

Notas e Discussões

## Exact solution for the nonlinear pendulum (Solução exata do pêndulo não linear)

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This paper deals with the nonlinear oscillation of a simple pendulum and presents not only the exact formula for the period but also the exact expression of the angular displacement as a function of the time, the amplitude of oscillations and the angular frequency for small oscillations. This angular displacement is written in terms of the Jacobi elliptic function  $sn(u;m)$  using the following initial conditions: the initial angular displacement is different from zero while the initial angular velocity is zero. The angular displacements are plotted using *Mathematica*, an available symbolic computer program that allows us to plot easily the function obtained. As we will see, even for amplitudes as high as  $0.75\pi$  ( $135^\circ$ ) it is possible to use the expression for the angular displacement, but considering the exact expression for the angular frequency  $\omega$  in terms of the complete elliptic integral of the first kind. We can conclude that for amplitudes lower than  $135^\circ$  the periodic motion exhibited by a simple pendulum is practically harmonic but its oscillations are not isochronous (the period is a function of the initial amplitude). We believe that present study may be a suitable and fruitful exercise for teaching and better understanding the behavior of the nonlinear pendulum in advanced undergraduate courses on classical mechanics.

**Keywords:** simple pendulum, large-angle period, angular displacement.

Este artigo aborda a oscilação não-linear de um pêndulo simples e apresenta não apenas a fórmula exata do período mas também a dependência temporal do deslocamento angular para amplitudes das oscilações e a frequência angular para pequenas oscilações. O deslocamento angular é escrito em termos da função elíptica de Jacobi  $sn(u;m)$  usando as seguintes condições iniciais: o deslocamento angular inicial é diferente de zero enquanto que a velocidade angular inicial é zero. Os deslocamentos angulares são plotados usando *Mathematica*, um disponível programa simbólico de computador que nos permite plotar facilmente a função obtida. Como veremos, mesmo para amplitudes tão grandes quanto  $0,75\pi$  ( $135^\circ$ ) é possível usar a expressão para o deslocamento angular mas considerando a expressão exata para a frequência angular  $w$  em termos da integral elíptica completa de primeira espécie. Concluímos que, para amplitudes menores que  $135^\circ$ , o movimento periódico exibido por um pêndulo simples é praticamente harmônico, mas suas oscilações não são isócronas (o período é uma função da amplitude inicial). Acreditamos que o presente estudo possa tornar-se um exercício conveniente e frutífero para o ensino e para uma melhor compreensão do pêndulo não-linear em cursos avançados de mecânica clássica na graduação.

**Palavras-chave:** pêndulo simples, período a grandes ângulos, deslocamento angular.

Perhaps one of the nonlinear systems most studied and analyzed is the simple pendulum [1-12], which is the most popular textbook example of a nonlinear system and is studied not only in advanced but also in introductory university courses of classical mechanics. The periodic motion exhibited by a simple pendulum is harmonic only for small angle oscillations [1]. Beyond this limit, the equation of motion is nonlinear: the simple harmonic motion is unsatisfactory to model the oscillation motion for large amplitudes and in such cases the period depends on amplitude. Application of Newton's second law to this physical system gives a dif-

ferential equation with a nonlinear term (the sine of an angle). It is possible to find the integral expression for the period of the pendulum and to express it in terms of elliptic functions. Although it is possible in many cases to replace the nonlinear differential equation by a corresponding linear differential equation that approximates the original equation, such linearization is not always feasible. In such cases, the actual nonlinear differential equation must be directly dealt with.

In this note we obtain analytical exact formulas for the period of a simple pendulum and we present some figures in which the angular displacement is plotted as

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a function of the time for different initial amplitudes. Moreover, we stress that one may show to the students the advantage of using available symbolic computer programs such as *Mathematica*.

As we can point out previously, one of the simplest nonlinear oscillating systems is the simple pendulum. This system consists of a particle of mass  $m$  attached to the end of a light inextensible rod, with the motion taking place in a vertical plane. The differential equation modelling the free undamped simple pendulum is

$$\frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta = 0, \quad (1)$$

where  $\theta$  is the angular displacement,  $t$  is the time and  $\omega_0$  is defined as

$$\omega_0 = \sqrt{\frac{g}{l}}. \quad (2)$$

Here  $l$  is the length of the pendulum and  $g$  is the acceleration due to gravity. Because of the presence of the trigonometric function  $\sin\theta$ , Eq. (1) is a nonlinear differential equation.

We consider that the oscillations of the pendulum are subjected to the initial conditions

$$\theta(0) = \theta_0 \quad \left(\frac{d\theta}{dt}\right)_{t=0} = 0 \quad (3)$$

where  $\theta_0$  is the amplitude of the oscillation. The system oscillates between symmetric limits  $[-\theta_0, +\theta_0]$ . The periodic solution  $\theta(t)$  of Eq. (1) and the angular frequency  $\omega$  (also with the period  $T = 2\pi/\omega$ ) depends on the amplitude  $\theta_0$ .

Equation (1), although straightforward in appearance, is in fact rather difficult to solve because of the nonlinearity of the term  $\sin\theta$ . In order to obtain the exact solution of Eq. (1), this equation is multiplied by  $d\theta/dt$ , so that it becomes

$$\frac{d\theta}{dt} \frac{d^2\theta}{dt^2} + \omega_0^2 \sin\theta \frac{d\theta}{dt} = 0 \quad (4)$$

which can be written as

$$\frac{d}{dt} \left[ \frac{1}{2} \left(\frac{d\theta}{dt}\right)^2 - \omega_0^2 \cos\theta \right] = 0 \quad (5)$$

Equation (5), which corresponds to the conservation of the mechanical energy, is immediately integrable, taking into account initial conditions in Eq. (3). From Eq. (5) we can obtain

$$\left(\frac{d\theta}{dt}\right)^2 = 2\omega_0^2 (\cos\theta - \cos\theta_0) \quad (6)$$

which can be written as follows

$$\left(\frac{d\theta}{dt}\right)^2 = 4\omega_0^2 \left[ \sin^2\left(\frac{\theta_0}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) \right], \quad (7)$$

where we use the trigonometric relation

$$\cos\theta = 1 - 2\sin^2\left(\frac{\theta}{2}\right). \quad (8)$$

Now let

$$y = \sin(\theta/2) \quad (9)$$

and

$$k = \sin^2(\theta_0/2). \quad (10)$$

From Eqs. (3), (9) and (10) we have

$$y(0) = \sqrt{k}. \quad (11)$$

It is easy to obtain the value of  $d\theta/dt$  as a function of  $dy/dt$  as follows. Firstly, from Eq. (10), we have

$$\frac{dy}{dt} = \frac{dy}{d\theta} \frac{d\theta}{dt} = \frac{1}{2} \frac{d\theta}{dt} \cos(\theta/2) \quad (12)$$

and secondly

$$\begin{aligned} \left(\frac{dy}{dt}\right)^2 &= \frac{1}{4} \cos^2\left(\frac{\theta}{2}\right) \left(\frac{d\theta}{dt}\right)^2 = \\ &= \frac{1}{4} [1 - \sin^2(\theta/2)] \left(\frac{d\theta}{dt}\right)^2 = \frac{1}{4} (1 - y^2) \left(\frac{d\theta}{dt}\right)^2. \end{aligned} \quad (13)$$

Then, we have

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{4}{1 - y^2} \left(\frac{dy}{dt}\right)^2. \quad (14)$$

Substituting Eqs. (9), (10) and (14) into Eq. (7), one get

$$\frac{4}{1 - y^2} \left(\frac{dy}{dt}\right)^2 = 4\omega_0^2 (k - y^2), \quad (15)$$

which can be rewritten as follows

$$\left(\frac{dy}{dt}\right)^2 = \omega_0^2 k (1 - y^2) \left(1 - \frac{y^2}{k}\right). \quad (16)$$

We do define new variables  $\tau$  and  $z$  as

$$\tau = \omega_0 t \quad \text{and} \quad z = \frac{y}{\sqrt{k}}. \quad (17)$$

Then Eq. (16) becomes

$$\left(\frac{dz}{d\tau}\right)^2 = (1 - z^2)(1 - kz^2), \quad (18)$$

where  $0 < k < 1$ , and

$$z(0) = 1 \quad \left(\frac{dz}{d\tau}\right)_{\tau=0} = 0. \quad (19)$$

Solving Eq. (18) for  $d\tau$  gives

$$d\tau = \pm \frac{dz}{\sqrt{(1 - z^2)(1 - kz^2)}}. \quad (20)$$

The time  $\tau$  to go from the point (1,0) to the point  $(z, dz/d\tau)$  in the lower half-plane of the graph of  $dz/d\tau$  as a function of  $z$  is

$$\tau = - \int_1^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k\zeta^2)}} \quad (21)$$

Equation (21) can be rewritten as follows

$$\tau = \int_0^1 \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k\zeta^2)}} - \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k\zeta^2)}}, \quad (22)$$

which allows us to obtain  $\tau$  as a function of  $z$  and  $k$  as

$$\tau(z) = K(k) - F(\arcsin z; k), \quad (23)$$

where  $K(m)$  and  $F(\varphi; m)$  are the complete and the incomplete elliptical integral of the first kind, defined as follows [13]

$$K(m) = \int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-mz^2)}} \quad (24)$$

$$F(\varphi; m) = \int_0^\varphi \frac{dz}{\sqrt{(1-z^2)(1-mz^2)}} \quad (25)$$

and  $z = \sin\varphi$ .

The period of oscillation  $T$  is four times the time taken by the pendulum to swing from  $\theta = 0$  ( $z = 0$ ) to  $\theta = \theta_0$  ( $z = 1$ ). Therefore

$$T = 4t(0) = \frac{4\tau(0)}{\omega_0} = \frac{4}{\omega_0} K(k) = \frac{2}{\pi} T_0 K(k), \quad (26)$$

where

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{l}{g}} \quad (27)$$

is the period of the pendulum for small oscillations. Equation (23) can be written as follows

$$F(\arcsin z; k) = K(k) - \tau \quad (28)$$

which can be written in terms of the Jacobi elliptic function  $sn(u; m)$  [14]

$$z = sn(K(k) - \tau; k). \quad (29)$$

In terms of the original variables and taking into account Eqs. (9), (10), (17) and (26), Eq. (29) becomes

$$\sin(\theta/2) = \sin \frac{\theta_0}{2} sn \left[ K \left( \sin^2 \frac{\theta_0}{2} \right) - \omega_0 t; \sin^2 \frac{\theta_0}{2} \right] \quad (30)$$

which allows us to express  $\theta$  as a function of  $t$  as

$$\theta(t) = 2 \arcsin \left\{ \sin \frac{\theta_0}{2} sn \left[ K \left( \sin^2 \frac{\theta_0}{2} \right) - \omega_0 t; \sin^2 \frac{\theta_0}{2} \right] \right\} \quad (31)$$

In Figs. 1-6 we have plotted the exact angular displacement  $\theta$  as a function of  $\omega_0 t$  (Eq. (31)) for different values of the initial amplitude  $\theta_0$ . Plots have been obtained using the *Mathematica* program. In these figures we have also included the angular displacement  $\theta(t) = 2 \arcsin \left\{ \sin \frac{\theta_0}{2} sn \left[ K \left( \sin^2 \frac{\theta_0}{2} \right) - \omega_0 t; \sin^2 \frac{\theta_0}{2} \right] \right\}$ . As we can see, for amplitudes  $\theta_0 < 0.75\pi$  ( $135^\circ$ ) it would be possible to use the approximate expression

$$\theta(t) = 2 \arcsin \left\{ \sin \frac{\theta_0}{2} sn \left[ K \left( \sin^2 \frac{\theta_0}{2} \right) - \omega_0 t; \sin^2 \frac{\theta_0}{2} \right] \right\} \quad (32)$$

for the angular displacement of the pendulum and Eq. (32) corresponds to a simple harmonic oscillator, but with the following exact expression for the angular frequency

$$\omega(\theta_0) = \frac{\pi\omega_0}{2K(\sin^2[\theta_0/2])}. \quad (33)$$

However, for larger initial amplitudes it is easy to see that the angular displacement does not correspond to a simple harmonic oscillator and the exact expression must be considered.

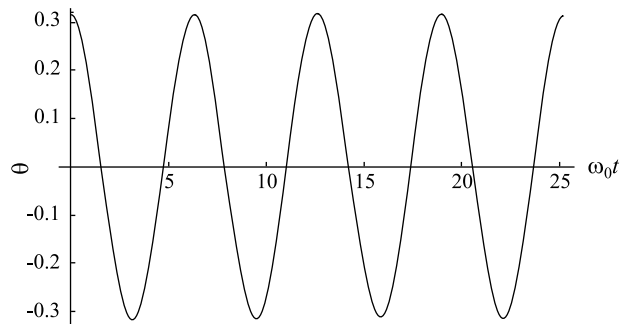


Figure 1 - Angular displacement  $\theta$  as a function of  $\omega_0 t$  for  $\theta_0 = 0.1\pi$ . Equation (31), bold line. Equation (32), thin line.

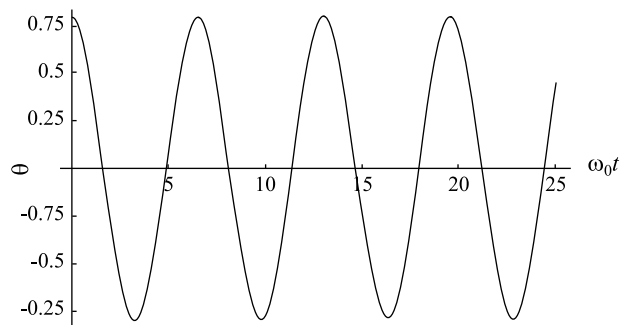


Figure 2 - Angular displacement  $\theta$  as a function of  $\omega_0 t$  for  $\theta_0 = 0.25\pi$ . Equation (31), bold line. Equation (32), thin line.

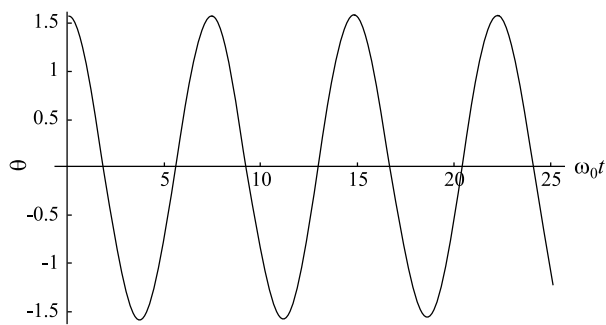


Figure 3 - Angular displacement  $\theta$  as a function of  $\omega_0 t$  for  $\theta_0 = 0.5\pi$ . Equation (31), bold line. Equation (32), thin line.

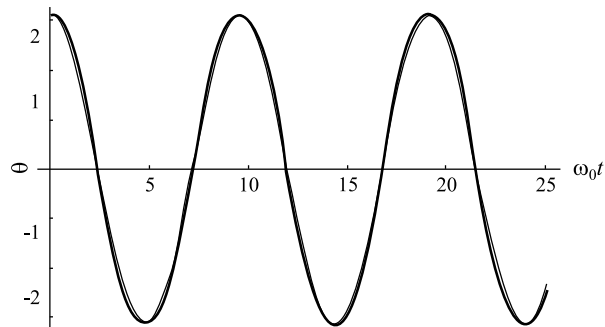


Figure 4 - Angular displacement  $\theta$  as a function of  $\omega_0 t$  for  $\theta_0 = 0.75\pi$ . Equation (31), bold line. Equation (32), thin line.

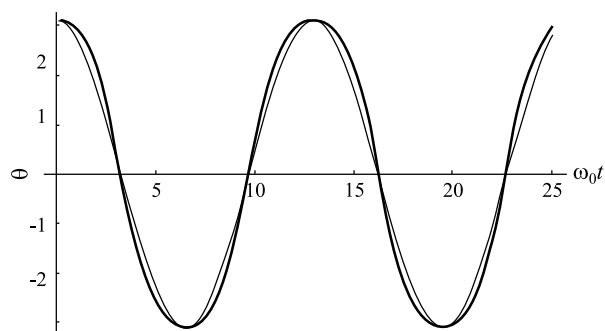


Figure 5 - Angular displacement  $\theta$  as a function of  $\omega_0 t$  for  $\theta_0 = 0.9\pi$ . Equation (31), bold line. Equation (32), thin line.

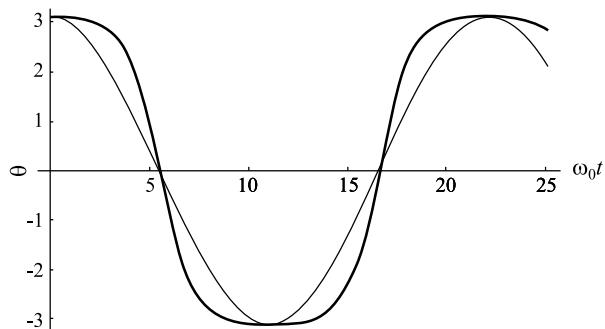


Figure 6 - Angular displacement  $\theta$  as a function of  $\omega_0 t$  for  $\theta_0 = 0.99\pi$ . Equation (31), bold line. Equation (32), thin line.

We can conclude that for amplitudes as high as  $135^\circ$  the effect of the nonlinearity is seen only in the fact that the angular frequency of the oscillation  $\omega$  depends on the amplitude  $\theta_0$  of the motion. For these amplitudes the harmonic function provides an excellent approximation to the periodic solution of Eq. (1) and the periodic motion exhibited by a simple pendulum is practically harmonic but its oscillations are not isochronous (the period is a function of the amplitude of oscillations). Finally, we point out that in the process of plotting the exact displacement one may show to the students the advantage of using available symbolic computer programs such as *Mathematica*.

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## References

- [1] J.B. Marion, *Classical Dynamics of Particles and Systems* (Harcourt Brace Jovanovich, San Diego, 1970).
- [2] L.P. Fulcher and B.F. Davis, *Am. J. Phys.* **44**, 51 (1976).
- [3] W.P. Ganley, *Am. J. Phys.* **53**, 73 (1985).
- [4] L.H. Cadwell and E.R. Boyco, *Am. J. Phys.* **59**, 979 (1991).
- [5] M.I. Molina, *Phys. Teach.* **35**, 489 (1997).
- [6] R.B. Kidd and S.L. Fogg, *Phys. Teach.* **40**, 81 (2002).
- [7] L.E. Millet, *Phys. Teach.* **41**, 162-3 (2003).
- [8] R.R. Parwani, *Eur. J. Phys.* **25**, 37 (2004).
- [9] G.E. Hite *Phys. Teach.* **43**, 290 (2005).
- [10] A. Beléndez, A. Hernández, A. Márquez, T. Beléndez and C. Neipp, *Eur. J. Phys.* **27**, 539 (2006).
- [11] A. Beléndez, A. Hernández, T. Beléndez, C. Neipp and A. Márquez, *Eur. J. Phys.* **28**, 93 (2007).
- [12] R.E. Mickens, *Oscillations in Planar Dynamics Systems* (World Scientific, Singapore, 1996).
- [13] L.M. Milne-Thomson, in M. Abramowitz and I.A. Stegun (eds) *Handbook of Mathematical Functions* (Dover Publications, Inc., Nova Iorque, 1972).
- [14] L.M. Milne-Thomson in M. Abramowitz and I.A. Stegun (eds) *Handbook of Mathematical Functions* (Dover Publications, Inc., Nova Iorque, 1972).