-1} \varepsilon_{klm} B^m. \end{aligned} \quad (\text{IV.80})$$

The Ansatz (IV.8) has to be generalised to

$$\begin{aligned} \xi^i &= \alpha q^i + \alpha' B^i (C_k q^k) - \beta \frac{1}{2} \varepsilon^{ijk} p_j C_k; \\ \pi_k &= \alpha p_k + \alpha' (p_i B^i) C_k + \beta \frac{1}{2} \varepsilon_{klm} B^l q^m. \end{aligned} \quad (\text{IV.81})$$

For α, β similar equations as in (IV.9) are obtained:

$$\alpha^2 - \alpha\beta - \frac{\vartheta}{4} \beta^2 = 0, \quad \alpha^2 + \vartheta(\alpha\beta) - \frac{\vartheta}{4} \beta^2 = \chi, \quad (\text{IV.82})$$

with a the same solution (χ assumed to be strictly positive):

$$\alpha = \sqrt{u}; \beta = \frac{1}{\sqrt{u}}; u = \frac{1}{2}(1 + \sqrt{\chi}). \quad (\text{IV.83})$$

Furthermore, there is an additional equation for α' :

$$\chi \left(\vartheta \alpha'^2 + 2\alpha\alpha' \right) + \left(\alpha^2 - \alpha\beta + \frac{1}{4} \beta^2 \right) = 0. \quad (\text{IV.84})$$

Substituting (IV.83), one obtains

$$\vartheta \alpha'^2 + 2\sqrt{u}\alpha' + \frac{1}{4u} = 0,$$

with solution, remaining finite when $\vartheta \rightarrow 0$:

$$\alpha' = \sqrt{u}\gamma = \frac{(1 - \sqrt{u})}{\vartheta}. \quad (\text{IV.85})$$

The formulae (IV.81) are finally written as:

$$\begin{aligned} \xi^i &= \sqrt{u} \left(q^i + \gamma B^i (C_k q^k) - \frac{1}{2u} \varepsilon^{ijk} p_j C_k \right); \\ \pi_k &= \sqrt{u} \left(p_k + \gamma (p_i B^i) C_k + \frac{1}{2u} \varepsilon_{klm} B^l q^m \right). \end{aligned} \quad (\text{IV.86})$$

In old fashioned vector notation, this appears as:

$$\begin{aligned} \bar{\xi} &= \sqrt{u} \left(\bar{\mathbf{q}} + \gamma \underline{\mathbf{B}} (\underline{\mathbf{C}} \cdot \bar{\mathbf{q}}) - \frac{1}{2u} \underline{\mathbf{p}} \times \underline{\mathbf{C}} \right); \\ \underline{\pi} &= \sqrt{u} \left(\underline{\mathbf{p}} + \gamma (\underline{\mathbf{p}} \cdot \underline{\mathbf{B}}) \underline{\mathbf{C}} + \frac{1}{2u} \underline{\mathbf{B}} \times \bar{\mathbf{q}} \right). \end{aligned} \quad (\text{IV.87})$$

The inverse formulae of (IV.86) are obtained as:

$$\begin{aligned} q^i &= \frac{\sqrt{u}}{\sqrt{\chi}} \left(\xi^i + \gamma' B^i (C_k \xi^k) + \frac{1}{2u} \varepsilon^{ijk} \pi_j C_k \right); \\ p_k &= \frac{\sqrt{u}}{\sqrt{\chi}} \left(\pi_k + \gamma' C_k (\pi_l B^l) - \frac{1}{2u} \varepsilon_{klm} B^l \xi^m \right) \end{aligned} \quad (\text{IV.88})$$

Or, in vector notation:

$$\begin{aligned} \bar{\mathbf{q}} &= \frac{\sqrt{u}}{\sqrt{\chi}} \left(\bar{\xi} + \gamma' \underline{\mathbf{B}} (\underline{\mathbf{C}} \cdot \bar{\xi}) + \frac{1}{2u} \underline{\pi} \times \underline{\mathbf{C}} \right); \\ \underline{\mathbf{p}} &= \frac{\sqrt{u}}{\sqrt{\chi}} \left(\underline{\pi} + \gamma' \underline{\mathbf{C}} (\underline{\pi} \cdot \underline{\mathbf{B}}) - \frac{1}{2u} \underline{\mathbf{B}} \times \bar{\xi} \right), \end{aligned} \quad (\text{IV.89})$$

where

$$\gamma' = \frac{\sqrt{\chi} - \sqrt{u}}{\vartheta \sqrt{u}}. \quad (\text{IV.90})$$

Again, for sake of simplicity, we consider a configuration space which is Euclidean $Q = \mathbf{E}^3$ with metric $\langle \bar{\mathbf{v}}; \bar{\mathbf{w}} \rangle = \delta_{ij} v^i w^j = (\underline{\mathbf{v}} \cdot \bar{\mathbf{w}})$ such that $v_i = \delta_{ij} v^j$. Substitution of (IV.88) in a Hamiltonian of the form (IV.2), leads to a Hamiltonian quadratic in (ξ, π) and to a system of linear evolution equations. In the case when $\bar{\mathbf{B}}$ and $\underline{\mathbf{C}}$ point in the same direction:

$$\bar{\mathbf{B}} = B \bar{\mathbf{e}}_Z; \quad \underline{\mathbf{C}} = C \mathbf{e}_Z, \quad (\text{IV.91})$$

a particularly simple Hamiltonian is obtained. Parallel coordinates are defined by ξ^3 , π_3 and transverse coordinate vectors

by $\bar{\xi}_\perp = \bar{\xi} - \xi^3 \bar{\mathbf{e}}_Z$ and $\underline{\pi}_\perp = \underline{\pi} - \pi_3 \mathbf{e}_Z$. Indeed, eq. (IV.88) becomes

$$\begin{aligned} q^1 &= \frac{\sqrt{u}}{\sqrt{\chi}} \left(\xi^1 + \frac{1}{2u} \pi_2 C \right), & p_1 &= \frac{\sqrt{u}}{\sqrt{\chi}} \left(\pi_1 + \frac{1}{2u} \xi^2 B \right), \\ q^2 &= \frac{\sqrt{u}}{\sqrt{\chi}} \left(\xi^2 - \frac{1}{2u} \pi_1 C \right), & p_2 &= \frac{\sqrt{u}}{\sqrt{\chi}} \left(\pi_2 - \frac{1}{2u} B \xi^1 \right), \\ q^3 &= \xi^3, & p_3 &= \pi_3. \end{aligned} \quad (\text{IV.92})$$

The Hamiltonian is:

$$\mathcal{H}(\xi, \pi) = \left(\frac{1}{2m_\perp} (\underline{\pi}_\perp)^2 + \frac{k_\perp}{2} (\bar{\xi}_\perp)^2 \right) + \left(\frac{1}{2m} (\pi_3)^2 + \frac{k}{2} (\xi^3)^2 \right) + \mathcal{H}_{int}(\xi, \pi). \quad (\text{IV.93})$$

The transverse degrees of freedom are seen to have a renormalised[25] mass and elasticity constant which are given by the same expressions as in (IV.18):

$$\frac{1}{m_\perp} = \frac{1}{m} \frac{u}{\chi} \left(1 + \frac{c^2}{4u^2} \right); \quad \kappa_\perp = \kappa \frac{u}{\chi} \left(1 + \frac{b^2}{4u^2} \right), \quad (\text{IV.94})$$

where

$$b = \frac{B}{\sqrt{m\kappa}}; \quad c = C\sqrt{m\kappa}.$$

The fields $\bar{\mathbf{B}}$ and $\underline{\mathbf{C}}$ induce a sort of magnetic moment interaction along the Z -axis with the same Larmor frequency as before:

$$\tilde{\mathcal{H}}_{ind}(\xi, \pi) = -\omega'_L \Lambda_3, \quad (\text{IV.95})$$

where $\Lambda_3 = \xi^1 \pi_2 - \xi^2 \pi_1$. Actually, the condition (IV.91) reduces the $(N=3)$ case to a sum $(N=2) \oplus (N=1)$. The three relevant frequencies of our oscillator are:

$$\omega_3 = \sqrt{k/m}; \quad \omega_\perp = \sqrt{k_\perp/m_\perp}; \quad \omega'_L = \frac{1}{\chi} \omega_0 (b-c). \quad (\text{IV.96})$$

The spectrum of the quantum Hamiltonian is easily obtained as

$$E(n_{(+)}, n_{(-)}, n_3) = \hbar \omega_{(+)} (n_{(+)} + 1/2) +$$

$$\hbar \omega_{(-)} (n_{(-)} + 1/2) + \hbar \omega_3 (n_3 + 1/2), \quad (\text{IV.97})$$

where $n_{(\pm)}, n_3$ are nonnegative integers. Corresponding eigenvectors are denoted by $|n_{(+)}, n_{(-)}, n_3\rangle$.

V. SYMMETRIES

For Euclidean configuration space $Q \equiv \mathbf{E}^N$, with metric δ_{ij} , an infinitesimal rotation is written as:

$$\varphi: q^i \rightarrow q'^i = q^i + \frac{1}{2} \delta \epsilon^{\alpha\beta} (M_{\alpha\beta})^i_j q^j, \quad (\text{V.98})$$

where $(M_{\alpha\beta})^i_j = \delta^i_\alpha \delta_{\beta j} - \delta^i_\beta \delta_{\alpha j}$ are the generators of the rotation group obeying the Lie algebra relations:

$$[M_{\alpha\beta}, M_{\mu\nu}] = -\delta_{\alpha\mu} M_{\beta\nu} + \delta_{\alpha\nu} M_{\beta\mu} - \delta_{\beta\nu} M_{\alpha\mu} + \delta_{\beta\mu} M_{\alpha\nu}. \quad (\text{V.99})$$

This induces the push forward in $T^*(Q)$:

$$\begin{aligned} \tilde{\varphi}: T^*(Q) &\rightarrow T^*(Q): (q^i, p_k) \rightarrow (q'^i, p'_k), \\ q'^i &= q^i + \frac{1}{2} \delta \epsilon^{\alpha\beta} (M_{\alpha\beta})^i_j q^j; \\ p'_k &= p_k - \frac{1}{2} \delta \epsilon^{\alpha\beta} p_l (M_{\alpha\beta})^l_k. \end{aligned} \quad (\text{V.100})$$

In a basis[26] $\{\mathbf{e}_{\alpha\beta}\}$ of $\mathcal{L}(SO(N))$, let $\mathbf{u} = (1/2)\mathbf{e}_{\alpha\beta} u^{\alpha\beta}$ denote a generic element. With $\mathcal{R}(\mathbf{u}) = \exp\{\frac{1}{2} u^{\alpha\beta} M_{\alpha\beta}\}$, finite rotations are written as

$$q^i \rightarrow q'^i = \mathcal{R}(\mathbf{u})^i_j q^j; \quad p_k \rightarrow p'_k = p_l \mathcal{R}^{-1}(\mathbf{u})^l_k. \quad (\text{V.101})$$

The vector field $\mathbf{X}_\mathbf{u}$ (see appendix A) is given by its components:

$$(X_\mathbf{u})^i = \frac{1}{2} u^{\alpha\beta} (M_{\alpha\beta})^i_j q^j; \quad (X_\mathbf{u})_k = -\frac{1}{2} u^{\alpha\beta} p_l (M_{\alpha\beta})^l_k. \quad (\text{V.102})$$

It conserves the canonical symplectic potential and two-form:

$$\mathcal{L}_{X_\mathbf{u}} \theta_0 = 0; \quad \mathcal{L}_{X_\mathbf{u}} \omega_0 = 0.$$

The action is in fact Hamiltonian for the *canonical symplectic structure*. With the notation of appendix A, we have

$$\begin{aligned} \mathbf{X}_{\mathbf{u}} &= \omega_0^\sharp(\mathbf{d}\Xi(\mathbf{u})), \\ \Xi(\mathbf{u}) &= \frac{1}{2} u^{\alpha\beta} \mathcal{J}_{\alpha\beta}^0(q, p), \\ \mathcal{J}^0 : T^*(Q) &\rightarrow \mathcal{L}^*(SO(N)) : (q, p) \rightarrow \frac{1}{2} \mathcal{J}_{\alpha\beta}^0(q, p) \mathbf{e}^{\alpha\beta}, \\ \mathcal{J}_{\alpha\beta}^0(q, p) &= p_k (M_{\alpha\beta})^k_j q^j. \end{aligned} \quad (\text{V.103})$$

In terms of the momenta $\mathcal{J}_{\alpha\beta}^0$, the rotation (V.98) reads

$$\delta q^i = \frac{1}{2} \delta \epsilon^{\alpha\beta} \{q^i, \mathcal{J}_{\alpha\beta}^0\}_0; \quad \delta p_k = \frac{1}{2} \delta \epsilon^{\alpha\beta} \{p_k, \mathcal{J}_{\alpha\beta}^0\}_0. \quad (\text{V.104})$$

The Lie algebra relations (V.99) become Poisson brackets:

$$\left\{ \mathcal{J}_{\alpha\beta}^0, \mathcal{J}_{\mu\nu}^0 \right\}_0 = -\delta_{\alpha\mu} \mathcal{J}_{\beta\nu}^0 + \delta_{\alpha\nu} \mathcal{J}_{\beta\mu}^0 - \delta_{\beta\nu} \mathcal{J}_{\alpha\mu}^0 + \delta_{\beta\mu} \mathcal{J}_{\alpha\nu}^0. \quad (\text{V.105})$$

Naturally, for the modified symplectic structure (III.1), the action (V.100) will be symplectic if, and only if, the magnetic fields obey:

$$\begin{aligned} F_{ki}(q) &= F_{ij}(\mathcal{R}(\mathbf{u})q) (\mathcal{R}(\mathbf{u}))^i_k (\mathcal{R}(\mathbf{u}))^j_l, \quad (\text{V.106}) \\ G^{kl}(p) &= (\mathcal{R}^{-1}(\mathbf{u}))^k_i (\mathcal{R}^{-1}(\mathbf{u}))^l_j G^{ij}(p \mathcal{R}^{-1}(\mathbf{u})). \end{aligned} \quad (\text{V.107})$$

For constant magnetic fields, this holds if $\mathcal{R}(\mathbf{u})$ belongs to the intersection of the isotropy groups of \mathbf{F} and \mathbf{G} , which, in three dimensions, is not empty if both magnetic fields are along the same axis. A rotation along this "z-axis" is then symplectic. However, in general it will not be Hamiltonian and there will be no momentum \mathcal{J}_Z such that $\delta q = \{q, \mathcal{J}_Z\}$. Again the discussion simplifies when one of the charges r or e vanishes. If the potentials \mathbf{A} or $\tilde{\mathbf{A}}$ are invariant under $\mathcal{R}(\mathbf{u})$, then the action is Hamiltonian[27] with momentum defined by the symplectic potentials (III.13) or (III.18) as

$$\langle \mathcal{J}(q, p) | \mathbf{u} \rangle = \langle \theta_{(e,0)} | X_{\mathbf{u}} \rangle \text{ or } \langle \theta_{(0,r)} | X_{\mathbf{u}} \rangle. \quad (\text{V.108})$$

Obviously there is always an $SO(N)$ group action on the (ξ, π) coordinates which is Hamiltonian with respect to (III.1) and momentum given by:

$$\mathcal{J}_{\alpha\beta}(\xi, \pi) = \pi_k (M_{\alpha\beta})^k_j \xi^j. \quad (\text{V.109})$$

However, the hamiltonian (IV.2), looking apparently $SO(N)$ symmetric, is explicitly seen not to be so when expressed in the (ξ, π) variables.

VI. FINAL COMMENTS

The symplectic structure in cotangent space, $T^*(Q) \xrightarrow{\kappa} \mathcal{Q}$, was modified through the introduction of a closed two-form \mathbf{F} on T^*Q , which has the geometric meaning of the pull-back of the magnetic field F , a closed two-form on Q : $\mathbf{F} = \kappa^*(F)$. A first caveat warns us that the other closed two-form \mathbf{G} does not have such an intrinsic interpretation. Indeed, it is obvious that

a mere change of coordinates in Q will spoil the form (III.1) of ω . This means that our approach must be restricted to configuration spaces with additional properties, which have to be conserved by coordinate changes. The most simple example is a flat linear[28] space $Q = \mathbf{E}^N$, when (III.1) is assumed to hold in linear coordinates. Obviously, a linear change in coordinates will then conserve this particular form. Although the restriction to constant fields \mathbf{F} and \mathbf{G} is a severe limitation[29], it allowed us to find explicit Darboux coordinates (IV.8) when $N = 2$ and (IV.81) when $N = 3$.

Finally, when $\det\{\mathbf{1} - r\mathbf{G}e\mathbf{F}\} = 0$, the closed two-form ω is degenerate with constant rank and defines a pre-symplectic structure on $T^*(Q)$. Its null-foliation decomposes $T^*(Q)$ in disjoint leaves and on the space of leaves, ω projects to a unique symplectic two-form. In two dimensions, the representations of the corresponding quantum algebra in Hilbert space and its reduction in the degeneracy case were studied in [11–14, 18].

APPENDIX A: ESSENTIAL SYMPLECTIC MECHANICS

Let $\{\mathcal{M}, \omega\}$ be a symplectic manifold with symplectic structure defined by a two-form ω which is closed, $\mathbf{d}\omega = 0$, and nondegenerate such that the induced mapping $\omega^\flat : T(\mathcal{M}) \rightarrow T^*(\mathcal{M}) : \mathbf{X} \rightarrow \iota_{\mathbf{X}}\omega$ has an inverse $\omega^\sharp : T^*(\mathcal{M}) \rightarrow T(\mathcal{M}) : \alpha \rightarrow \omega^\sharp(\alpha)$. The paradigm of a (non-compact) symplectic manifold is a cotangent bundle $T^*(Q)$ of a differential configuration space Q . In a coordinate system $\{q^i\}$ of Q , a cotangent vector may be written as $\alpha_q = p_i \mathbf{d}q^i$. This defines coordinates $z \Rightarrow \{q^i, p_k\}$ of points $z \in \mathcal{M} \equiv T^*(Q)$ and an associated holonomic basis $\{\mathbf{d}p_k, \mathbf{d}q^i\}$ of $T_z^*(\mathcal{M})$. The canonical one-form is defined as $\theta_0 \doteq p_i \mathbf{d}q^i$. Obviously, the exact two-form $\omega_0 \doteq -\mathbf{d}\theta_0 = \mathbf{d}q^i \wedge \mathbf{d}p_i$ is symplectic. To each observable, which is a differentiable function f on $\{\mathcal{M}, \omega\}$, the symplectic structure associates a *Hamiltonian vector field*:

$$\mathbf{X}_f \doteq \omega^\sharp(\mathbf{d}f) \quad \text{or} \quad \iota_{\mathbf{X}_f}\omega = \mathbf{d}f. \quad (\text{A.1})$$

Such a vector field generates a one-parameter (local) transformation group: $\mathcal{T}_f(t) : \mathcal{M} \rightarrow \mathcal{M} : z_0 \rightarrow z(t)$, solution of $\mathbf{d}z(t)/\mathbf{d}t = \mathbf{X}_f(z(t))$, $z(0) = z_0$.

In particular, *the* Hamiltonian \mathcal{H} generates the dynamics of the associated mechanical system. With the usual interpretation of time, $\mathbf{X}_{\mathcal{H}}$ is assumed to be complete such that its flux is defined for all $t \in [-\infty, +\infty]$. Transformations, induced by an Hamiltonian vector field \mathbf{X}_f , conserve the symplectic structure[30]:

$$\mathcal{T}_f(t)^*\omega = \omega \text{ or, locally: } \mathcal{L}_{\mathbf{X}_f}\omega = 0. \quad (\text{A.2})$$

More generally, the transformations conserving the symplectic structure form the group $\text{Sympl}(\mathcal{M})$ of *symplectomorphisms* or *canonical transformations*. Vector fields obeying $\mathcal{L}_{\mathbf{X}}\omega = 0$, generate canonical transformations and are called *locally Hamiltonian*, since [31] $\mathbf{d}\iota_{\mathbf{X}}\omega = 0$ implies that, locally in some $U \subset \mathcal{M}$, there exists a function f such that $\mathbf{d}f|_U = (\iota_{\mathbf{X}}\omega)|_U$.

The *Darboux theorem* guarantees the existence of local charts $U \subset \mathcal{M}$ with coordinates $\{q^i, p_k\}$ such that, in each U , ω is written as:

$$\omega|_U = \mathbf{d}q^i \wedge \mathbf{d}p_i. \quad (\text{A.3})$$

In the natural basis $\{\partial/\partial q^i, \partial/\partial p_k\}$ of $T_z(\mathcal{M})$, the Hamiltonian vector fields corresponding to f reads

$$\mathbf{X}_f = \frac{\partial f}{\partial p_i} \frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}.$$

The *Poisson bracket* of two observables is defined by: $\{f, g\} \doteq \omega(\mathbf{X}_f, \mathbf{X}_g)$, with the following properties:

$$\begin{aligned} \{f_1, f_2\} &= -\{f_2, f_1\} \\ \{f_1, g_1 \cdot g_2\} &= \{f_1, g_1\} \cdot g_2 + g_1 \cdot \{f_1, g_2\} \\ \{f, \{g_1, g_2\}\} &= \{\{f, g_1\}, g_2\} + \{g_1, \{f, g_2\}\} \end{aligned}$$

These properties, relating the pointwise product $g_1 \cdot g_2$ with the bracket $\{f, g\}$, are said to endow the set of differentiable functions on \mathcal{M} with the structure of a *Poisson algebra* $\mathcal{P}(\mathcal{M})$. In a coordinate system (z^A) , where $\omega = \frac{1}{2} \omega_{AB} \mathbf{d}z^A \wedge \mathbf{d}z^B$, it is given by:

$$\{f, g\} = \frac{\partial f}{\partial z^A} \Lambda^{AB} \frac{\partial g}{\partial z^B}, \quad (\text{A.4})$$

where Λ is minus ω^{-1} . In Darboux coordinates it reads:

$$\{f, g\}_0 = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i}. \quad (\text{A.5})$$

The Poisson brackets of the Darboux coordinates themselves are:

$$\{q^i, q^j\}_0 = 0, \{q^i, p_l\}_0 = \delta^i_l, \{p_k, q^j\}_0 = -\delta_k^j, \{p_k, p_l\}_0 = 0. \quad (\text{A.6})$$

The dynamical evolution of an observable is given by:

$$\frac{\mathbf{d}f}{\mathbf{d}t} = \overrightarrow{\mathbf{X}}_{\mathcal{H}}(f) = \iota_{\mathbf{X}_{\mathcal{H}}} \mathbf{d}f = \iota_{\mathbf{X}_{\mathcal{H}}} \iota_{\mathbf{X}_f} \omega = \omega(\mathbf{X}_f, \mathbf{X}_{\mathcal{H}}) = \{f, \mathcal{H}\}. \quad (\text{A.7})$$

A Lie group G acts as a symmetry group on a symplectic manifold \mathcal{M} , if there is a group homomorphism $\mathcal{T} : G \rightarrow \text{Symp}(\mathcal{M}) : g \rightarrow \mathcal{T}(g)$. An infinitesimal action defined by a Lie algebra element $\mathbf{u} \in \mathcal{G}$ is given by the locally Hamiltonian vector field

$$\mathbf{X}_{\mathbf{u}}(z) = \frac{d}{dt} (\mathcal{T}(\exp(t\mathbf{u}))z)|_{t=0}. \quad (\text{A.8})$$

When each $\mathbf{X}_{\mathbf{u}}$ is Hamiltonian, the group action is said to be *almost Hamiltonian* and $\{\mathcal{M}, \omega\}$ is called a *symplectic G-space*. In such a case, a linear map $\Xi : \mathcal{G} \rightarrow \mathcal{P}(\mathcal{M}) : \mathbf{u} \rightarrow \Xi(\mathbf{u})$ can always be constructed such that $\mathbf{X}_{\mathbf{u}} = \omega^\sharp(\mathbf{d}\Xi(\mathbf{u}))$. When there is a Ξ which is also a Lie algebra homomorphism: $\Xi([\mathbf{u}, \mathbf{v}]) = \{\Xi(\mathbf{u}), \Xi(\mathbf{v})\}$, the group is said to have a *Hamiltonian action* and $\{\mathcal{M}, \omega, \Xi\}$ is called a *Hamiltonian*

G-space. Since Ξ is linear in \mathcal{G} , it defines a *momentum mapping* J from \mathcal{M} to the dual \mathcal{G}^* of the Lie algebra defined by: $\langle J(z)|\mathbf{u} \rangle = \Xi(\mathbf{u}, z)$. When \mathcal{M} is a Hamiltonian G -space, the momentum mapping is equivariant under the action of G on \mathcal{M} and its co-adjoint action on \mathcal{G}^* .

In general there may be topological obstructions to such a Lie algebra homomorphism. However, when G acts on Q : $\varphi : G \rightarrow \text{Diff}(Q) : g \rightarrow \varphi(g) : q \rightarrow q' = \varphi(g)q$, the action is extended to a symplectic action in $\{\mathcal{M} = T^*(Q), \omega_0\}$: $\tilde{\varphi} : G \rightarrow \text{Symp}(\mathcal{M}) : g \rightarrow \tilde{\varphi}(g) : (q, p) \rightarrow (q', p')$, where p' is defined by $p = (\varphi(g))_q^* p'$. It follows that $\tilde{\varphi}(g)^* \theta_0 = \theta_0$; $\tilde{\varphi}(g)^* \omega_0 = \omega_0$. The infinitesimal action is given by $\mathbf{X}_{\mathbf{u}}(z) = (\mathbf{d}\tilde{\varphi}(\exp(t\mathbf{u}))z/dt)|_{t=0}$ and $\mathcal{L}_{\mathbf{X}_{\mathbf{u}}} \theta_0 = 0$; $\mathcal{L}_{\mathbf{X}_{\mathbf{u}}} \omega_0 = 0$. From $\omega_0^\flat(\mathbf{X}_{\mathbf{u}}) = \mathbf{d}\langle \theta_0 | \mathbf{X}_{\mathbf{u}} \rangle$, it follows that the action is almost Hamiltonian with $\Xi(\mathbf{u}) = \langle \theta_0 | \mathbf{X}_{\mathbf{u}} \rangle$. Moreover, since $\langle \theta_0 | \mathbf{X}_{[\mathbf{u}, \mathbf{v}]}\rangle = \omega_0(\mathbf{X}_{\mathbf{u}}, \mathbf{X}_{\mathbf{v}}) = \{\Xi(\mathbf{u}), \Xi(\mathbf{v})\}$, the action is Hamiltonian and $\{T^*(Q), \omega_0, \Xi\}$ is a Hamiltonian G -space.

APPENDIX B: PRESYMPLECTIC MECHANICS

A manifold \mathcal{M}_1 , endowed with a closed but degenerate[32] 2-form ω , with constant rank, is said to be presymplectic. The mapping ω^\flat has a nonvanishing kernel, given by those nonzero vector fields \mathbf{X}_0 obeying $\omega^\flat(\mathbf{X}_0) \doteq \iota_{\mathbf{X}_0} \omega = 0$. The fundamental dynamical equation

$$\omega^\flat(\mathbf{X}) = \mathbf{d}\mathcal{H}, \quad (\text{B.1})$$

has then a solution if

$$\langle \mathbf{d}\mathcal{H} | \mathbf{X}_0 \rangle = 0 \quad ; \quad \forall \mathbf{X}_0 \in \mathcal{Ker}(\omega^\flat). \quad (\text{B.2})$$

If this is nowhere satisfied on \mathcal{M}_1 , the hamiltonian \mathcal{H} does not define any dynamics on \mathcal{M}_1 . When (B.2) is identically satisfied, a particular solution \mathbf{X}_P of (B.1) is defined in the entire manifold \mathcal{M}_1 and so is the general solution obtained summing the general solution of the homogeneous equation $\iota_{\mathbf{X}_0} \omega = 0$, i.e $\mathbf{X}_G = \mathbf{X}_P + \mathbf{X}_0$, which will contain arbitrary functions. When (B.2) is satisfied for some points $z \in \mathcal{M}_1$, we shall assume they form a submanifold, called the secondary constrained submanifold with injection $\iota_2 : \mathcal{M}_2 \hookrightarrow \mathcal{M}_1$. The particular solution \mathbf{X}_P of (B.1) is now defined in \mathcal{M}_2 and so is the general solution \mathbf{X}_G . Requiring that \mathbf{X}_G conserves the constraints amounts to ask that \mathbf{X}_G is tangent to \mathcal{M}_2 :

$$\mathbf{X}_G = \iota_{2*}(\mathbf{X}_2) ; \mathbf{X}_2 \in \Gamma(\mathcal{M}_2, T\mathcal{M}_2). \quad (\text{B.3})$$

Again, when there are no points where this tangency condition is satisfied, (B.1) is meaningless. Another possibility is that some of the arbitrary functions in \mathbf{X}_0 become determined and the tangency condition is obeyed on the entire \mathcal{M}_2 . The general solution then still contains some arbitrary functions. Finally it may happen that the conditions (B.3) are only satisfied on some \mathcal{M}_3 with $\iota_3 : \mathcal{M}_3 \hookrightarrow \mathcal{M}_2$. The story then goes on until one of the first two alternatives are reached.

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- [19] See e.g. [8]
- [20] Well known in symplectic mechanics, see e.g.[5, 6, 9].
- [21] Observe that $\Phi_k^\ell = \delta_k^\ell - e\mathbf{F}_{kj}r\mathbf{G}^{j\ell}$ and $\Psi_j^i = \delta_j^i - r\mathbf{G}^{i\ell}e\mathbf{F}_{\ell j}$ are mutually transposed and that the matrices $\Psi_k^j r\mathbf{G}^{j\ell} = r\mathbf{G}^{kj}\Phi_j^\ell$ and $\Phi_k^j e\mathbf{F}_{j\ell} = e\mathbf{F}_{kj}\Psi_j^\ell$ are antisymmetric.
- [22] In the limit $\chi \rightarrow 0$, we have $m'\omega'_0 = \sqrt{m'\kappa'} \rightarrow |B|$.
- [23] Recall that with complex variables $q = q^1 + \mathbf{i}q^2$, the differentials $dq = dq^1 + \mathbf{i}dq^2$ and $dq^\dagger = dq^1 - \mathbf{i}dq^2$ have local dual vector fields $\{\partial/\partial q = (\partial/\partial q^1 - \mathbf{i}\partial/\partial q^2)/2; \partial/\partial q^\dagger = (\partial/\partial q^1 + \mathbf{i}\partial/\partial q^2)/2$ and similarly for the $p = p_1 + \mathbf{i}p_2$ variables.
- [24] The ($N = 3$) case wil only be examined in the nondegenerate case $\chi > 0$.
- [25] Due to $\kappa^2 + \kappa'^2 (rC)^2 (eB)^2 + 2\kappa\kappa' rCeB = 1$, the mass and elastic constant of the z degrees of freedom, as expected, are not renormalised.
- [26] with dual basis $\{\mathbf{e}^{\alpha\beta}\}$ in $\mathcal{L}^*(SO(N))$.
- [27] Exercise 4.2A in [6], defining a (generalized) Poincaré momentum.
- [28] Quantum mechanics on a noncommutative sphere S^2 and on general noncommutative Riemann surfaces was examined in ([12, 13]).
- [29] In the case $e = 0$, Darboux coordinates are given by (III.17) and in [16] such model was considered with the possibility of having a monopole in p -space!
- [30] $\mathcal{T}_f(t)^*$ denotes the pull-back of $\mathcal{T}_f(t)$ and \mathcal{L} is the Lie derivative along \mathbf{X}_f .
- [31] We use $\mathcal{L}_{\mathbf{X}} = \mathbf{d}\iota_{\mathbf{X}} + \iota_{\mathbf{X}}\mathbf{d}$ on differential forms.
- [32] \mathcal{M}_1 is the primary constrained manifold, arising e.g. from a degenerate Lagrangian.