Fractal Propagators in QED and QCD and Implications for the Problem of Confinement

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We show that QED radiative corrections change the propagator of a charged Dirac particle so that it acquires a fractional anomalous exponent connected with the fine structure constant. The result is a nonlocal object which represents a particle with a roughened trajectory whose fractal dimension can be calculated. This represents a significant shift from the traditional Wigner notions of asymptotic states with sharp well-defined masses. Non-abelian long-range fields are more difficult to handle, but we are able to calculate the effects due to Newtonian gravitational corrections. We suggest a new approach to confinement in QCD based on a particle trajectory acquiring a fractal dimension which goes to zero in the infrared as a consequence of self-interaction, representing a particle which, in the infrared limit, cannot propagate.

I. INTRODUCTION

In [1], a gauge-invariant calculation was presented showing that the propagator for a charged particle acquires an interesting fractal structure from its self-interaction in quantum electrodynamics. In simple terms, the usual Dirac propagator $S(k)$ is replaced by the nonlocal expression

$$S(k) = (\kappa - k\!\!\!/)\mathcal{D}(k),$$
$$\mathcal{D}(k) = \left(\frac{\kappa}{i\Lambda}\right)^{\alpha/\pi} \left\{ \frac{\Gamma\left(1 + (\alpha/\pi)\right)}{\left[k^2 + \kappa^2 - i0^+\right]^{\left(1 + (\alpha/\pi)\right)}} \right\}, \quad (1)$$

where $\hbar\kappa = mc$, Λ is a short distance length scale, the fractional exponent γ is a function of the coupling strength $\alpha = (e^2/\hbar c)$ which we find to be α/π and the usual Gamma function is

$$\Gamma(z) = \int_0^\infty e^{-s} s^z \frac{ds}{s} \quad \text{with} \quad \Re e(z) > 0. \quad (2)$$

This represents a rather radical departure from the usual textbook discussions of particle propagators (with the notable exception of [2]) and from the usual Wigner classification of elementary particles in terms of a sharp mass and spin (see, for example, [3]). This is despite the fact that it is well-known that particles coupled to long-range massless fields cannot have sharp masses[4–6]. A good review of the basic issues can be found in [7].

Aside from having been neglected in textbooks, the actual exponent in the above expression has been the subject of some controversy. There are calculations in the literature based on either infinite sums of logarithmic Feynman diagrams[15] or non-perturbative Schwinger[8, 9, 11] computations which argue that such an electron propagator should be of the *form* we obtained:

$$S(k) = \left(\frac{\kappa}{i\Lambda}\right)^{\gamma}\Gamma(1+\gamma) \left\{ \frac{\kappa - k\!\!\!/}{\left[k^2 + \kappa^2 - i0^+\right]^{(1+\gamma)}} \right\}, \quad (3)$$

.

In the literature there has not been full agreement about what constitutes the correct function $\gamma(\alpha)$, nor has there been a consensus as to whether or not it can be set to zero by a suitable choice of gauge.

Appelquist and Carazzone[12] argued that $\gamma = -(\alpha/\pi) + \dots$ to leading order, in conflict with earlier work based on summing logarithms[15]. Reference [12] does not actually derive the exponent, rather citing the earlier work, so there is a possibility that a typographical error may be involved.

The fourth volume of the Landau and Lifschitz course of theoretical physics[10] starts off in agreement with $\gamma \neq 0$ but ultimately sets $\gamma = 0$ by giving a small mass to the photon. Such a photon mass explicitly breaks both gauge invariance and conformal/scale invariance and must be rejected on physical grounds. Intuitively, what goes wrong is that the fractal structure of the propagator for a charged particle must be due to the fact that ever increasing wavelengths one can produce more and more soft photons without limit. A photon mass implies a maximum photon wavelength and a loss of scale invariance.

There have been other approaches in the literature, often tackling only the case of charged scalar fields, or again arguing that the anomalous dimension can somehow be set to zero. For example, a path integral[11] approach using the Schwinger[8] proper time representation of the propagator and some work by Bloch and Nordsieck[16] on soft photon emission gives the same sort of result as ours, but with the final answer given only for charged scalar fields. These results are argued to be gauge invariant in such a way that the singularity structure can be returned to a simple pole by a choice of gauge. As argued above, this is unphysical as there is a real meaning to a fractional exponent and the failure of charged particles to have sharp masses. Fried also discusses this problem[17] as do Johnson and Zumino[18], and Stefanis and collaborators[19]. Batalin, Fradkin and Schvartsman have made a similar gauge dependent calculation for scalar particles[20].

In addition, there are possible experimental consequences of the radical change in the nature of the singularity in the propagator. Handel has argued[21] that the change from a pole into a branch point has measurable physical implications for "$1/\omega$" noise in the Schrödinger (non-relativistic) limit.

The result is clearly not analytic in α and requires a non-perturbative approach. Such non-analyticity in α for an all-orders or non-perturbative calculation could be anticipated on physical grounds from the arguments of Dyson against the convergence of perturbative expansions in QED[13].

Given the apparent confusion in the literature, we present a simple derivation which preserves gauge invariance throughout. With this in hand, we then discuss the physical interpretation of the result, extend it to include Newtonian gravity, discuss the fractal dimension one can associate with paths involved, and finally introduce a new way to think about confinement in these terms.

II. DERIVATION OF THE DRESSED ELECTRON PROPAGATOR

For completeness, we rederive the expression for the dressed electron propagator presented in [1].

For an electron in an external electromagnetic field, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the Dirac propagator

$$(-i\slashed{d} + \kappa)\, G(x,y;A) = \delta(x-y) \ ,$$

$$d_\mu = \partial_\mu - i\left(\frac{eA_\mu}{\hbar c}\right) \ , \qquad (4)$$

may be solved employing the function $\Delta(x,y;A)$;

$$\Delta(x,y;A) = \int \gamma_5 G(x,z;A)\gamma_5 G(z,y;A)d^4z \ ,$$
$$G(x,y;A) = (i\slashed{d} + \kappa)\,\Delta(x,y;A) \ ,$$
$$\left(\slashed{d}^2 + \kappa^2\right)\Delta(x,y;A) = \delta(x-y) \ ,$$
$$\slashed{d}^2 = -d^\mu d_\mu - \frac{e}{2\hbar c}\sigma^{\mu\nu}F_{\mu\nu} \ . \qquad (5)$$

The Hamiltonian of the electron can be written

$$\mathcal{H}_{tot} = \mathcal{H} + \mathcal{H}_{spin} \ ,$$
$$\mathcal{H} = \frac{1}{2m}\left\{\left(p - \frac{e}{c}A\right)^2 + m^2c^2\right\} \ ,$$
$$\mathcal{H}_{spin} = -\left(\frac{e\hbar}{4mc}\right)\sigma^{\mu\nu}F_{\mu\nu} \qquad (6)$$

and with $p_\mu = -i\hbar\partial_\mu$, one may define the amplitude for the electron to go from y to x in a proper time τ as the matrix element

$$\mathcal{G}(x,y,\tau;A) = \langle x| e^{-i\mathcal{H}_{tot}\tau/\hbar} |y\rangle . \qquad (7)$$

From Eqs.(5), (6) and (7) follows the electron propagator expression

$$\Delta(x,y;A) = \frac{i\hbar}{2m}\int_0^\infty \mathcal{G}(x,y,\tau;A)d\tau,$$
$$\hbar G(x,y;A) = \left(mc - \slashed{p} + \frac{e}{c}\slashed{A}(x)\right)\Delta(x,y;A) \ . \qquad (8)$$

One also has a Lagrangian

$$\mathcal{L}(v,x;A) = \frac{1}{2}m\left(v^\mu v_\mu - c^2\right) + \frac{e}{c}v^\mu A_\mu(x),$$
$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial v^\mu}\right) = \left(\frac{\partial\mathcal{L}}{\partial x^\mu}\right); \qquad (9)$$

As we are interested here in the infrared limit, spin-flips are suppressed and we can neglect the corresponding term in the Hamiltonian. This corresponds to usual Bloch-Nordsieck replacement of $c\gamma^\mu$ by four velocity v^μ – one simply thinks of the Dirac spinor having only one nonvanishing component representing an electron of given spin state and allows no coupling to the other spin state (and, of course, no coupling to positrons).

With this approximation, Eq. (7) which gives the propagation amplitude for an electron can be written in the Lagrangian *path integral* formulation

$$\mathcal{G}(x,y,\tau;A) \approx \int_{X(0)=y}^{X(\tau)=x} e^{iS[X;A]/\hbar}\prod_\sigma dX(\sigma),$$
$$S[X;A] = \int_0^\tau \mathcal{L}(\dot{X}(\sigma),X(\sigma);A)d\sigma. \qquad (10)$$

This expression deserves some comments. The paths $X(\sigma)$ which are integrated over represent virtual histories for the electron. The proper time τ for each path in the sum is not given by the classical expression $c^2\tau^2 \neq -(x-y)^2$, so in integrating over paths one is integrating here over all proper times.

For each one of these paths, with its own proper time, the interaction between the electron and an elecromagnetic vector potential A^μ is described by the action

$$S_{\text{int}}(P;A) = \int_0^\tau \mathcal{L}_{\text{int}}(\dot{X}(\sigma),X(\sigma);A)d\sigma,$$
$$S_{\text{int}}(P;A) = \frac{e}{c}\int_0^\tau A_\mu(X(\sigma))\dot{X}^\mu(\sigma)d\sigma,$$
$$S_{\text{int}}(P;A) = \frac{e}{c}\int_P A_\mu(X)dX^\mu, \qquad (11)$$

where the integral is along the worldline P.

For the case of no external field, we want the action due to self-interaction (interaction with vacuum fluctuations), so one wants to apply the rule

$$e^{iS_{\text{int}}(P;A)/\hbar} \rightarrow \langle 0| e^{iS_{\text{int}}(P;\hat{A})/\hbar} |0\rangle_+ ,$$
$$e^{iS_{\text{int}}(P;A)/\hbar} \rightarrow e^{iS_{self}(P)/\hbar},$$
$$S_{self}(P) = \frac{\hbar\alpha}{2}\int_P\int_P D_{\mu\nu}(x_1 - x_2)dx_1^\mu dx_2^\nu. \qquad (12)$$

In the above Eq.(12), the subscript "+" denotes time ordering, $\hat{A}_\mu(x)$ denotes the operator vector potential field and the photon propagator is given by

$$D_{\mu\nu}(x_1 - x_2) = \frac{i}{\hbar c}\langle 0|\hat{A}_\mu(x_1)\hat{A}_\nu(x_2)|0\rangle_+ . \qquad (13)$$

The action form in Eq.(12) is of a well known form[22]. As noted earlier, we have bypassed the usual Bloch-Nordsieck replacement of $c\gamma^\mu$ by four velocity v^μ by simply evaluating a phase in the soft photon infrared limit that we are considering here.

The propagator may be written

$$D_{\mu\nu}(x-y) = \left(\eta_{\mu\nu} - (1-\xi)\frac{\partial_\mu\partial_\nu}{\partial^2}\right)D(x-y),$$

$$D(x-y) = \int \frac{4\pi}{k^2 - i0^+}e^{ik\cdot(x-y)}\frac{d^4k}{(2\pi)^4},$$

$$D(x-y) = \frac{i}{\pi}\left\{\frac{1}{(x-y)^2 + i0^+}\right\}. \qquad (14)$$

where the parameter ξ fixes a gauge.

Because the worldline of the electron never begins nor ends the partial derivative terms in Eq.(14) do not contribute to the self action in Eq.(12). This requirement that the worldline have no endpoints is required by charge conservation (or, equivalently, by gauge invariance).

Independent of any choice of the gauge parameter ξ then, we have

$$S_{self}(P) = \frac{\hbar\alpha}{2}\int_P\int_P D(x_1 - x_2)dx_{1\mu}dx_2^\mu. \qquad (15)$$

In the absence of any external field (other than that due to vacuum fluctuations) we have now derived expressions for the renormalized vacuum electron propagator

$$\tilde{G}(x-y) = \int S(k)e^{ik\cdot(x-y)}\frac{d^4k}{(2\pi)^4},$$

$$\tilde{G}(x-y) = \langle 0|G(x,y;\hat{A})|0\rangle_+,$$

$$\tilde{G}(x-y) = (i\slashed{\partial} + \kappa)\tilde{\Delta}(x-y),$$

$$\tilde{\Delta}(x-y) = \frac{i\hbar}{2m}\int_0^\infty \tilde{G}(x-y,\tau)d\tau. \qquad (16)$$

The functional integral expression for $\tilde{G}(x-y,\tau)$ is given by

$$\tilde{G}(x-y,\tau) = \int_{X(0)=y}^{X(\tau)=x}e^{i\tilde{S}[X;A]/\hbar}\prod_\sigma dX(\sigma),$$

$$\tilde{S}[X] = \int_0^\tau \mathcal{L}_0(\dot{X}(\sigma))d\sigma + S_{self}[X], \qquad (17)$$

wherein the *free* electron Lagrangian is

$$\mathcal{L}_0(\dot{X}) = \frac{1}{2}m_0(\dot{X}^\mu\dot{X}_\mu - c^2), \qquad (18)$$

and the self action is given by Eqs.(14) and (15) as

$$S_{self}[X] = \frac{i\hbar\alpha}{2\pi}\int_0^\tau\int_0^\tau\frac{\dot{X}^\mu(\sigma_1)\dot{X}_\mu(\sigma_2)d\sigma_1d\sigma_2}{(X(\sigma_1)-X(\sigma_2))^2 + i0^+}. \qquad (19)$$

The divergent piece of the self action

$$\Re eS_{self}[X] = \frac{\Delta m}{2}\int_0^\tau(\dot{X}^\mu(\sigma)\dot{X}_\mu(\sigma) - c^2)d\sigma,$$

$$|\Delta m| = \infty. \qquad (20)$$

The formally infinite self-mass can be absorbed into a redefinition of the finite physical mass $0 < m = (m_0 + \Delta m) < \infty$.

Thus, Eq.(17) is renormalized to

$$\tilde{G}(x-y,\tau) = \int_{X(0)=y}^{X(\tau)=x}e^{i\tilde{S}[X;A]/\hbar}\prod_\sigma dX(\sigma),$$

$$\tilde{S}[X] = \int_0^\tau \mathcal{L}_m(\dot{X}(\sigma))d\sigma + iW[X]$$

$$\mathcal{L}_m(\dot{X}) = \frac{1}{2}m(\dot{X}^\mu\dot{X}_\mu - c^2)$$

$$W[X;\tau] = \Im mS_{self}[X],$$

$$W[X;\tau] = \frac{\hbar\alpha}{2\pi}\int_0^\tau\int_0^\tau\frac{\dot{X}^\mu(\sigma_1)\dot{X}_\mu(\sigma_2)d\sigma_1d\sigma_2}{(X(\sigma_1)-X(\sigma_2))^2}. \qquad (21)$$

For a straight-line path $X^\mu(\sigma) = V^\mu\sigma$ with $V^\mu V_\mu = -c^2$, one finds

$$W(\tau) = \frac{\hbar\alpha}{2\pi}\int_0^\tau\int_0^\tau\frac{d\sigma_1d\sigma_2}{(\sigma_1 - \sigma_2)^2}. \qquad (22)$$

This expression as it stands is infinite and requires regularization. Using differential regularization[23] we have

$$\frac{d^2W(\tau)}{d\tau^2} = \frac{\hbar\alpha}{\pi\tau^2}. \qquad (23)$$

The solution to the differential equation Eq.(23) with a logarithmic cut-off Λ is

$$W(\tau) = -\left(\frac{\hbar\alpha}{\pi}\right)\ln\left(\frac{c\tau}{2\Lambda}\right). \qquad (24)$$

From Eqs.(21) and (24), one finds

$$\tilde{G}(x-y,\tau) \approx e^{-W(\tau)/\hbar}\tilde{G}_m(x-y,\tau) \qquad (25)$$

wherein $\tilde{G}_m(x-y,\tau)$ is the proper time Green's function for a particle of mass m with the corresponding free Lagrangian $\mathcal{L}_m(\dot{X})$. To exponentially lowest order in α, one then has

$$\tilde{G}_m(x-y,\tau) = \int\left\{e^{-i\hbar(k^2+\kappa^2)\tau/2m}e^{ik\cdot(x-y)}\right\}\frac{d^4k}{(2\pi)^4},$$

$$\tilde{G}(x-y,\tau) = \left(\frac{c\tau}{2\Lambda}\right)^{\alpha/\pi}\tilde{G}_m(x-y,\tau). \qquad (26)$$

Eqs.(16) and (26) then imply

$$\tilde{\Delta}(x-y) = \int \mathcal{D}(k)e^{ik\cdot(x-y)}\frac{d^4k}{(2\pi)^4},$$

$$\tilde{G}(x-y) = \int S(k)e^{ik\cdot(x-y)}\frac{d^4k}{(2\pi)^4}. \qquad (27)$$

and we obtain Eq. (1).

We chose differential regularization as simple and convenient, and preserving gauge invariance, but other regularizations can also be used as long as they are also gauge invariant. We plan to return to this issue in a later publication in more detail. In a sense, the form of the answer is intuitively clear since, on dimensional grounds, the expression to be regularized can only be logarithmic. The sign, as we shall see below, has a physically sensible interpretation, and is in agreement with reference [2] and with reference [12] if one assumes that there was a typographical error going from reference [15].

III.   A PHYSICAL INTERPRETATION

It is interesting to consider what the physical interpretation of the non-integer exponent in the radiatively corrected Dirac propagator means. First of all, the fact that the exponent is non-integer means that the renormalized Dirac operator is non-local[24]. This was, of course, to be expected since the electromagnetic field has infinite range.

Non-locality has been previously introduced ad-hoc[25, 26] as a regularization tool. Here it appears naturally, suggesting that some form of regularization of otherwise formally divergent expressions may be implicit in at least some quantum field theories with massless fields, but appearing only when one goes beyond perturbation theory. We will also see later that there is evidence that the corrections to the exponent can be changed when other fields are added, and need not always be of the same sign as in pure QED. The appearance of this sort of non-locality also makes the analytic but rarely used regularization proposed by Speer[27] more physically motivated.

So far we have only been able to treat the long-wavelength approximation since we expect a quenched approximation in which we can ignore electron loops to be a reasonable one. The physical picture would be one where, as one backs away from the worldline of an electron, one continues to see photons radiated and absorbed, but now of longer and longer wavelength. This would suggest a fractal[28] structure, which is made precise by the above derivation. Such notions of scaling and fractality are not new in QED and in quantum field theory in general, but are often considered in the high energy, ultraviolet limit[29, 30]. This continues without limit at longer and longer distance scales since there is no minimum energy photon (photons are massless) and this self-similarity reflects the scale invariance of Maxwell's equations.

If the photon is given a mass, however small, this structure will break down asymptotically, since now there is a minimum energy required to create a virtual photon, and at distances greater than the corresponding Compton wavelength one will get the non-interacting Dirac propagator. This was done by Lifshitz *et al.*[10], and this argument makes clear how breaking gauge invariance, *i.e.* including a photon mass, removes the anomalous scaling behavior here derived for gauge invariant QED.

The fact that a particle is non-localizable, at least in part due to its electromagnetic field which extends over all space, is interesting. The feeling of this calculation is such that at least part of what one thinks of as quantum-mechanical about an otherwise point-particle (its lack of localizability) may arise from the non-perturbative quantum mechanics of its self-interaction[31].

At shorter distance scales one might again expect an anomalous dimension, but now the calculations are more complicated since one must imagine more and more electron (and other charged particle) loops contributing, with an ever "frothier" structure at smaller and smaller distance scales. In fact, one would not even expect a constant exponent which is independent of momentum since one expects a running α with values which change with momentum scale. In the infrared limit

one simply goes to $\alpha(q^2 = 0)$ which is the Thompson value and there are no additional complications. In the ultraviolet limit, one expects a continuously changing exponent and thus a multifractal as opposed to fractal structure. In addition, the lack of asymptotic freedom leads one to expect trouble at very short distance scales unless, as noted above, other fields enter significantly.

IV.   NEWTONIAN QUANTUM GRAVITY

For quantum gravity one can do the analysis in much the same way as for quantum electrodynamics. The ultraviolet divergences need not worry us since we are dealing with a strictly infrared problem. There should be graviton-graviton interactions, but if we neglect these as small compared to graviton-electron interactions we can just repeat what was done for electrostatics but now with the Newtonian limit of gravity.

This approximation would not be as reasonable in the QCD case for a quark propagator since gluon-gluon couplings are comparable to gluon-quark couplings, but there is evidence from other calculations of the appearance of anomalous dimensions for infrared propagators. (For a review see [32]. Very recent calculations can also be found in references [33].)

Perhaps the simplest way to think of this is to look at Eq. (15) and regard the (singular) double integral as the limit of two paths which must be taken as approaching each other arbitrarily closely. This is an alternative to the differential regularization we used in Eq.(23) and physically corresponds to two electron worldlines interacting via the exchange of an arbitary number of photons between them and at all points of their paths. In the limit of the paths coinciding, this becomes self-interaction, with the electron continuously exchanging photons with itself along its worldline.

For gravity (at least in the Newtonian limit) we want to replace the repulsive electrostatic self-interaction, say $+(e^2/r)$, with the attractive gravitational self-interaction, say $-Gm^2/r$, suggesting an asymptotic form of the Dirac propagator exponent

$$\gamma \approx \frac{1}{\pi\hbar c}(e^2 - Gm^2) + \dots . \tag{28}$$

If $m = |e/\sqrt{G}|$, which is the ADM[34, 35] mass of charged shell of charge e, regularized by its own gravity, then one recovers an effectively free propagator. Since one generally makes measurements using the electromagnetic interaction, and the suggestion that quantum mechanics might be linked to self-interaction[31], it is interesting to consider what this might imply for the role of gravity in the quantum measurement problem[36]. In particular, since one has a connection between mass and charge which is non-perturbative in Newton's G, and implies a mass near the Planck mass $\sim 10^{-5}$ gm, which may be thought to be in the neighborhood of a putative classical-quantum boundary.

Intuitively, electromagnetic interactions roughen the path of an electron, making it "spread out more", which is natural since the electromagnetic force between two identical charges

(or an electron and itself) is repulsive. Gravitational interactions, on the other hand, make it "spread out less", since the gravitational force between two identical objects (here an electron and itself) is attractive.

V. FRACTAL PATHS

The notion of a path somehow "spreading out" can be made precise with the notion of fractal dimensions.

The fractal nature of particle paths has been discussed in the literature (see, for example [37, 38]). Abbott and Wise[39], working within nonrelativistic quantum mechanics, found 2 as the dimension of a quantum mechanical path, as opposed to 1 for a classical path. Cannata and Ferrari[40] extended this work for spin-1/2 particles and find different results not only in the classical and quantum mechanical limits, but also in the non-relativistic and relativistic limits.

Intuitively one can understand the dimension 2 result for the nonrelativistic quantum mechanical case by thinking of the Schrödinger equation as a diffusion equation in imaginary time[41]. For diffusion one has the distance r a particle covers in time t satisfying a relationship of the form $t \propto r^d$ wherein d is the fractal dimension of the "path". For example, $t \propto r^2$ in the diffusion limit, and $t \propto r$ in the ballistic (simple path) limit[42].

Here we have a closely analogous situation but with a 4-dimensional Hamiltonian $\mathcal{H}$ and with fractal diffusion in proper time, and as was showed in [1], one has

$$d = 2(1+\gamma) \approx 2 + \frac{2\alpha}{\pi} + \dots , \qquad (29)$$

All previous discussions of fractal propagators in quantum field theory and quantum mechanics have, to the best of our knowledge, ignored the effects of self-interaction via long-range fields.

The fact that self-interaction can qualitatively change the nature of a propagating particle even when coupled to a weak (small coupling constant) Abelian gauge field naturally leads one to wonder whether or not coupling to a strong (large coupling constant) non-Abelian gauge field might have significant physical effects. We turn now to a discussion of possible implications of this sort of phenomenon for an understanding of confinement in QCD.

VI. QCD

In this section we consider how propagators with anomalous dimensions could shed light on the confinement mechanism in QCD. Precise and rigorous calculations are beyond the scope of this paper, but we do indicate approaches to such calculations and what one would expect qualitatively.

Quantum chromodynamics is an incredibly difficult theory in which to say anything precise. The fundamental degrees of freedom carry strong (colour) charges, interact via highly nonlinear interactions, and (by hypothesis), are not even observable asymptotically.

Perturbative treatments make QCD look very much like QED, and involve propagators which look pretty much like those of QED with the main difference being in the nonabelian nature of the gauge fields and their associated self-couplings. The question which we now raise is whether or not one can imagine a situation whereby anomalous exponents in the infrared will give rise to such strong qualitative changes in how propagators behave that the originally postulated degrees of freedom will simply not propagate (at least at low energies) – that is, that some form of confinement appears.

What could confinement mean in terms of an anomalous dimension? If in Eq. 29 we had a γ which drove d to zero, one would have a zero-dimensional path, which is no path at all! This would correspond to a confined particle.

Let us see how far we can argue that such a phenomenon would occur. One could start from Eq.(15) with an appropriate form of $D(x_1 - x_2)$ for QCD valid for any separation. Unfortunately, we do not have an exact expression for $D(x_1 - x_2)$ in the infrared limit and in fact every expectation is that (even aside from colour indices) it is very different from the QED one – presumably having a confining term, roughly linear in separation. Of course this would be *assuming* confinement.

A simpler approach is to imagine that $d = 2(1+\gamma)$ where $\gamma \propto \alpha_s$ and $\gamma < 0$. If the strong coupling constant α_s is large enough then one could get $d \to 0$. Let us follow this line of argument more closely. The first thing to check is that the energy of self-interaction of a quark with itself is indeed of the same sign as that due to gravity (and opposite to that due to electromagnetism). While the interaction energy between a quark and an antiquark in the singlet state separated by a distance r is given perturbatively by $\frac{4}{3}\frac{\alpha_s(r)}{r}$, the corresponding quark-quark expression in the triplet is $-\frac{2}{3}\frac{\alpha_s(r)}{r}$. In other words, it is of the correct sign to lead to confinement.

Now let us revisit Eq.(15) and regard the (singular) double integral as the limit of two paths which must be taken as approaching each other arbitrarily closely as we did in section IV. We are now looking at two paths of a red, say, quark, exchanging an arbitrary number of gluons. If we now argue as we did for gravity, we have $d = 2(1 - \frac{2}{3}\frac{\alpha_s(r)}{\pi})$. It is well-known[43, 44] (and can be shown on very general grounds from dispersion relations) that α_s must increase without bound as r goes to infinity (or squared momentum transfer q^2 goes to zero). Happily here one only needs α_s to reach a value making $d = 0$. At the corresponding value of q^2 the dimension of a quark path goes to zero and the quark is confined. Lower values q^2 (and negative dimensions) make no sense since the integrals in Eq. (15) now have no paths any more as soon as d hits zero. Note that this approach to thinking about confinement does not require any singular (i.e. infinite) values of q^2 (or length) but should happen at some well-defined value of q^2 (or length) presumably related to the confinement scale.

A rigorous calculation along these lines would be difficult, but a few points can be made in defense of this overall picture. First of all, why would one imagine in the regularized view of Eq. (15) that one was always looking at a red quark interacting

with another red quark? Surely each emitted gluon changes the colour of the particle and complicates matters. This is indeed true, but if one has defined the quark as being red (in some gauge) then, even with various colours of gluons being continuously emitted, one expects that it would remain *on average* red. Any tendency to be green or blue would be a violation of $SU(3)$ symmetry since it would indicate a preferred direction in colour space other than "red" (The effective value of the colour charge would be expected to be scale-dependent, but this is exactly what a running coupling constant is meant to describe.) This sort of argument could be the basis for a more rigourous mean field theory approach. Second α_s is usually studied as a function of q^2, but its form is not even known at long distance scales and thus a rather difficult (!) function to work out in coordinate space to use in Eq.(15). The point we want to make here is that it *is* very generally known that α increases in the infrared limit, and this alone is enough to argue for confinement. Even without an explicitly known confining form for $D(x-y)$, all one needs to know is that α continues to rise. At some point it will rise to a value high enough to drive d to zero. Beyond that, the quark is non-propagating (our version of "confined") and there is no need to evelute paths in Eq.(15) since the quark has none! The expected results from asymptotic freedom are, of course, reproduced, since as α_s goes to zero the propagator goes over to its free form (aside from whatever corrections remain due to charge and mass as discussed earlier – particles coupled to long-range fields never really get free!).

A similar argument would be expected to hold for any coloured objects, including gluons and hypothetical objects in other representations of $SU(3)$ colour.

The physical picture is an interesting one and rather different from the assumption that quarks are bound due to strong attractive forces to other quarks which increase linearly with separation in some approximation. Here, below some energy scale (that at which the dimension d goes to zero), coloured objects interact so much with their own glue that they get "stuck" in the sense that their would-be paths are reduced to dimension zero. At higher energy scales they partially escape, with a dimension which rises as the effective coupling strength drops until they become, to a good approximation, free. It is interesting to note that the view of confinement suggested here applies to any coloured object with no need for it to have neighbours present to "bind" to it and ensure that a colour singlet state propagates – here interaction of a single coloured object with its own colour field is enough to confine it.

VII. CONCLUSIONS

We have reviewed the simple and intuitive path integral description of how the propagator for a charged Dirac particle is modified by soft self-energy radiative corrections as shown in reference [1]. The result is a self-similar (fractal) object with the non-locality one would expect for a particle carrying an infinite range field. Arguments are made for a similar, but qualitatively different, effect due to attractive self-interactions such as gravity and a calculation made for Newtonian gravity. The results are linked to the fractal dimensions of the paths that particles take in quantum field theory, and the effects of repulsive (QED) and attractive (gravity) self-interactions are discussed. Finally an attempt is made to estimate what effects would be expected in QCD, with a link made to confinement at a finite energy scale in terms of a fractal dimension which goes to zero.

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