Colors from polarizers and birefringent films
(Cores em polarizadores e filmes birrefringentes)

Fernando Fuzinatto Dall'Agnol[1] and Daniel den Engelsen

Centro de Tecnologia da Informação Renato Archer, Campinas, SP, Brasil

In this paper we analyze the interesting artwork of Austine W. Comarow that creates colored images from colorless birefringent films. This artwork consists of birefringent polymer films sandwiched between two polarizers, similar to a polariscope. We explain quantitatively the main features observed in Comarow's work. We derive the equations that predict the colors generated in the system as a function of the thickness and the angles between the films. We illustrate the phenomena with simple homemade devices.
Keywords: birefringence, retarders, polariscope, polarimeter, Polage Art®.

Neste artigo analisamos o interessante trabalho artístico de Austine W. Comarow que cria imagens coloridas a partir de filmes birrefringentes incolores. Este trabalho artístico consiste em sanduichar um filme polimérico birrefringente entre dois polarizadores, similarmente a um polariscópio. Explicamos quantitativamente as principais características observadas no trabalho de Comarow. Desenvolvemos as equações que prevêem as cores geradas no sistema como função da espessura e dos ângulos entre os filmes. Ilustramos o fenômeno com dispositivos simples montados por nós mesmos.
Palavras-chave: birrefringência, retardadores, polariscópio, polarímetro, Polage Art®.

1. Introduction

This paper is inspired by the art work of Austine W. Comarow. She created a technique called Polage Art® [1] that can be admired at the entrance hall of the Boston Museum of Science [2]; it is an unusual optical system that generates saturated colors from colorless materials. Basically it is a polarizer-birefringent-analyzer stack (PBAS), similar to a *polariscope* [3]. Figure 1 shows an image of the Brazilian flag using this technique. The flag appears in vivid colors only when the analyzer covers the birefringent layers. The dazzling properties of a PBAS are a challenge to undergraduate students to understand light propagation in birefringent media; a phenomenon of great importance due to many applications. Birefringent materials are used in many measuring instruments and devices, *e.g.* in LCDs to turn the pixels on and off [4]; in 3D movie cinemas, where it is used to generate circularly polarized light clockwise and counter clockwise to address the images to the right and left eyes [5]; it is explored for optical storage research [6] and to measure the thickness of thin films [7]. The polarization microscope, a special embodiment of a polariscope, may be used to visualize stress regions in plastics and glasses as well as birefringence in crystals [8].

This article is an extension of the work presented in Ref [9]. Here we present a simple algorithm to predict the colors that emerge from a PBAS as a function of parameters such as the thickness of the birefringent layer, the number of layers and the relative azimuth between the layers in the stack. These analyses unweave Comarow's artistic work by explaining the features observed. To explain the resulting colors perceived in the PBAS we use the CIE-31 diagram that will be reviewed briefly in this paper as well. Most figures in this article have meaningful colors. We recommend the reader to read this article in the online colored version.

2. Theoretical basis

Birefringence is a property of anisotropic materials that have more than one refractive index [10]. For our purpose here, we consider thin birefringent layers that are characterized by only two indexes of refraction n_o and n_e, as shown in Fig. 2. Birefringence is defined as $\Delta n = |n_o - n_e|$. If polarized light enters a birefringent material at an angle θ (defined in Fig. 2) it is decomposed into two waves which are called the ordinary wave and the extraordinary wave. These waves have planes of polarization perpendicular to one another. Since the

propagation of the ordinary and extra ordinary wave inside the birefringent material is different, a phase difference between these waves is gradually built up. If linearly polarized light impinges the birefringent layer, then one of the components (the ordinary wave or the extraordinary wave) will be retarded: for this reason birefringent layers are also called *retarders*. As a result, the light will become elliptically polarized as will be shown in more detail in the following section. The evolution of the electric field components of the light is dependent on the wavelength, although n_o and n_e are regarded as constants here. Finally, when the analyzer, being the second linear polarizer, covers the retarder at angle ϕ (defined in Fig. 2) the components of the electric field in direction **a** will be selected. Thus, the wavelength band that maximizes its electric field in the direction **a** will be selected, causing a color effect.
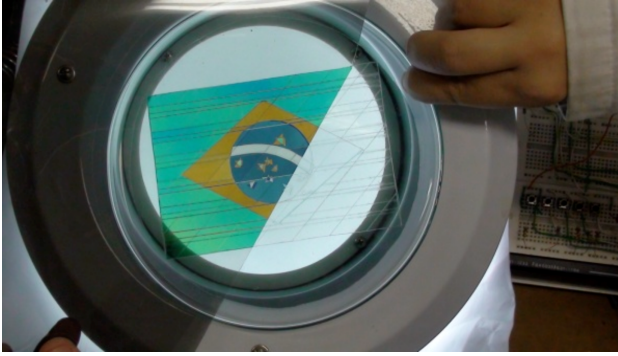

Figure 1 - Colors generated in a PBAS forming the image of the Brazilian flag. In the region where the colorless birefringent film is covered by the analyzer, the colors appear vividly. Elsewhere, the image is not seen.
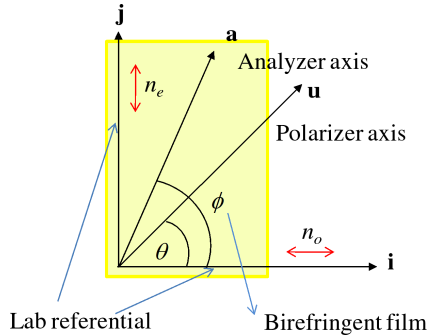

Figure 2 - Film with refractive indices n_o in the horizontal axis and n_e in the vertical axis.

We start deriving the equations to account for the electric field in a single birefringent layer. Then we shall generalize these to N birefringent layers.

2.1. Monochromatic light

The vector components of the electric field and their interaction with the polarizers and the retarder can be conveniently expressed in the frequently used Jones' matrix formalism [11]. However, the vector formalism

can also conveniently be applied for more than one retarder by deriving recurrent relations, as will be shown in section 4.

First we shall treat the simple case in which a beam of monochromatic light impinges on the PBAS. After passing the polarizer, the light is linearly polarized and it reaches the birefringent medium with field components given by

$$\mathbf{E}_o(0) = E_0 \cos\theta \ \mathbf{i}, \tag{1}$$
$$\mathbf{E}_e(0) = E_0 \sin\theta \ \mathbf{j}, \tag{2}$$

where E_0 is the norm of the electric field, the **i** unit vector is horizontal and the **j** is vertical as defined in Fig. 2. In the birefringent layer the field components will evolve differently and will reach the analyzer elliptically polarized with components

$$\mathbf{E}_o(z) = E_0 \cos\theta \exp(-ik_o z) \ \mathbf{i}, \tag{3}$$
$$\mathbf{E}_e(z) = E_0 \sin\theta \ \exp(-ik_e z)\mathbf{j}, \tag{4}$$

where $k_o = 2\pi n_o/\lambda_0$ is and $k_e = 2\pi n_e/\lambda_0$ are the wave numbers for the ordinary and extraordinary waves respectively, λ_0 is the wavelength in vacuum and z is the coordinate perpendicular to the plane of the film. The transmitted field is the projection of Eqs. (3) and (4) in the direction of the analyzer's transmittance, **a**, given by

$$\mathbf{E_a}(z) = ([\mathbf{E}_o(z) + \mathbf{E}_e(z)].\mathbf{a})\mathbf{a}. \tag{5}$$

where **a** is given by

$$\mathbf{a} = \mathbf{i}\cos\phi + \mathbf{j}\sin\phi. \tag{6}$$

The transmittance is given by

$$T_r = \frac{\mathbf{E_a^*}(z).\mathbf{E_a}(z)}{E_0^2}, \tag{7}$$

where $\mathbf{E}_a{}^*$ is the complex conjugated of $\mathbf{E}_a$ Replacing $\mathbf{E}_a$ by expression (5), $\mathbf{E}_o$ and $\mathbf{E}_e$ by expressions (3) and (4) into (7) Eq. (7), T_r can be written as

$$T_r = \sin^2\theta \sin^2\phi + \cos^2\theta \cos^2\phi +$$
$$\frac{1}{2}\sin 2\theta \sin 2\phi \cos(\Delta k z), \tag{8}$$

where $\Delta k = k_o - k_e = \Delta n/\lambda_0$ and Δn is the birefringence, defined above. A value of 5×10^{-3} will be used for Δn in the simulations. This birefringence value is typical for commercial retarder films like adhesive tapes.

The change from Eq. (7) to Eq. (8) is a straightforward algebraic simplification. This equation provides a good insight in the nature of the transmitted light as we

vary the angles θ or ϕ and the phase shift Δkz. It can be seen that T_r is a function of the wavelength: it changes periodically when the wavelength λ_0 is changed. Note that θ and ϕ are interchangeable in Eq. (8). This means that it doesn't matter what is the polarizer and what is the analyzer: the transmittance does not change if the PBAS is flipped upside down.

2.2.   Color coordinates

For creating color effects in a PBAS one needs white light. We shall simulate color effects in the following sections with a white light source that has an equal-energy spectrum, being an imaginary or artificial spectrum that provides the same radiant power at all wavelengths named "illuminant E0", thus representing a "full" spectrum. Beware not to confuse the illuminant class E0 with the norm of the electric field E_0 that appears in Eqs. (1) to (4). E0 is a constant radiance for all wavelengths. The value of this constant is chosen arbitrarily as 1 W/m^2 in this article.

The transmittance in Eq. (8) yields a color with coordinates (x,y) that can be obtained by calculating the functions X, Y and Z as follows.

$$X = \int_{380}^{780} \bar{x}(\lambda)I(\lambda)d\lambda, \qquad (9)$$

$$Y = \int_{380}^{780} \bar{y}(\lambda)I(\lambda)d\lambda, \qquad (10)$$

and

$$Z = \int_{380}^{780} \bar{z}(\lambda)I(\lambda)d\lambda, \qquad (11)$$

where the limits of the integrals are in nanometers (visible range) $I(\lambda)$ is the irradiance after passing PBAS. As we are assuming the uniform spectrum illuminant E0 then $I(\lambda)$ can be written as

$$I(\lambda) = Tr(\lambda)\text{E0}. \qquad (12)$$

The functions represented by Eqs. (9), (10) and (11) are the tristimulus values in the XYZ color space [12, 13], which was chosen in 1931 by the Commission Internationale d'Éclairage (CIE) to represent the color viewing capability of the standard human observer. The functions $\bar{x}$, $\bar{y}$ and $\bar{z}$ in these equations are the so-called *color matching functions* [14] that have been determined experimentally in the 1920s.

The more the spectrum of incoming light concentrates under the peak of $\bar{x}(\lambda)$, the more the color of the object will be perceived as red. Analogously, $\bar{y}$ is

for green and $\bar{z}$ for blue. These spectra are shown in Fig. 5. Note that the functions vanish at the limits of the visible range. Finally the color coordinates are obtained normalizing X, Y by the sum $X + Y + Z$.

$$x = \frac{X}{X + Y + Z}, \qquad (13)$$

$$y = \frac{Y}{X + Y + Z}. \qquad (14)$$

Colors are conveniently represented by the color coordinates x and y as shown in Fig. 4. In the subsequent sections we will represent the results of the simulations in the (x,y) chromaticity diagram.
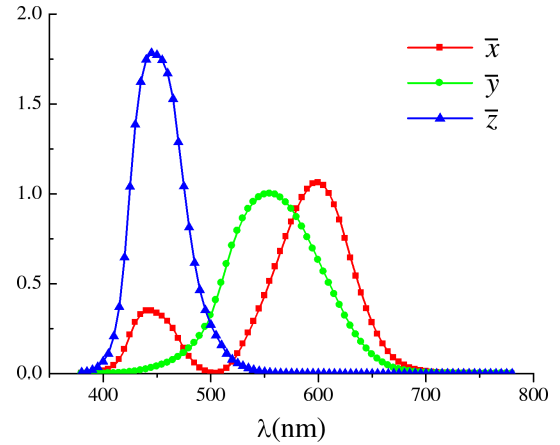

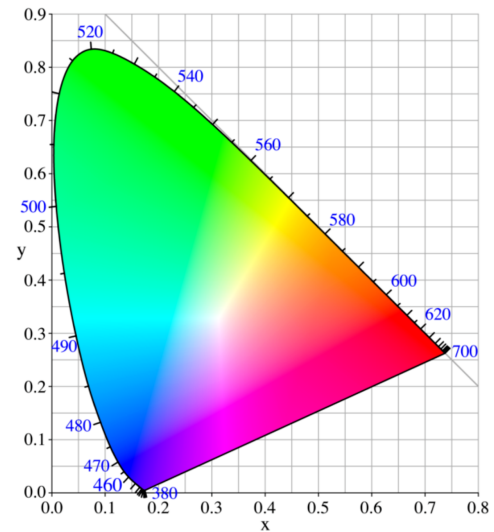Figure 3 - The color matching functions of the XYZ color space.

Figure 4 - CIE-31 chromaticity diagram. All visible colors are represented by the (x, y) coordinates. Monochromatic colors are represented by the line that envelopes the figure from 380 nm to 700 nm.

3.   One birefringent layer

3.1.   Varying the phase difference Δkz

Setting $\theta = \phi = 45°$ (aligned polarizers) and varying Δkz, the colors are generated with coordinates as shown

in Fig. 5. As the thickness of the birefringent layer z is increased, the phase difference in the PBAS changes and various colors can be made. The coordinates of those colors are represented in the 1931 CIE chromaticity diagram [12]. The R, G and B points are primaries measured in a typical LCD display.

The points L_1 and L_2 are close, which means that they have almost the same color. The only difference is their relative luminance, which differs in this particular case by a factor of 8. So, for some colors it is possible to choose a thickness that gives not only the desired color but also a more preferred luminance. Point L_2 is a little more to the periphery than L_1 implying that L_2 is more pure. However, in this example it is convenient to sacrifice the color purity to get greater luminance. This is a common tradeoff in color theory and applications.

In the intersection points, as point M, there are two different spectra with the same color coordinate. This propriety is called *metamerism*; different spectra that have the same color [15]. The two transmittance spectra at point M are shown in Fig. 6.
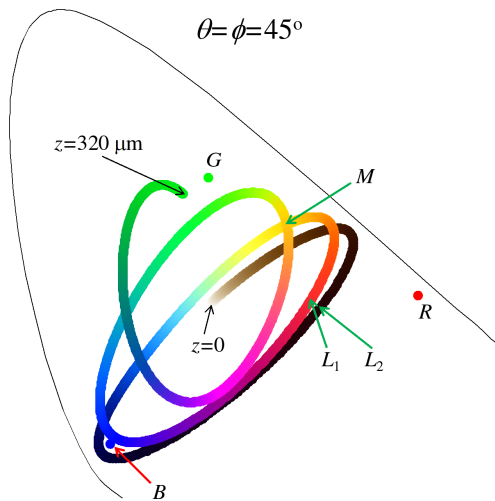

Figure 5 - Color path as one varies the thickness of the birefringent film from 0 to 320 μm. The neighboring points L_1 and L_2 have almost the same color although L_1 is more luminous.
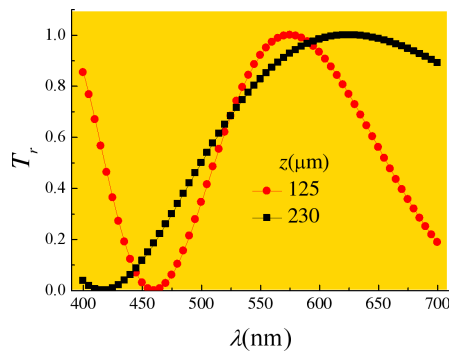

Figure 6 - Metamerism at M in Fig. 5. These two spectra are seen as the same color shown in the background of the graphics.

3.2. Varying θ (or ϕ)

Fig. 7 shows that complementary colors can be obtained by rotating only one of the polarizers in the PBAS. The colors swing between the complementary colors crossing the white point. The white is obtained whenever the polarizer is aligned with one of the axes of the retarder. In this figure the thicknesses to obtain the "best" green (or magenta), blue (or yellow) and red (or cyan) are indicated. What we regard as "best" color is a subjective balance between saturation and luminosity.
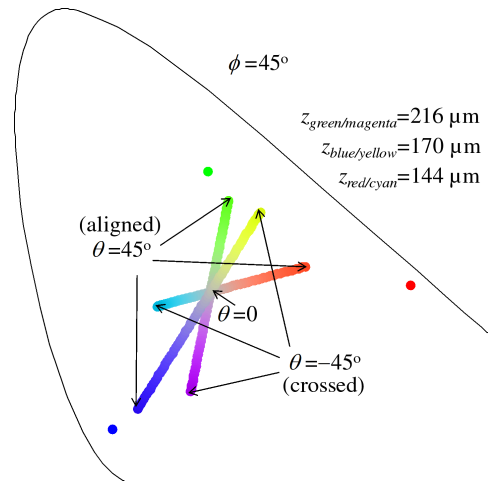

Figure 7 - By rotating one of the polarizers complementary colors can be obtained for a given thickness z.

The effect described above is illustrated in Fig. 8.

$\theta = \phi = 45°$ (aligned polarizers)
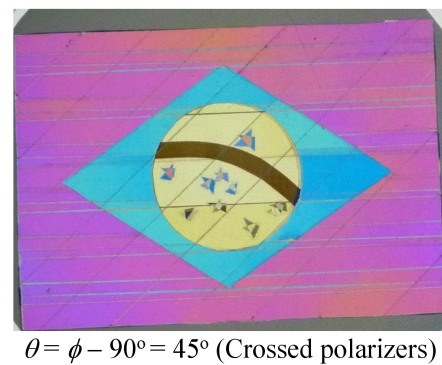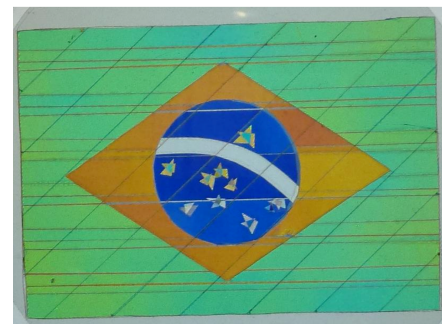

$\theta = \phi - 90° = 45°$ (Crossed polarizers)

Figure 8 - Complementary colors are transmitted if one of the polarizers is rotated 90° in a PBAS.

Note that for $\Delta n = 5 \times 10^{-3}$ chosen for these analyses, the colors dramatically change by varying the thickness: with 144 μm the PBAS is red (or cyan), with 170 μm it is already blue (or yellow) and with 216 μm it is green (or magenta). If Δn is too large then fluctuations in the thickness cause non-uniformities in the color. Some non-uniformity can be seen in Figs. 1, 8 and 12.

3.3. Rotating the birefringent layer

The third characteristic we like to describe is the effect of rotating the retarder. Let us consider the polarizer and analyzer being aligned ($\theta = \phi$) just as an example. Then, rotating the polarizers together is equivalent to rotate the retarder. In this case, the colors shift from saturated to neutral to saturated every 90° as shown in Fig. 9. The effect is similar to what happens when one of the polarizer is rotated. The difference is that the colors do not swing between their complementary pair.
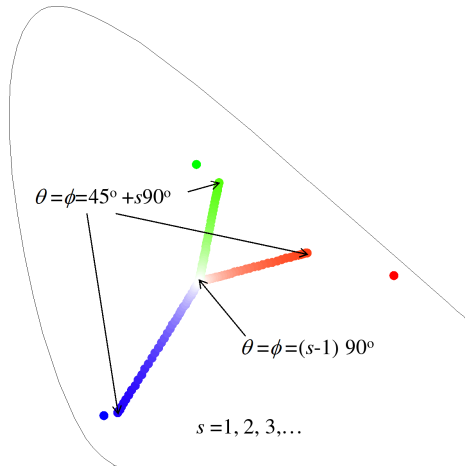

Figure 9 - Colors change periodically from saturated to neutral as the birefringent film is rotated in multiples of 90°.

4. N birefringent layers

To describe the behavior of N retarders ($N > 1$) analytically is rather complicated, while numerically it is simple. So, we are not deriving a function as in Eq. (8), instead, we derive a recurrent relation.

Consider Fig. 10, where the orientation of the mth retarder is indicated. The field components $\mathbf{E}o_m$ and $\mathbf{E}e_m$ entering the mth retarder are given by the projection of the out coming field from the $(m-1)^{th}$ retarder in the directions $\mathbf{i}_m$ and $\mathbf{j}_m$ respectively. Then the phase of each component evolves according to the refractive indices, resulting in the following recurrent relations for the out coming field from the mth retarder

$$\mathbf{E}o_m(z_m) = (\mathbf{E}o_{m-1}.\mathbf{i}_m)\exp(-ik_oz_m)\mathbf{i}_m, \qquad (15)$$

$$\mathbf{E}e_m(z_m) = (\mathbf{E}e_{m-1}.\mathbf{j}_m)\exp(-ik_ez_m)\mathbf{j}_m, \qquad (16)$$

where

$$\mathbf{i}_m = \mathbf{i}\cos\xi_m + \mathbf{j}\sin\xi_m, \qquad (17)$$

$$\mathbf{j}_m = -\mathbf{i}\sin\xi_m + \mathbf{j}\cos\xi_m, \qquad (18)$$

and z_n is the thickness of the mth retarder and ξ_m is the angle of the mth retarder ordinary axis with respect to the lab referential. The field that impinges the analyzer is the out coming field from the last retarder, of which the component in direction $\mathbf{a}$ is simply the projection

$$\mathbf{E_{a}}_N = [(\mathbf{E}o_N + \mathbf{E}e_N).\mathbf{a}]\mathbf{a}, \qquad (19)$$

with $\mathbf{a}$ given in Eq. (6). Finally the transmission is

$$T_{r_N} = \frac{\mathbf{E_{a}^*}_N.\mathbf{E_{a}}_N}{E_0^2}, \qquad (20)$$

where N is the number of birefringent layers. The starting condition for the field in Eqs. (15) and (16) are

$$\mathbf{E}o_0(0) = E_0\cos\theta\ \mathbf{i}, \qquad (21)$$

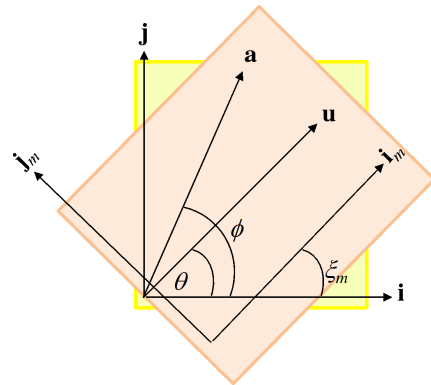$$\mathbf{E}e_0(0) = E_0\sin\theta\ \mathbf{j}. \qquad (22)$$

Figure 10 - Axis definitions on the mth retarder.

One can create many colors by changing the variables in Eq. (20). We shall elaborate a little bit on a feature that is predicted by Eq. (20) for $N = 2$: when the retarders have their axes fixed at 45° and they are rotated together, the color generated from the first retarder goes from saturated to neutral while the second is in counter phase going from neutral to saturated. Fig. 11 shows an example in which the color of the system swings between red and green when the polarizers are aligned or between cyan and magenta when the polarizers are crossed.
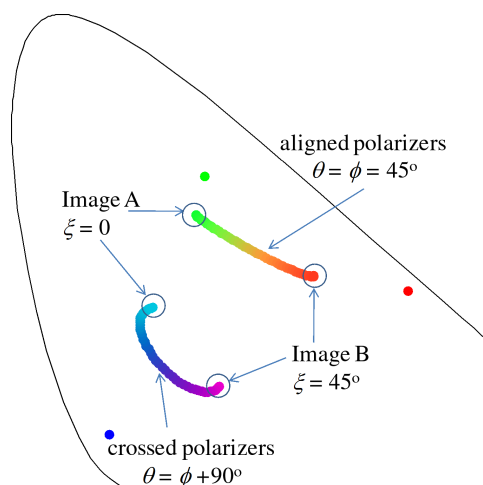
Figure 11 - Example of color change by rotating a stack of two retarders.

The applicability of this effect is illustrated in Fig. 12. This figure refers to a stack of two retarders. In the first we constructed the Brazilian flag and in the second there is the logo of CTI, our research center. The different colors are obtained by varying the thickness. For instance, the flag's blue is obtained with three layers of a stretched plastic tape; the yellow is obtained with a superposition of two layers and green with five layers. White is observed at positions where the retarder has been removed. Thus, each image is made of several retarder parts, but all parts have the ordinary axis in the same direction, *i.e.* $\mathbf{i}_1$ for image A (CTI logo) and $\mathbf{i}_2$ for image B (Brazilian flag). In Fig. 12, the images axes are fixed at $45°$ to each other, so the angle between $\mathbf{i}_1$ and $\mathbf{i}_2$ is $45°$. In other words $\xi_2 = \xi_1 + 45°$. Also, the polarizers are aligned, so $\theta = \phi = 0°$. When the CTI logo is at $\xi_2 = 45°$ with the polarizers it appears in saturated color, whereas the Brazilian flag is aligned to the polarizers ($\xi_1 = 0$); so it cannot be seen. When the retarders are rotated $45°$ clockwise, the CTI image's extraordinary axis (axis at direction $\mathbf{j}_2$) is aligned with the polarizers, so it cannot be seen, while the flag is at $\xi_1 = -45°$ and the images are switched. At intermediate angles both symbols can be partially seen. Watch the videos in Ref. [16] for more.

5. Conclusion

In this article we have described mathematically the beautiful artistic work of A. W. Comarow, who utilized the properties of a Polarizer-Birefringent-Analyzer-Stack (PBAS). We derived expressions for the transmittance in a PBAS as a function of the angles of the elements in the stack and the thicknesses of the retarders. We made a couple of PBAS using only a pair of scissors and polarizers for illustrating this article. With these PBAS, we reproduced the characteristics observed in Comarow's work such as the switching images effect.
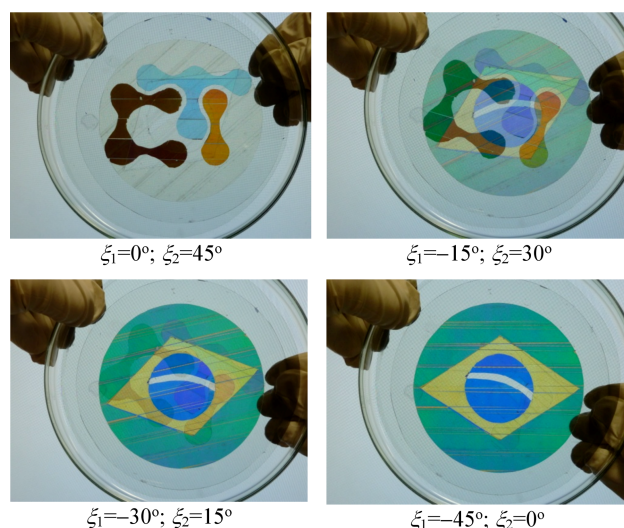

Figure 12 - Two birefringent layers, each one with a different image, are at $45°$ with respect to each other ($\xi_2=\xi_1+45^o$). As they are rotated together the images are switched.

There are still additional optical and mathematical analyses required to understand the scope of the generalized expression (20) for $N > 2$, which may reveal more interesting features in images made with a PBAS.

The visual appeal of the system is expected to challenge professors and students to comprehend the optical phenomena involved and stimulate them building their own PBAS-systems.

References

[1] http://www.youtube.com/watch?v=lavOfYw6y-g&NR=1.

[2] http://www.youtube.com/watch?v=UEU-aoFHIRk&feature=related.

[3] http://www.lexic.us/definition-of/polariscope.

[4] Jiun-Haw Lee, David N. Liu and Shin-TsonWu, *Introduction to Flat Panal Dispalay* (Wiley-SID Series in Display Technology, 2008), chapter 4.

[5] http://en.wikipedia.org/wiki/3-D_film.

[6] Z. Sekkat and W. Knoll, *Photoreactive Organic Thin Films* (Elsevier Publication, San Diego, 2002).

[7] D. den Engelsen, J. Opt. Soc. Am. **61**, 1460 (1971).

[8] http://www.physicstogo.org/features/featuresummary.cfm?FID=987; http://www.youtube.com/watch?v=GNPBSkPpXSE; http://www.meteorite-times.com/Back_Links/2009/may/Micro_Visions.htm.

[9] F.F. Dall'Agnol, *Proceedings of Latin Display* (2010).

[10] Eugene Hecht, *Optics* (Adisson Wesley, Reading, 2002), 4th ed., chapter 8, p. 325.

[11] http://en.wikipedia.org/wiki/Jones_calculus.

[12] Jiun-Haw Lee, David N. Liu and Shin-TsonWu, *Introduction to Flat Panal Dispalay* (Wiley-SID Series in Display Technology, 2008), chapter 2, p. 18.

[13] L.D. Silverstein and R.M. Marrifield, *The Development and Evaluation of Color Systems for Airborne Applications* ((SAE Technical Paper, 1985)) chapter 2, p. 9.

[14] CIE downloadable documents at `http://www.cie.co.at/main/freepubs.html`.

[15] Jiun-Haw Lee, David N. Liu and Shin-Tson Wu, *Introduction to Flat Panal Dispalay* (Wiley-SID Series in Display Technology, 2008), chapter 2, p. 27-28.

[16] `http://www.youtube.com/watch?v=-rSPqJqK5LY`.