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Motion of a falling drop with accretion using Newtonian methods

(Estudio mediante métodos Newtonianos del movimiento de una gota que cae y cuya masa crece por acreción)

G. Hernandez[1], G. del Valle[1], I. Campos[2] y J.L. Jimenez[3]

[1] *Area de Física Atómica y Molecular Aplicada, División de Ciencias Básicas e Ingeniería, Universidad Autónoma Metropolitana, Azcapotzalco, D.F., México*

[1] *Departamento de Física, Facultad de Ciencias, Universidad Nacional Autónoma de México, D.F., México*

[1] *Departamento de Física, División de Ciencias Básicas e Ingeniería, Universidad Autónoma Metropolitana, Iztapalapa, D.F., México*

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The motion of a falling drop whose mass grows by accretion is studied with Newtonian methods to the point of finding the position as a function of time. The equation of motion applied is the equation of motion of continuum mechanics in its Eulerian or space formulation. We study three examples of laws of accretion: mass growing linearly with time, mass growing linearly with the surface of the drop and mass growing proportionally to the product of surface and velocity. These examples are sometimes left as exercises, without further discussion, asking only for $v(z)$, or the final velocity. We also show that the solutions have the correct limit of a particle of constant mass in free fall and of such a particle with friction linear in the velocity.
Keywords: variable mass systems, accretion.

Se estudia, mediante métodos Newtonianos, el movimiento de una gota que cae y cuya masa crece por acreción; se encuentra detalladamente la posición como función del tiempo. La ecuación de movimiento aplicada es la ecuación de movimiento de la mecánica de medios contínuos en su forma Euleriana o espacial. Estudiamos tres ejemplos de leyes de acreción: masa incrementándose linealmente con el tiempo, masa incrementándose propocionalmente a la superficie de la gota e incremento de la masa proporcional al producto de la superficie por la velocidad. También mostramos que las soluciones tienen el límite correcto, el de una partícula, con masa constante, en caída libre y el de esa partícula con fricción lineal en la velocidad.
Palabras-clave: sistemas de masa variable, acreción.

1. Introduction

The motion of systems with variable mass has conceptual and mathematical difficulties that make its treatment a challenge for teachers and students alike. The typical example is the rocket, discussed in many texts without beginning from an equation of motion, and rather applying conservation of momentum in a clever way. Other examples are the motion of a rope falling from a table, a conveyor on which sand is dropped, and a raindrop whose mass grows by accretion. In this work we solve in detail this last problem by Newtonian methods considering three specific laws of accretion. The relevance of this problem in several fields of science is pointed out by Krane [1]. The present work will be useful for those interested in conceptual problems in physics and specifically graduates and beginning graduate students, as well as for teachers, interested in the conceptual problems that variable mass systems exhibit.

Among the conceptual difficulties that these problems present is the equation of motion to be applied. Sometimes the equation used is $\frac{d\mathbf{p}}{dt} = \mathbf{F}$, as if it were Newton's second law, assuming now that $\mathbf{p} = m(t)\mathbf{v}$, but we must recall that this law applies to a constant mass particle on which only external forces act. Thus Tiersten [2] shows that this equation holds only because other terms of the general equation are zero. Also Krane [1] points out in a note that this equation is a particular case of a more general equation that we discuss here. However, by their very nature, variable mass systems are composed of many "particles" and the system is modeled as a continuum, where the "particle" is a small part of it on which now body forces as well as surface forces act. Tiersten [2] and Krane [1] have pointed out that there must be a more general equation of motion for dealing with variable mass systems, and we propose that such equation of motion is the equation of motion of continuum mechanics.

[1] E-mail: gpe@correo.azc.uam.mx.

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We find that there are two expressions for the generalization of Newton's law applicable to a continuum [2, 3-5]. One is the material or Lagrangian form

$$\rho\frac{d\mathbf{v}}{dt}=\rho\mathbf{b}+\nabla\cdot\overleftrightarrow{T}. \quad (1)$$

The other is the spatial or Eulerian form

$$\frac{\partial\left(\rho\mathbf{v}\right)}{\partial t}=\rho\mathbf{b}+\nabla\cdot\left(\overleftrightarrow{T}-\rho\mathbf{v}\mathbf{v}\right). \quad (2)$$

In these equations ρ is the mass density, $\mathbf{b}$ is the body force per unit mass, $\overleftrightarrow{T}$ is the stress tensor,and $\mathbf{v}$ is the velocity with respect to our reference frame. In the material or Lagrangian description, the system is a given material particle. Therefore the system has constant mass and this description is not appropriate to deal with variable mass system. In the spatial or Eulerian description the system is a particular volume of a continuum. Thus mass can enter or leave this system (sometimes called "control volume") and therefore this description is appropriate to deal with variable mass systems. By the way, the usual continuity equation for mass conservation, $\frac{\partial\rho}{\partial t}-\nabla\cdot(\rho\mathbf{v})\ =0$, is given in the Eulerian description. The stress tensor gives the force on the surface of a region of the continuum. This is the way Cauchy conceptualized the force which the rest of continuum exerts on a small part of it. The relation between both descriptions is given in the appendix .The first equation can be obtained from the second taking into account mass conservation. We propose that the equation of motion to be applied to variable mass systems is the Eulerian formulation since now the system is a particular volume, fixed or in motion with respect to our reference frame, in which mass may enter or leave carrying or not momentum. It is in this formulation that momentum flux must be considered.

We analyze three different laws of accretion: mass growing in proportion to time, mass growing in proportion to surface, equivalent to assuming that the radius of the drop grows linearly with time, and mass growing in proportion to the product of surface and velocity, equivalent to assuming that the radius of the drop grows proportionally to the distance travelled in falling. In all cases we find the correct limit of a constant mass particle freely falling.

2. Newtonian formulation

We solve the problem by Newtonian methods, which imply knowing all the forces acting on the system, and from these calculating the trajectory of the system in physical space. In the present case we find that this is the most difficult part of solving the problem, which explains why in texts it is asked usually to find only the velocity as function of height.

We use as equation of motion the volume integral of the Eulerian expression, which after a volume integration takes the form

$$\frac{d\left(m\mathbf{v}\right)}{dt}=\mathbf{F}-\mathbf{\Phi}+\oint_S\overleftrightarrow{T}\cdot\hat{n}dS, \quad (3)$$

where F is the body force, in our case gravity and friction, and $\mathbf{\Phi}$ is the momentum flux given by

$$\mathbf{\Phi}=\oint_S\rho\mathbf{v}\mathbf{v}\cdot\hat{n}dS. \quad (4)$$

In the case that the considered volume is in motion, like in the rocket, it is convenient to distinguish the relative velocity of the particles respect to the moving volume, so that $\mathbf{v}=\mathbf{v_r}+\mathbf{u}$ where $\mathbf{v_r}$ is the velocity of the particles with respect to the volume, and $\mathbf{u}$ is the velocity of the volume. Then the momentum flux $\mathbf{\Phi}$ is expressed as [4]

$$\mathbf{\Phi}=\oint_S\rho\mathbf{v}\mathbf{v_r}\cdot\hat{n}dS. \quad (5)$$

The last integral in Eq. (3) is the force that the surrounding medium exerts on the mass enclosed by the surface.

In our case the momentum flux is zero, since the velocity of the mass sticking to the drop is zero. Also, the surface integral of the stress, corresponding to surface tension, is zero, because of the spherical symmetry. The surface integral of the pressure gives the buoyancy force, that we discard assuming a small drop. Then our equation of motion is as a particular case of Eq. (3)

$$\frac{d\left(m\mathbf{v}\right)}{dt}=\mathbf{F}, \quad (6)$$

which can be written in the form

$$\frac{dv}{dt}+\frac{1}{m}\frac{dm}{dt}v=\frac{F}{m}. \quad (7)$$

Equation (6) seems the usual expression of Newton's second law, but it is not so. It is a particular case of Eq. (3) and has the same structure of the equation of motion of a particle subject to a friction linear in the velocity. In other cases, as in the rocket [6, 7] or in the rope falling of a table [8], the particular expression of this equation is different.

We need to specify the mass as function of time in order to have an equation of motion to solve. We study three usual cases, the mass being proportional to:

a) the time

b) the surface of the spherical drop, and

c) surface times the velocity

3. Accretion proportional to time

In this case we have

$$m\left(t\right)=m_o+bt, \quad (8)$$

where m_0 is the initial mass and b is a constant. If we take gravity as the only body force our equation is

$$\frac{dv}{dt}+\frac{bv}{m}=-g. \quad (9)$$

It is easy to take into account a friction force of the form

$$f = -kv, \tag{10}$$

since then our equation is of the same form

$$\frac{dv}{dt} + \frac{(b+k)v}{m} = -g. \tag{11}$$

A friction quadratic in v is more difficult to treat, since then we have a differential equation of Riccati's type. We consider only friction linear in the velocity.

Now we proceed to solve the differential Eq. (9) with the initial condition

$$v\,(t=0) = 0. \tag{12}$$

This equation is of type

$$\frac{dv}{dt} + P\,(t)\,v = Q\,(t)\,. \tag{13}$$

Then the solution can be obtained with an integrating factor of the form $e^{\int P(t)dt}$. That is

$$v(t) = e^{-\int P(t)dt} \int Q(t)e^{\int P(t)dt} + ce^{-\int P(t)dt}, \tag{14}$$

where $P(t) = \dfrac{(b+k)}{m\,(t)}$, $Q = -g$ and c is a constant determined by initial conditions.

The solution satisfying the initial condition $v(0) = 0$ is

$$v\,(t) = -\frac{g}{b}\frac{m}{(\lambda+1)} + \frac{g}{b}\frac{m_o^{\lambda+1}m^{-\lambda}}{(\lambda+1)}, \tag{15}$$

where

$$\lambda = 1 + \frac{k}{b}. \tag{16}$$

This solution, Eq. (15), if correct must contain the case of a constant mass particle in free fall as a limit when $k = 0$ and $b \to 0.$ It must contain also the case of a constant mass particle falling with friction linear in the velocity. Some authors [1, 9, 10] consider the limit $m_o \to 0$,which for the case without friction $(\lambda = 1)$ gives $v = \dfrac{-gt}{2}.$ As we show below, this limit gives in other laws of accretion $v = \dfrac{-gt}{n}$, with some integer (see section 4 below and [1]). This result seems strange and may be that the limit $m_o \to 0$ is rather formal, with uncertain physical meaning.

Now, to show that Eq. (15) has the correct limits we proceed by cases, first without friction $(k = 0, b \to 0)$ and then with friction $(k \neq 0, b \to 0)$.

The frictionless case is given by $k = 0$ or $\lambda = 1$,then

$$v\,(t) = -\frac{g}{2b}m\,(t) + \frac{g}{2b}\frac{m_o^2}{m\,(t)}. \tag{17}$$

It seems that this solution diverges for $b \to 0$ $(m(t) \to m_o)$, but writing

$$v\,(t) = -\frac{g}{2b}\,(m_o + bt) + \frac{gm_o}{2b}\left(1 + \frac{bt}{m_o}\right)^{-1}, \tag{18}$$

we see that with the binomial theorem we obtain the correct limit

$$\begin{aligned} v\,(t) &\to -\frac{g}{2b}\,(m_o + bt) + \frac{gm_o}{2b}\left(1 - \frac{bt}{m_o}\right) \\ &= -gt. \end{aligned} \tag{19}$$

The same result can be obtained by writing the solution Eq. (17) in the form

$$v\,(t) = -\frac{g}{2}\left(\frac{m - m_o^2/m}{b}\right), \tag{20}$$

and applying l'Hopital rule to this indeterminate limit, since $b \to 0$ implies $m \to m_o.$

Now for the case $k \neq 0, b \to 0$ we must proceed carefully, since $\lambda \to \infty.$

First we notice that $m(t)^{-\lambda}$ can be written as

$$m(t)^{-\lambda} = (m_o + bt)^{-\lambda} = m_o^{-\lambda}\left(1 + \frac{bt}{m_o}\right)^{-\lambda}. \tag{21}$$

With the change of variable

$$x = \frac{bt}{m_o}, \tag{22}$$

we obtain

$$\begin{aligned} m(t)^{-\lambda} &= m_o{}^{-\lambda}\,(1+x)^{-(1+k/b)} \\ &= m_o{}^{-\lambda}\,(1+x)^{-1}\,(1+x)^{-\frac{kt}{m_o x}} \\ &= m_o{}^{-\lambda}\,(1+x)^{-1}\left[(1+x)^{\frac{1}{x}}\right]^{-\frac{kt}{m_o}}. \end{aligned} \tag{23}$$

But

$$\lim_{x\to 0}\,(1+x)^{1/x} = e, \tag{24}$$

thus

$$\lim_{b\to 0} m(t)^{-\lambda} = \lim_{x\to 0} m(t)^{-\lambda} = m_o^{-\lambda}e^{-kt/m_o}. \tag{25}$$

With this result it is easy to see from Eq. (15), that

$$\lim_{b\to 0} v(t) = \lim_{b\to 0}\left\{-\frac{gm_o}{2b+k} + \frac{gm_o^{\lambda+1}m_o^{-\lambda}}{2b+k}\left(1+\frac{b}{m_o}t\right)^{-\lambda}\right\}, \tag{26}$$

because $b(\lambda+1) = 2b+k$, and using Eq. (24)

$$\lim_{b\to 0} v(t) = -\frac{gm_o}{k}\left(1-e^{-\frac{kt}{m_o}}\right), \tag{27}$$

which is the expected result.

A further integration of Eq. (15) gives the solution for the height of the center of mass of the drop, satisfying the initial condition $y(0) = h$ (Fig. 1).

$$y(t) = h - \frac{gm_o^2}{2b^2(-\lambda+1)} - \frac{gm^2}{2b^2(\lambda+1)} + \frac{gm_o^{\lambda+1}m^{-\lambda+1}}{b^2(-\lambda^2+1)}.$$

For the case without friction $(\lambda = 1)$ this result seems to diverge, but we can see that it is not so.

First we rewrite Eq. (28) as

$$y(t) = h - \frac{gm_o^2}{2b^2}\left(\frac{1-\frac{2m_o^{\lambda-1}m^{-\lambda+1}}{(\lambda+1)}}{(-\lambda+1)}\right) - \frac{gm^2}{2b^2(\lambda+1)}. \tag{28}$$

We define

$$\Delta_1 = 1 - \frac{\frac{2m_o^{\lambda-1}m^{-\lambda+1}}{(\lambda+1)}}{(-\lambda+1)}, \tag{29}$$

where it is necessary to take the limit $\lambda \to 0.$ Using l'Hopital rule again, we obtain

$$\lim_{x\to 1}\Delta_1 = \frac{2\frac{d}{d\lambda}\left(\frac{m_o^{\lambda-1}m^{-\lambda+1}}{(\lambda+1)}\right)}{\frac{d}{d\lambda}(-\lambda+1)}\rfloor_{\lambda=1} \tag{30}$$
$$= -2\frac{d}{d\lambda}\left(\frac{\left(\frac{m_o}{m}\right)^{\lambda-1}}{\lambda+1}\right)\rfloor_{\lambda=1}.$$

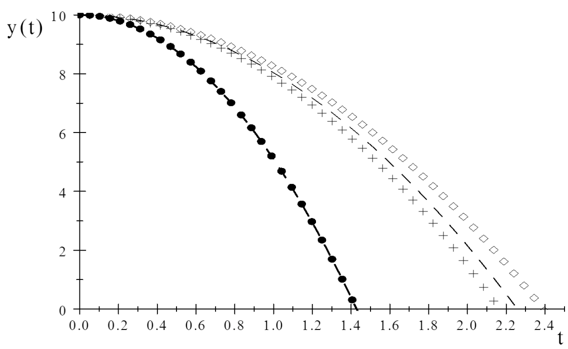

Figura 1 - Accretion proportional to time $m_o = 1\times 10^{-4}$, b = 9.1, free fall $(\cdot -)$, $\lambda = 1.3\,(\times\times)$, $\lambda = 1.5\,(--)$, $\lambda = 1.8\,(\diamond\diamond)$.

Using

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx},$$

we can write

$$\lim_{x\to 1}\Delta_1 = 2\left[\frac{(\lambda+1)\left(\frac{m_o}{m}\right)^{\lambda-1}\ln\left(\frac{m_o}{m}\right)+\left(\frac{m_o}{m}\right)^{\lambda-1}}{(\lambda+1)^2}\right]_{\lambda=1} \tag{31}$$
$$= \ln\left(\frac{m}{m_o}\right) - \frac{1}{2}.$$

On the other hand, if we define

$$\Delta_2 = \frac{gm^2}{2b^2(\lambda+1)}, \tag{32}$$

it is evident that

$$\lim_{\lambda\to 1}\Delta_2 = \frac{gm^2}{4b^2}. \tag{33}$$

Using Eqs. (32) and (34) we get

$$\lim_{\lambda\to 1} y(t) = h - \frac{gm_o^2}{2b^2}\left(\ln\left(\frac{m}{m_o}\right)-\frac{1}{2}\right) - \frac{gm^2}{4b^2} \tag{34}$$
$$= h - \frac{gm_o^2}{2b^2}\ln\left(\frac{m}{m_o}\right) - \frac{gm_o}{2b} - \frac{gt^2}{4}.$$

The particle of constant mass in free fall is obtained expanding $\ln\left(\frac{m}{m_o}\right)$, that is

$$\ln\left(\frac{m}{m_o}\right) = \ln\left(1+\frac{bt}{m_o}\right) \approx \frac{bt}{m_o} - \frac{1}{2}\left(\frac{bt}{m_o}\right)^2. \tag{35}$$

Then

$$y(t) = h - \frac{gm_o^2}{2b^2}\left(\frac{bt}{m_o} - \frac{1}{2}\left(\frac{bt}{m_o}\right)^2\right) - \frac{gm_o}{2b} - \frac{gt^2}{4} \tag{36}$$
$$= h - \frac{gt^2}{2}.$$

The expected result (Fig. 2).

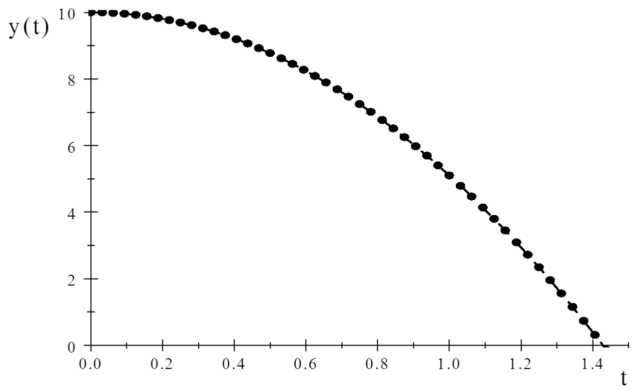

Figura 2 - Free fall.

4. Accretion proportional to the surface of the drop

Now we make the assumption that

$$\frac{dm}{dt} = \alpha 4\pi r^2. \tag{37}$$

Then the equation of motion for this case is

$$\frac{dv}{dt} + \frac{1}{m}\left(\alpha 4\pi r^2\right) v = -g. \tag{38}$$

The assumption, Eq. (38), is equivalent to the hypothesis that the radius of the drop grows linearly with time, since

$$m = \rho \frac{4\pi}{3} r^3, \tag{39}$$

and then

$$\frac{dm}{dt} = \rho 4\pi r^2 \frac{dr}{dt}. \tag{40}$$

Therefore $r\,(t) = r_o + \alpha t$ implies that $\frac{dr}{dt} = \alpha$, and we obtain the usual assumption that mass grows proportionally to the surface of the drop.

Our equation of motion is then

$$\frac{dv}{dt} + \frac{3v\alpha}{r_o + \alpha t} = -g. \tag{41}$$

This differential equation can be solved by the same method used for the case of accretion proportional to time, and the solution with the initial condition $v(0) = 0$ is

$$v = -\frac{g}{4\alpha}\left(r - \frac{r_o^4}{r^3}\right). \tag{42}$$

In the formal limit $m_o \to 0$, or $r_o \to 0$, $r = \alpha t$ and then we obtain

$$v\,(t) = \frac{gt}{4}; \tag{43}$$

as we mentioned previously, it is a strange result.

With the binomial theorem we can write this solution as

$$v = -\frac{g}{4\alpha}\left[r_o + \alpha t - r_o\left(1 - \frac{3\alpha t}{r_o} + \cdots\right)\right], \tag{44}$$

obtaining the free fall case, $v = -gt$, as $\alpha \to 0$.

An integration of Eq. (44), with the initial condition $y(0) = h$, gives (Fig. 3)

$$y\,(t) = h - \frac{g}{8\alpha^2}\left[(r_o + \alpha t)^2 + \frac{r_o^4}{(r_o + \alpha t)^2} - 2r_o^2\right]. \tag{45}$$

Again, an expansion with the binomial theorem to second order in t shows that we can obtain the free fall case as $\alpha \to 0$.

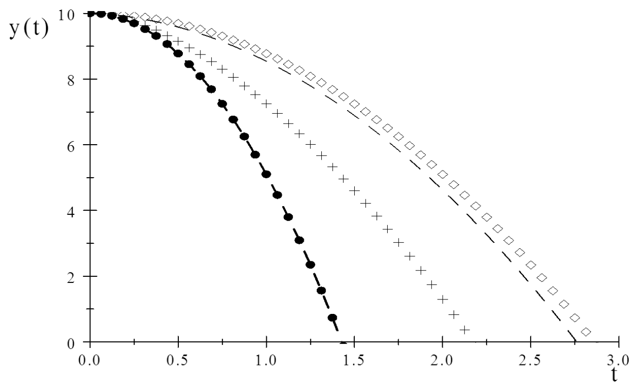

Figura 3 - Accretion proportional to the surface of the drop: $r_o = 1 \times 10^{-4}$, free fall ($\cdot -$), $\alpha = 1 \times 10^{-4}$ ($\times\times$), $\alpha = 1 \times 10^{-3}$ ($--$), $\alpha = 0.4$ ($\diamond\diamond$).

5. Accretion proportional to the surface times the velocity

The assumption that

$$\frac{dm}{dt} = -\beta 4\pi r^2 v, \tag{46}$$

seems more natural if we notice that it is equivalent to assuming that

$$r\,(t) = r_o + \beta\,(h - y\,(t)). \tag{47}$$

That is, from $m\,(t) = m\,(r\,(y\,(t))) = \rho\frac{4\pi}{3}r^3$ we find that

$$\frac{dm}{dt} = \rho 4\pi r^2 \frac{dr}{dy}\frac{dy}{dt} = -\beta 4\pi r^2 v. \tag{48}$$

Therefore our equation of motion is now the non linear equation

$$\frac{dv}{dt} - \frac{3\beta v^2}{r_o + \beta\,(h - y\,(t))} = -g. \tag{49}$$

This equation can be transformed with the identity

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = -\beta v\frac{dv}{dr} = -\frac{1}{2}\beta\frac{d\left(v^2\right)}{dr}. \tag{50}$$

Then the equation of motion is

$$\frac{d(v^2)}{dr} + \frac{6v^2}{r} = \frac{2g}{\beta}. \tag{51}$$

Now we have a differential equation for v^2 of the same type we have solved before, and the solution, with the initial condition $v(0) = 0$, is

$$v^2 = \frac{2g}{7\beta}\left[r - \frac{r_o^7}{r^6}\right]. \tag{52}$$

We can show with the binomial theorem that this solution reduces to the free fall case as $\beta \to 0$.

The solution can be put in terms of y, obtaining

$$v^2 = \frac{2g}{7}\left[(h_o - y) - \frac{(\frac{r_o}{\beta})^7}{(h_o - y)^6}\right]. \tag{53}$$

Here h_0 is defined by

$$h_o = h + \frac{r_o}{\beta}. \tag{54}$$

Then it is obvious that

$$\frac{dy}{dt} = v = -\sqrt{\frac{2g}{7}}\frac{\sqrt{(h_o - y)^7 - (\frac{r_o}{\beta})^7}}{(h_o - y)^3}, \tag{55}$$

and in order to obtain $y(t)$ we have to solve the integral

$$t = -\sqrt{\frac{7}{2g}}\int_h^y \frac{(h_o - \acute{y})^3\, d\acute{y}}{\sqrt{(h_o - \acute{y})^7 - (\frac{r_o}{\beta})^7}}. \tag{56}$$

From this equation it is easy to obtain the formal limit $r_o \to 0$, considered by some authors [11]. In this case $h_o = h$ and Eq. (56) becomes

$$t = -\sqrt{\frac{7}{2g}}\int_h^y (h - y)^{-\frac{1}{2}}\, d\acute{y}, \tag{57}$$

which after integration and some simplifications results

$$y = h - \frac{gt^2}{14}. \tag{58}$$

This result implies that

$$v = \frac{gt}{7}, \tag{59}$$

that as we mention before, is strange.

This integral is not immediate and it is convenient to define $z_o = \frac{r_o}{\beta}$ and $\acute{z} = h_o - y$, in order to transform the integral to the form

$$t = \sqrt{\frac{7}{2g}}\int_{z_o}^{z} \frac{\acute{z}^3 dz'}{\sqrt{\acute{z}^7 - z_o^7}}. \tag{60}$$

We need another change of variable

$$\theta = \frac{z^7}{z_o^7} - 1, \tag{61}$$

so that we have now the integral

$$t = \sqrt{\frac{7}{2g}}\frac{\sqrt{z_o}}{7}\int_0^\theta \acute{\theta}^{-\frac{1}{2}}\left(1 + \acute{\theta}\right)^{-\frac{3}{7}} d\acute{\theta}, \tag{62}$$

This integral is found in standard tables [12] and is given as

$$t = \sqrt{\frac{2z_o}{7g}}\,\theta^{\frac{1}{2}}\,{}_2F_1\left[\frac{3}{7}, \frac{1}{2}, \frac{3}{2}, -\theta\right], \tag{63}$$

where ${}_2F_1$ is the hypergeometric function.

In terms of the original variables we have

$$t = \sqrt{\frac{2\left(\frac{r_o}{\beta}\right)}{7g}}\left((h_o - y)^7\left(\frac{\beta}{r_o}\right)^7 - 1\right)^{\frac{1}{2}} \times$$
$${}_2F_1\left[\frac{3}{7}, \frac{1}{2}, \frac{3}{2}, -\left((h_o - y)^7\left(\frac{\beta}{r_o}\right)^7 - 1\right)\right], \tag{64}$$

that with a Taylor expansion around $y = h$ results in

$$t = -\sqrt{\frac{2}{g}}(h - y)^{\frac{1}{2}} + \frac{(h - y)^{\frac{3}{2}}}{\sqrt{2g}}\left(\frac{\beta}{r_o}\right) +$$
$$\frac{(h - y)^{\frac{5}{2}}}{\sqrt{32g}}\left(\frac{\beta}{r_o}\right)^2 + O(h - y)^{\frac{7}{2}}. \tag{65}$$

Finally, inverting this series we find

$$(h - y)^{\frac{1}{2}} = -\sqrt{\frac{g}{2}}t - \frac{g^{\frac{3}{2}}}{4\sqrt{2}}\left(\frac{\beta}{r_o}\right)t^3 -$$
$$\frac{7g^{\frac{5}{2}}}{32\sqrt{2}}\left(\frac{\beta}{r_o}\right)^2 t^5 + O(t)^7, \tag{66}$$

which after squaring gives $y(t)$ (Fig. 4) as

$$y(t) = h - \frac{g}{2}t^2 - \frac{g^2}{4}\left(\frac{\beta}{r_o}\right)t^4 + O\left(t^6\right). \tag{67}$$

This time it is obvious that we get the free fall case as $\beta \to 0$.

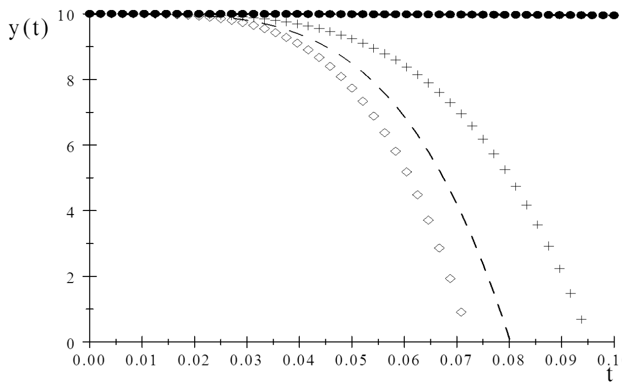

Figura 4 - Accretion proportional to the surface times the velocity: $r_o = 1 \times 10^{-4}$, free fall $(\cdot -)$, $\beta = 0.5$ $(\times\times)$, $\beta = 1.0\,(--)$, $\beta = 1.5\,(\diamond\diamond)$.

6. Conclusions

We have solved in a general way the problem of the motion of a falling drop whose mass grows by accretion according to a specific law of accretion. We have considered three specific laws of accretion, and have solved the problem by Newtonian methods. Then we had to apply a generalization of Newton's second law, which we took as the Eulerian formulation of the equation of motion for a continuum. Specifying clearly the hypothesis that lead to the particular equation of motion, these examples were solved to the point of getting the path of the center of mass of the falling drop, which is the aim of the Newtonian method. As a check of the solution obtained, we obtained in all cases the correct limit of a constant mass particle in free fall and that of a particle falling with friction linear in the velocity.

Appendix

The natural generalization of Newton´s second law for a continuum is given by the Lagrangian description, since a small part of this continuum, a "particle" of constant mass, is followed through its motion under the action of external forces, the body force and the surface force given by a surface integral of the stress tensor.

Then the equation of motion in this description is

$$\rho \frac{d\mathbf{v}}{dt} = \rho f_b + \nabla \cdot \overleftrightarrow{T},$$

where $\frac{d\mathbf{v}}{dt}$ is the total derivative, also called material derivative, given by

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \mathbf{v} \ .$$

Then

$$\rho \frac{d\mathbf{v}}{dt} = \rho \frac{\partial \mathbf{v}}{\partial t} + \rho\, (\mathbf{v} \cdot \nabla)\, \mathbf{v} \ .$$

The term $(\rho \mathbf{v} \cdot \nabla)\, \mathbf{v}$ can be developed with the aid of the tensor identity

$$\nabla \cdot (\rho \mathbf{v}\mathbf{v}) = (\rho \mathbf{v} \cdot \nabla)\, \mathbf{v} + \mathbf{v}\, (\nabla \cdot \rho \mathbf{v}) \, ,$$

and the continuity equation for conservation of mass in a given volumen,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0 \ .$$

Thus

$$(\rho \mathbf{v} \cdot \nabla)\, \mathbf{v} = \nabla \cdot (\rho \mathbf{v}\mathbf{v}) + \mathbf{v} \frac{\partial \rho}{\partial t} \ .$$

Then

$$\rho \frac{d\mathbf{v}}{dt} = \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) = \frac{\partial\, (\rho \mathbf{v})}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v}) \ .$$

Finally, the equation of motion for matter in a given volume is

$$\frac{\partial\, (\rho \mathbf{v})}{\partial t} = \rho \mathbf{f}_b + \nabla \cdot \left(\overleftrightarrow{T} - \rho \mathbf{v}\mathbf{v} \right),$$

which is the Eulerian description of motion, the last term representing the momentum flux.

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