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Periodic boundary value problems for impulsive neutral differential equations with multi-deviation arguments

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Abstract. We develop the impulsive inequality and the classical lower and upper solutions, and establish the comparison principles. By using these results and the monotone iterative technique, we obtain the existence of solutions of periodic boundary value problems for a class of impulsive neutral differential equations with multi-deviation arguments. An example is given to demonstrate our main results.

Mathematical subject classification: Primary: 34A37; Secondary: 34k10.

Key words: Impulsive neutral differential equation, periodic boundary value, impulsive inequality, lower and upper solutions, monotone iterative technique, extremal solution.

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1 Introduction

Impulsive differential equations have become more important in recent years in some mathematical models of real phenomena, especially in control, biological or medical domains (see, for example, [1-5]). As to periodic boundary value problems for impulsive differential equations, many authors have obtained

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excellent existence results; see, for instance, [7-11] for impulsive differential equations, [15-18] for impulsive neutral functional differential equations, [19] for abstract impulsive neutral functional differential equations. In [15-19], the characters of their neutral types are

$$\frac{d}{dt}[x(t) - g(t, x_t)], \quad \frac{d}{dt}(u(t) + g(t, u_t)), \quad \frac{d}{dt}(u(t) + F(t, u_t)),$$

$$\frac{d}{dt}\left[x(t) - g\left(t, x_t, \int_0^t a(t, s, x_s)ds\right)\right] \quad \text{and} \quad \frac{d^2}{dt^2}(x(t) - g(t, x_t)),$$

respectively. In this paper, however, the character of its neutral type is $(u(\theta(t)))'$. The character is different from the previous ones. Consider the following periodic boundary value problems for impulsive neutral differential equations with multi-deviation arguments of the form

$$\begin{cases} (u(\theta(t)))' = f(t, u(t), u(\varphi_1(t)), \dots, u(\varphi_q(t))), \\ \qquad\qquad t \in J = [0, T],\ t \neq \zeta_k, \\ \Delta u(t_k) = I_k(u(t_k)), \quad k = 1, \dots, p, \\ u(0) = u(T), \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < \cdots < t_p < t_{p+1} = T$; $\theta \in C^1(J, \mathbb{R})$, θ is monotone increasing with $0 \le \theta(t) \le t$ $(t \in J), \theta(0) = 0, \theta(T) = T$, and set $\theta(\zeta_k) = t_k$ $(k = 1, \dots, p)$, $J_0 = J\backslash\{t_1, \dots, t_p\}$, $J_1 = J\backslash\{\zeta_1, \dots, \zeta_p\}$; $f\colon J\times \mathbb{R}^{q+1} \to \mathbb{R}$ is continuous almost everywhere, and $\varphi_i\colon J \to \mathbb{R}$ continuous with $\varphi_i(J) \subseteq J$ $(i = 1, \dots, q)$; and $I_k \in C(\mathbb{R}, \mathbb{R})$, $\Delta u(t_k) = u(t_k^+) - u(t_k)$. Denote by $PC(X, Y)$, where $X \subset \mathbb{R}, Y \subset \mathbb{R}$, the set of all functions $u\colon X \to Y$ which are piecewise continuous in X with points of discontinuity of the first kind at the points $t_k \in X$, i.e., there exist the limits $u(t_k^+) < \infty$ and $u(t_k^-) = u(t_k) < \infty$. $PC^1(X, Y)$ denotes the set of all functions $u \in PC(X, Y)$, that are continuously differentiate for $t \in X$, $t \neq t_k$. Let $\Omega = PC([0, T], \mathbb{R}) \bigcap PC^1([0, T], \mathbb{R})$.

Definition. We say that the functions $\alpha, \beta \in \Omega$ are lower and upper solutions of (1.1), respectively, if there exist $M > 0$ and $0 \le L_k < 1$ such that

$$\begin{cases} (\alpha(\theta(t)))' \le f(t, \alpha(t), \alpha(\varphi_1(t)), \dots, \alpha(\varphi_q(t))) - a(t), \quad t \in J_1, \\ \Delta\alpha(t_k) \le I_k(\alpha(t_k)) - L_k a_k, \quad k = 1, \dots, p, \end{cases}$$

where

$$a(t)=\begin{cases}0, & \alpha(0)\le\alpha(T),\\ \frac{\theta'(t)+Mt+\sum_{i=1}^{q}N_i\varphi_i(t)}{T}(\alpha(0)-\alpha(T)), & \alpha(0)>\alpha(T),\end{cases}\tag{1.2}$$

$$a_k=\begin{cases}0, & \alpha(0)\le\alpha(T),\\ \frac{t_k}{T}(\alpha(0)-\alpha(T)), & \alpha(0)>\alpha(T),\end{cases}\tag{1.3}$$

and

$$\begin{cases}(\beta(\theta(t)))'\ge f(t,\beta(t),\beta(\varphi_1(t)),\ldots,\beta(\varphi_q(t)))+b(t), & t\in J_1,\\ \Delta\beta(t_k)\ge I_k(\beta(t_k))+L_kb_k, \quad k=1,\ldots,p,\end{cases}$$

where

$$b(t)=\begin{cases}0, & \beta(0)\ge\beta(T),\\ \frac{\theta'(t)+Mt+\sum_{i=1}^{q}N_i\varphi_i(t)}{T}(\beta(T)-\beta(0)), & \beta(0)<\beta(T),\end{cases}\tag{1.4}$$

$$b_k=\begin{cases}0, & \beta(0)\ge\beta(T),\\ \frac{t_k}{T}(\beta(T)-\beta(0)), & \beta(0)<\beta(T).\end{cases}\tag{1.5}$$

The definitions of classical lower and upper solutions make reference to the case $\alpha(0)\le\alpha(T)$ and $\beta(0)\ge\beta(T)$.

2 Preliminaries

Lemma 1. *Let $s\in[0,T]$, $c_k\ge 0$, $\alpha_k, k=1,\ldots,p$ be constants, $p,q\in PC(J,\mathbb{R})$, $x\in PC^1(J,\mathbb{R})$ and θ be set by* (1.1). *If*

$$\begin{cases}(x(\theta(t)))'\le p(t)x(\theta(t))+q(t), & t\in[s,T),\ t\ne\zeta_k,\\ x(t_k^+)\le c_kx(t_k)+\alpha_k, & t_k\in[s,T),\end{cases}$$

then for $t\in[s,T]$,

$$x(\theta(t)) \;\le\; x(\theta(s^+))\left(\prod_{s<\zeta_k<t}c_k\right)\exp\left(\int_s^t p(u)du\right)+\int_s^t\left(\prod_{u<\zeta_k<t}c_k\right)$$
$$\times\exp\left(\int_u^t p(\tau)d\tau\right)q(u)du+\sum_{s<\zeta_k<t}\left(\prod_{\zeta_k<\zeta_i<t}c_i\right)\exp\left(\int_{\zeta_k}^t p(\tau)d\tau\right)\alpha_k.$$

This proof is similar to the one of [1], here we omit it.

Lemma 2. *Let $u \in \Omega$, $M > 0$, $N_i \geq 0$ $(i = 1, \dots, q)$, $0 \leq L_k < 1$ and θ be set by* (1.1)*, such that*

(A1) $(u(\theta(t)))' + Mu(t) + \sum_{i=1}^{q} N_i u(\varphi_i(t)) \leq 0, \quad t \in J_1$;

(A2) $\Delta u(t_k) \leq -L_k u(t_k), \quad k = 1, \dots, p$;

(A3) $u(0) \leq u(T)$;

(A4) $\left(M + \sum_{i=1}^{q} N_i\right) \int_0^T \prod_{s<\zeta_k<T} (1 - L_k) ds \leq \prod_{k=1}^{p} (1 - L_k)^2$.

Then $u \leq 0$ on J.

Proof. By (A1) and (A2), we have

$$(u(\theta(t)))' \leq -Mu(t) - \sum_{i=1}^{q} N_i u(\varphi_i(t)), \quad t \in J_1, \tag{2.1}$$

$$\frac{du(\tau)}{d\tau} \cdot \frac{d\tau}{dt} \leq -Mu(t) - \sum_{i=1}^{q} N_i u(\varphi_i(t)), \quad t \in J_1,\ \tau = \theta(t), \tag{2.2}$$

$$u\left(t_k^+\right) \leq (1 - L_k) u(t_k), \quad k = 1, \dots, p. \tag{2.3}$$

To prove $u(t) \leq 0$ on J, we shall consider the following two cases.

Case 1. $u(t) \geq 0$ for all $t \in J$. In this case, by (2.2) and (2.3) and the properties of θ, we get $u'(t) \leq 0$ on J_0 and $u(t_k^+) \leq u(t_k)$, $k = 1, \dots, p$. Therefore $u(t)$ is a non-increasing function on J. Then $u(0) \geq u(T)$. Since $u(0) \leq u(T)$, $u(t) \equiv c$ on J (c is a non-negative constant), and $u' = 0$ on J_0. This and (A1) imply $\left(M + \sum_{i=1}^{q} N_i\right)c \leq 0$. Then $u \equiv 0$ on J.

Case 2. There exists $t^* \in J$ such that $u(t^*) > 0$ and $u(t)$ can take negative values in J. Let $\theta(\zeta^*) = t^*$. Again let $\bar{\zeta} = \min\{t \in J, \inf_{s \in J} u(\theta(s)) = u(\theta(t)) = -\lambda, \lambda > 0\}$. Since (A3), $\bar{\zeta} \in [0, T)$. Without loss of generality, let $\bar{\zeta} \neq \zeta_k, \zeta_k^+$, $k = 1, \dots, p$ (If $\bar{\zeta} = \zeta_k$ or ζ_k^+, the proof is similar, here we omit.). In this case, we consider two subcases.

Subcase 1. $u(T) > 0$. From (2.1), we have

$$(u(\theta(t)))' \leq \lambda \left(M + \sum_{i=1}^{q} N_i \right), \quad t \in J_1. \tag{2.4}$$

In view of (2.4), (2.3), and Lemma 1, we can get for $t \in [\bar{\zeta}, T]$,

$$\begin{aligned} u(\theta(t)) \; &\leq \; u(\theta(\bar{\zeta})) \prod_{\bar{\zeta}<\zeta_k<t} (1 - L_k) + \int_{\bar{\zeta}}^{t} \prod_{s<\zeta_k<t} (1 - L_k)\lambda \left(M + \sum_{i=1}^{q} N_i \right) ds \\ &= \; -\lambda \prod_{\bar{\zeta}<\zeta_k<t} (1 - L_k) + \lambda \left(M + \sum_{i=1}^{q} N_i \right) \int_{\bar{\zeta}}^{t} \prod_{s<\zeta_k<t} (1 - L_k) ds. \end{aligned}$$

Let $t = T$. Then we have

$$\begin{aligned} u(\theta(T)) &\leq -\lambda \prod_{\bar{\zeta}<\zeta_k<T} (1 - L_k) + \lambda \left(M + \sum_{i=1}^{q} N_i \right) \int_{\bar{\zeta}}^{T} \prod_{s<\zeta_k<T} (1 - L_k) ds, \\ u(T) &\leq \lambda \left[\left(M + \sum_{i=1}^{q} N_i \right) \int_{0}^{T} \prod_{s<\zeta_k<T} (1 - L_k) ds - \prod_{k=1}^{p} (1 - L_k) \right]. \end{aligned} \tag{2.5}$$

Since $u(T) > 0$, we get

$$\begin{aligned} \prod_{k=1}^{p} (1 - L_k) &< \left(M + \sum_{i=1}^{q} N_i \right) \int_{0}^{T} \prod_{s<\zeta_k<T} (1 - L_k) ds, \\ \prod_{k=1}^{p} (1 - L_k)^2 &< \left(M + \sum_{i=1}^{q} N_i \right) \int_{0}^{T} \prod_{s<\zeta_k<T} (1 - L_k) ds, \end{aligned}$$

which is contradictory to (A4).

Subcase 2. $u(T) \leq 0$. In this subcase, then $\bar{\zeta} < \zeta^*$ or $\bar{\zeta} > \zeta^*$.

(i) $\bar{\zeta} < \zeta^*$. According to the same arguments as (2.5), we get

$$\begin{aligned} u(\theta(\zeta^*)) \leq -\lambda \prod_{\bar{\zeta}<\zeta_k<\zeta^*} (1 - L_k) + \lambda &\left(M + \sum_{i=1}^{q} N_i \right) \\ &\times \int_{\bar{\zeta}}^{\zeta^*} \prod_{s<\zeta_k<\zeta^*} (1 - L_k) ds. \end{aligned}$$

Multiplying both sides of the above inequality by $\prod_{\zeta^* \le \zeta_k < T}(1 - L_k)$ (If $\zeta^* > \zeta_p$, this reduction is not needed.), we obtain

$$\begin{aligned}
&u(\theta(\zeta^*)) \prod_{\zeta^* \le \zeta_k < T}(1 - L_k) \\
&\quad \le -\lambda \prod_{\bar{\zeta} < \zeta_k < T}(1 - L_k) + \lambda\left(M + \sum_{i=1}^{q} N_i\right) \int_{\bar{\zeta}}^{\zeta^*} \prod_{s < \zeta_k < T}(1 - L_k) ds \\
&\quad \le -\lambda \prod_{k=1}^{p}(1 - L_k) + \lambda\left(M + \sum_{i=1}^{q} N_i\right) \int_{0}^{T} \prod_{s < \zeta_k < T}(1 - L_k) ds \\
&\quad = \lambda\left[\left(M + \sum_{i=1}^{q} N_i\right) \int_{0}^{T} \prod_{s < \zeta_k < T}(1 - L_k) ds - \prod_{k=1}^{p}(1 - L_k)\right].
\end{aligned}$$

Since $u(\theta(\zeta^*)) > 0$, we have

$$\prod_{k=1}^{p}(1 - L_k) < \left(M + \sum_{i=1}^{q} N_i\right) \int_{0}^{T} \prod_{s < \zeta_k < T}(1 - L_k) ds,$$

$$\prod_{k=1}^{p}(1 - L_k)^2 < \left(M + \sum_{i=1}^{q} N_i\right) \int_{0}^{T} \prod_{s < \zeta_k < T}(1 - L_k) ds,$$

which is contradictory to (A4).

(ii) $\bar{\zeta} > \zeta^*$. By (A3), we have $u(0) \le 0$. This and the properties of θ imply $0 < \zeta^*$. According to the same arguments as (2.5), we get

$$\begin{aligned}
u(\theta(\zeta^*)) &\le u(\theta(0)) \prod_{0 < \zeta_k < \zeta^*}(1 - L_k) \\
&\quad + \lambda\left(M + \sum_{i=1}^{q} N_i\right) \int_{0}^{\zeta^*} \prod_{s < \zeta_k < \zeta^*}(1 - L_k), \\
u(\theta(\zeta^*)) &\le u(T) \prod_{0 < \zeta_k < \zeta^*}(1 - L_k) \\
&\quad + \lambda\left(M + \sum_{i=1}^{q} N_i\right) \int_{0}^{\zeta^*} \prod_{s < \zeta_k < \zeta^*}(1 - L_k) ds.
\end{aligned}$$

Since $u(\theta(\zeta^*)) > 0$, we have

$$u(T) \prod_{0 < \zeta_k < \zeta^*}(1 - L_k) \ge -\lambda\left(M + \sum_{i=1}^{q} N_i\right) \int_{0}^{\zeta^*} \prod_{s < \zeta_k < \zeta^*}(1 - L_k) ds. \tag{2.6}$$

By (2.5) and (2.6), we obtain

$$\prod_{0<\zeta_k<\zeta^*}(1-L_k)\prod_{\bar{\zeta}<\zeta_k<T}(1-L_k) \le \left(M+\sum_{i=1}^{q} N_i\right)$$
$$\times\left[\int_0^{\zeta^*}\prod_{s<\zeta_k<\zeta^*}(1-L_k)ds + \prod_{0<\zeta_k<\zeta^*}(1-L_k)\int_{\bar{\zeta}}^{T}\prod_{s<\zeta_k<T}(1-L_k)ds\right].$$

Multiplying both sides of the above inequality by $\prod_{\zeta^*\le\zeta_k<T}(1-L_k)$ (If $\zeta^*>\zeta_p$, this reduction is not needed.), we get

$$\prod_{0<\zeta_k<T}(1-L_k)\prod_{\bar{\zeta}<\zeta_k<T}(1-L_k)$$
$$\le\left(M+\sum_{i=1}^{q} N_i\right)\left[\prod_{\zeta^*\le\zeta_k<T}(1-L_k)\int_0^{\zeta^*}\prod_{s<\zeta_k<\zeta^*}(1-L_k)ds\right.$$
$$\left.+\prod_{0<\zeta_k<T}(1-L_k)\int_{\bar{\zeta}}^{T}\prod_{s<\zeta_k<T}(1-L_k)ds\right],$$

$$\prod_{k=1}^{p}(1-L_k)\prod_{\bar{\zeta}<\zeta_k<T}(1-L_k)$$
$$\le\left(M+\sum_{i=1}^{q} N_i\right)\left[\int_0^{\zeta^*}\prod_{s<\zeta_k<T}(1-L_k)ds+\int_{\bar{\zeta}}^{T}\prod_{s<\zeta_k<T}(1-L_k)ds\right],$$

$$\prod_{k=1}^{p}(1-L_k)\prod_{\bar{\zeta}<\zeta_k<T}(1-L_k)<\left(M+\sum_{i=1}^{q} N_i\right)\int_0^{T}\prod_{s<\zeta_k<T}(1-L_k)ds,$$

$$\prod_{k=1}^{p}(1-L_k)^2<\left(M+\sum_{i=1}^{q} N_i\right)\int_0^{T}\prod_{s<\zeta_k<T}(1-L_k)ds,$$

which is contradictory to (A4).

Thus, in either case $u\le 0$ on J. Therefore the proof of the Lemma is complete.

Lemma 3. *Let $u\in\Omega$, $M>0$, $N_i\ge 0$ $(i=1,\dots,q)$, $0\le L_k<1$ and θ be set by* (1.1)*, such that*

(B1) $(u(\theta(t)))' + Mu(t) + \sum_{i=1}^{q} N_i u(\varphi_i(t)) + \frac{\theta'(t)+Mt+\sum_{i=1}^{q} N_i\varphi_i(t)}{T}[u(0) - u(T)] \leq 0, \ t \in J_1$;

(B2) $\Delta u(t_k) \leq -L_k u(t_k) - L_k \times \frac{t_k}{T}[u(0) - u(T)], \quad k = 1, \ldots, p$;

(B3) $u(0) > u(T)$.

Also assume that $(A4)$ *holds. Then* $u \leq 0$ *on* J.

Proof. Set $m(t) = u(t) + t/T \cdot [u(0) - u(T)]$. Clearly, $m(0) = m(T)$. It follows that for $t \in J_0$,

$$(m(\theta(t)))' + Mm(t) + \sum_{i=1}^{q} N_i m(\varphi_i(t)) = (u(\theta(t)))' + Mu(t)$$

$$+ \sum_{i=1}^{q} N_i u(\varphi_i(t)) + \frac{\theta'(t)+Mt+\sum_{i=1}^{q} N_i\varphi_i(t)}{T}[u(0) - u(T)] \leq 0,$$

and for $t = t_k$,

$$\Delta m(t_k) = \Delta u(t_k) \leq -L_k u(t_k) - L_k \times t_k/T \cdot [u(0) - u(T)] = -L_k m(t_k).$$

By Lemma 2, we obtain $m(t) \leq 0$ on J, and so $u(t) \leq 0$ on J. Thus, we have completed the proof of the Lemma.

3 Existence for linear problem

In this section, we consider the linear problem of (1.1)

$$\begin{cases} (u(\theta(t)))' + Mu(t) + \sum_{i=1}^{q} N_i u(\varphi_i(t)) = \sigma(t), & t \in J_1, \\ \Delta u(t_k) = -L_k u(t_k) + \gamma_k, & k = 1, \ldots, p, \\ u(0) = u(T), \end{cases} \tag{3.1}$$

where $\sigma(t) \in PC(J, \mathbb{R})$, $\gamma_k \in \mathbb{R}$, $k = 1, \ldots, p$, $M > 0$, $N_i \geq 0$ $(i = 1, \ldots, q)$, $0 \leq L_k < 1$ and θ is set by (1.1). For α, $\beta \in \Omega$, set $[\alpha, \beta] = \{u | \alpha(t) \leq u(t) \leq \beta(t), t \in J\}$.

Theorem 1. *Suppose that there exist* $\alpha, \beta \in \Omega$ *such that*

(C1) $\alpha \leq \beta$ *on* J;

(C2) $$(\alpha(\theta(t)))' + M\alpha(t) + \sum_{i=1}^{q} N_i\alpha(\varphi_i(t)) \leq \sigma(t) - a(t), \quad t \in J_1,$$
$$\Delta\alpha(t_k) \leq -L_k\alpha(t_k) + \gamma_k - a_k, \quad k = 1, \ldots, p;$$
$$(\beta(\theta(t)))' + M\beta(t) + \sum_{i=1}^{q} N_i\beta(\varphi_i(t)) \geq \sigma(t) + b(t), \quad t \in J_1,$$
$$\Delta\beta(t_k) \geq -L_k\beta(t_k) + \gamma_k + b_k, \quad k = 1, \ldots, p,$$

where $a(t), b(t), a_k, b_k$ *are defined by* (1.2)-(1.5). *Also assume that* $(A4)$ *holds. Then there exists a unique solution* u *for* (3.1) *with* $u \in [\alpha, \beta]$.

Proof. We shall prove the Theorem in the following three steps.

Step 1. If u_1, u_2 are solutions of (3.1), set $v_1 = u_1 - u_2$ and $v_2 = u_2 - u_1$, then

$$\begin{cases} (v_1(\theta(t)))' + Mv_1(t) + \sum_{i=1}^{q} N_i v_1(\varphi_i(t)) = 0, & t \in J_1, \\ \Delta v_1(t_k) = -L_k v_1(t_k), & k = 1, \ldots, p, \\ v_1(0) = v_1(T), \end{cases}$$

and

$$\begin{cases} (v_2(\theta(t)))' + Mv_2(t) + \sum_{i=1}^{q} N_i v_2(\varphi_i(t)) = 0, & t \in J_1, \\ \Delta v_2(t_k) = -L_k v_2(t_k), & k = 1, \ldots, p, \\ v_2(0) = v_2(T). \end{cases}$$

By Lemma 2, we obtain $v_1 = u_1 - u_2 \leq 0$ and $v_2 = u_2 - u_1 \leq 0$. Thus $u_1 = u_2$. Then there exists a unique solution u for (3.1).

Step 2. We prove that if ω, γ are classical lower and upper solutions, respectively, for (3.1) with $\omega \leq \gamma$, then (3.1) has a solution $u \in [\omega, \gamma]$.

Let $u(\cdot, a)$ denote the unique solution of the following equation

$$\begin{cases} (u(\theta(t)))' + Mu(t) + \sum_{i=1}^{q} N_i u(\varphi_i(t)) = \sigma(t), & t \in J_1, \\ \Delta u(t_k) = -L_k u(t_k) + \gamma_k, & k = 1, \ldots, p, \\ u(0) = a. \end{cases} \tag{3.2}$$

Firstly, we show $\omega(0) \leq u(T, \omega(0))$ and $\gamma(0) \geq u(T, \gamma(0))$. Assume $\omega(0) > u(T, \omega(0))$. Let $v(t) = \omega(t) - u(t, \omega(0))$. Then the function v satisfies

$$\begin{cases} (v(\theta(t)))' + Mv(t) + \sum_{i=1}^{q} N_i v(\varphi_i(t)) \leq 0, & t \in J_1, \\ \Delta v(t_k) = -L_k v(t_k), & k = 1, \ldots, p, \\ v(0) = \omega(0) - \omega(0) < \omega(T) - u(T, \omega(0)) = v(T). \end{cases}$$

By Lemma 2, we have $v(t) \leq 0$ on J. This implies $v(T) = \omega(T) - u(T, \omega(0)) \leq 0$. Thus $\omega(0) \leq \omega(T) \leq u(T, \omega(0))$, which is contradictory to the above assumption. Then $\omega(0) \leq u(T, \omega(0))$. Similarly, we have $\gamma(0) \geq u(T, \gamma(0))$.

Next, we prove that there exists $c \in [\omega(0), \gamma(0)]$ such that $u(0, c) = u(T, c)$. Now, we consider two cases.

Case 1. $\omega(0) = \gamma(0)$. In this case, we get $\omega(0) \leq u(T, \gamma(0)) \leq \gamma(0) = \omega(0)$. Thus $u(T, \gamma(0)) = \omega(0)$. Then we choose $c = \omega(0)$, and so $u = u(\cdot, c)$ is a solution of (3.1).

Case 2. $\omega(0) < \gamma(0)$. In this case, we define the map $F\colon [\omega(0), \gamma(0)] \to \mathbb{R}$ by $F(s) = s - u(T, s)$. Clearly F is continuous. Since $F(\omega(0)) \leq 0 \leq F(\gamma(0))$, there must exist one point $c \in [\omega(0), \gamma(0)]$ such that $F(c) = 0$. Then $u = u(\cdot, c)$ is a solution of (3.1).

Finally, we claim $u \in [\omega, \gamma]$. Let $m_1(t) = \omega(t) - u(t, c)$ and $m_2(t) = u(t, c) - \gamma(t)$. It is evident that $m_1, m_2 \in \Omega$, and

$$\begin{cases} (m_1(\theta(t)))' + Mm_1(t) + \sum_{i=1}^{q} N_i m_1(\varphi_i(t)) \leq 0, & t \in J_1, \\ \Delta m_1(t_k) \leq -L_k m_1(t_k), & k = 1, \ldots, p, \\ m_1(0) = \omega(0) - u(0, c) \leq \omega(T) - u(T, c) = m_1(T), \end{cases}$$

and

$$\begin{cases} (m_2(\theta(t)))' + Mm_2(t) + \sum_{i=1}^{q} N_i m_2(\varphi_i(t)) \leq 0, & t \in J_1, \\ \Delta m_2(t_k) \leq -L_k m_2(t_k), & k = 1, \ldots, p, \\ m_2(0) = u(0, c) - \gamma(0) \leq u(T, c) - \gamma(T) = m(T). \end{cases}$$

Using Lemma 2, we obtain $m_1 \leq 0$ and $m_2 \leq 0$ on J. Thus $\omega \leq u(\cdot, c) \leq \gamma$ on J.

Step 3. We prove that $\bar{\alpha}(t)$, $\bar{\beta}(t)$ are classical lower and upper solutions, respectively, for (3.1) with $\bar{\alpha} \le \bar{\beta}$, moreover $[\bar{\alpha}, \bar{\beta}] \subseteq [\alpha, \beta]$, where

$$\bar{\alpha}(t) = \begin{cases} \alpha(t), & \alpha(0) \le \alpha(T), \\ \alpha(t) + t/T \cdot [\alpha(0) - \alpha(T)], & \alpha(0) > \alpha(T), \end{cases} \tag{3.3}$$

and

$$\bar{\beta}(t) = \begin{cases} \beta(t), & \beta(0) \ge \beta(T), \\ \beta(t) - t/T \cdot [\beta(T) - \beta(0)], & \beta(0) < \beta(T). \end{cases} \tag{3.4}$$

It is evident that $\alpha \le \bar{\alpha}$ and $\bar{\beta} \le \beta$ on J. Thus $\bar{\alpha}(0) = \alpha(0) \le \bar{\alpha}(T)$ and $\bar{\beta}(0) = \beta(0) \ge \bar{\beta}(T)$.

(i) If $\alpha(0) \le \alpha(T)$, then

$$\begin{aligned}(\bar{\alpha}(\theta(t)))' + M\bar{\alpha} + \sum_{i=1}^{q} N_i \bar{\alpha}(\varphi_i(t)) &= (\alpha(\theta(t)))' \\ &\quad + M\alpha(t) + \sum_{i=1}^{q} N_i \alpha(\varphi_i(t)) \le \sigma(t), \ t \in J_1, \\ \Delta\bar{\alpha}(t_k) = \Delta\alpha(t_k) &\le -L_k\alpha(t_k) + \gamma_k - a_k \le -L_k\alpha(t_k) + \gamma_k - L_k a_k \\ &= -L_k[\alpha(t_k) + a_k] + \gamma_k = -L_k\bar{\alpha}(t_k) + \gamma_k, \quad k = 1, \ldots, p,\end{aligned}$$

and add to $\bar{\alpha}(0) \le \bar{\alpha}(T)$.

(ii) If $\alpha(0) > \alpha(T)$, then

$$\begin{aligned}(\bar{\alpha}(\theta(t)))' + M\bar{\alpha}(t) + \sum_{i=1}^{q} N_i \bar{\alpha}(\varphi_i(t)) &= (\alpha(\theta(t)))' + M\alpha(t) \\ &\quad + \sum_{i=1}^{q} N_i \alpha(\varphi_i(t)) + \frac{\theta'(t) + Mt + \sum_{i=1}^{q} N_i \varphi_i(t)}{T}(\alpha(0) - \alpha(T)) \\ &\quad \le \sigma(t), \ t \in J_1,\end{aligned}$$

$$\Delta\bar{\alpha}(t_k) = \Delta\alpha(t_k) \le -L_k\bar{\alpha}(t_k) + \gamma_k, \quad k = 1, \ldots, p,$$

and add to $\bar{\alpha}(0) \le \bar{\alpha}(T)$.

Thus, in either case, $\bar{\alpha}$ is a classical lower solution for (3.1). The same arguments show that $\bar{\beta}$ is a classical upper solution for (3.1).

Now, we consider the function $m = \bar{\alpha} - \bar{\beta}$.

$$
\begin{aligned}
&(m(\theta(t)))' + Mm(t) + \sum_{i=1}^{q} N_i m(\varphi_i(t)) \\
&\quad = \left[(\bar{\alpha}(\theta(t)))' + M\bar{\alpha}(t) + \sum_{i=1}^{q} N_i \bar{\alpha}(\varphi_i(t))\right] \\
&\qquad - \left[(\bar{\beta}(\theta(t)))' + M\bar{\beta}(t) + \sum_{i=1}^{q} N_i \bar{\beta}(\varphi_i(t))\right] \\
&\quad \le \sigma(t) - \sigma(t) = 0, \quad t \in J_1, \\
&\Delta m(t_k) = \Delta\bar{\alpha}(t_k) - \Delta\bar{\beta}(t_k) \le [-L_k\bar{\alpha}(t_k) + \gamma_k] - [-L_k\bar{\beta}(t_k) + \gamma_k] \\
&\qquad\quad = -L_k m(t_k), \quad k = 1, \ldots, p, \\
&m(0) = \bar{\alpha}(0) - \bar{\beta}(0) \le \bar{\alpha}(T) - \bar{\beta}(T) = m(T).
\end{aligned}
$$

Using Lemma 2, we get $m \le 0$ on J, i.e., $\bar{\alpha} \le \bar{\beta}$ on J.

Thus, we have completed the proof of Theorem 1.

4 Existence for nonlinear problem

In this section, we establish the existence criteria for solutions of (1.1) by the lower and upper solutions and the monotone iterative technique.

Theorem 2. *Suppose that there exist* $\alpha, \beta \in \Omega$ *such that*

(D1) α *and* β *are lower and upper solutions for* (1.1) *with* $\alpha \le \beta$;

(D2) $f(t, x_2, y_{12}, \ldots, y_{q2}) - f(t, x_1, y_{11}, \ldots, y_{q1}) \ge -M(x_2 - x_1) - \sum_{i=1}^{q} N_i(y_{i2} - y_{i1})$ *for every* $t \in J_1, \alpha \le x_1 \le x_2 \le \beta$, $\alpha(\varphi_i(t)) \le y_{i1}(\varphi_i(t)) \le y_{i2}(\varphi_i(t)) \le \beta(\varphi_i(t))$ $(i = 1, \ldots, q)$;

(D3) $I_k(x) - I_k(y) \ge -L_k(x - y)$ *for* $\alpha(t_k) \le y(t_k) \le x(t_k) \le \beta(t_k)$, $k = 1, \ldots, p$.

Also assume that $(A4)$ *holds. Then there exist monotone sequence* $\{\bar{\alpha}_n(t)\}$, $\{\bar{\beta}_n(t)\}$ *with* $\bar{\alpha}_0 = \bar{\alpha}, \bar{\beta}_0 = \bar{\beta}$*, where* $\bar{\alpha}, \bar{\beta}$ *are defined by* (3.3) *and* (3.4)*, such that* $\lim_{n\to\infty} \bar{\alpha}_n(t) = \rho(t)$ *and* $\lim_{n\to\infty} \bar{\beta}_n(t) = \psi(t)$ *uniformly hold on* J*, where* $\rho(t), \psi(t)$ *are minimal and maximal solutions of* (1.1)*, respectively.*

Proof. We shall prove the Theorem in the following three steps.

Step 1. It is evident that $\alpha \le \bar{\alpha}$ and $\beta \le \bar{\beta}$ on J. Thus $\alpha(0) = \bar{\alpha}(0) \le \bar{\alpha}(T)$ and $\bar{\beta}(0) = \beta(0) \ge \bar{\beta}(T)$.

Let the function $m = \bar{\alpha} - \bar{\beta}$, then $m(0) = \bar{\alpha}(0) - \bar{\beta}(0) \le \bar{\alpha}(T) - \bar{\beta}(T) = m(T)$. Next, we consider two cases.

Case 1. $\alpha(0) > \alpha(T)$ and $\beta(0) < \beta(T)$.

Firstly, by (D2), we get

$$\begin{aligned}
&(m(\theta(t)))' + Mm(t) + \sum_{i=1}^{q} N_i m(\varphi_i(t)) \\
&= \left[(\alpha(\theta(t)))' + M\alpha(t) + \sum_{i=1}^{q} N_i \alpha(\varphi_i(t)) + \frac{\theta'(t)+Mt+\sum_{i=1}^{q} N_i\varphi_i(t)}{T}(\alpha(0)-\alpha(T)) \right] \\
&\quad - \left[(\beta(\theta(t)))' + M\beta(t) + \sum_{i=1}^{q} N_i \beta(\varphi_i(t)) - \frac{\theta'(t)+Mt+\sum_{i=1}^{q} N_i\varphi_i(t)}{T}(\beta(T)-\beta(0)) \right] \\
&\le \left[f(t,\alpha(t),\alpha(\varphi_1(t)),\ldots,\alpha(\varphi_q(t))) - f(t,\beta(t),\beta(\varphi_1(t)),\ldots,\beta(\varphi_q(t))) \right] \\
&\quad - \left[M(\beta(t)-\alpha(t)) + \sum_{i=1}^{q} N_i(\beta(\varphi_i(t)) - \alpha(\varphi_i(t))) \right] \\
&\le 0, \ \ t \in J_1.
\end{aligned}$$

Again, by (D3), we obtain

$$\begin{aligned}
\Delta m(t_k) &= \Delta\bar{\alpha}(t_k) - \Delta\bar{\beta}(t_k) = \Delta\alpha(t_k) - \Delta\beta(t_k) \\
&\le [I_k(\alpha(t_k)) - L_k a_k] - [I_k(\beta(t_k)) + L_k b_k] \\
&\le -L_k[\alpha(t_k) - \beta(t_k)] - L_k a_k - L_k b_k \le -L_k m(t_k), \quad k = 1, \ldots, p.
\end{aligned}$$

Finally, add to $m(0) \le m(T)$. Using Lemma 2, $m(t) \le 0$ on J, i.e., $\bar{\alpha} \le \bar{\beta}$ on J.

It follows that

$$\begin{aligned}
(\bar{\alpha}(\theta(t)))' &= (\alpha(\theta(t)))' + \tfrac{\theta'(t)}{T}[\alpha(0)-\alpha(T)] \\
&\le f(t,\alpha(t),\alpha(\varphi_1(t)),\ldots,\alpha(\varphi_q(t))) - \frac{Mt+\sum_{i=1}^{q} N_i\varphi_i(t)}{T}[\alpha(0)-\alpha(T)].
\end{aligned}$$

Since $\alpha \leq \bar{\alpha} \leq \beta$, by (D2), we get

$$f(t, \bar{\alpha}(t), \bar{\alpha}(\varphi_1(t)), \ldots, \bar{\alpha}(\varphi_q(t))) - f(t, \alpha(t), \alpha(\varphi_1(t)), \ldots, \alpha(\varphi_q(t)))$$
$$\geq -M[\bar{\alpha}(t) - \alpha(t)] - \sum_{i=1}^{q} N_i[\bar{\alpha}(\varphi_i(t)) - \alpha(\varphi_i(t))].$$

Then

$$\begin{aligned}
(\bar{\alpha}(\theta(t)))' &\leq f(t, \bar{\alpha}(t), \bar{\alpha}(\varphi_1(t)), \ldots, \bar{\alpha}(\varphi_q(t))) + M[\bar{\alpha}(t) - \alpha(t)] \\
&\quad + \sum_{i=1}^{q} N_i[\bar{\alpha}(\varphi_i(t)) - \alpha(\varphi_i(t))] - \frac{Mt + \sum_{i=1}^{q} N_i\varphi_i(t)}{T}[\alpha(0) - \alpha(T)] \\
&= f(t, \bar{\alpha}(t), \bar{\alpha}(\varphi_1(t)), \ldots, \bar{\alpha}(\varphi_q(t))) \\
&\quad + \frac{Mt + \sum_{i=1}^{q} N_i\varphi_i(t)}{T}[\alpha(0) - \alpha(T)] - \frac{Mt + \sum_{i=1}^{q} N_i\varphi_i(t)}{T}[\alpha(0) - \alpha(T)] \\
&= f(t, \bar{\alpha}(t), \bar{\alpha}(\varphi_1(t)), \ldots, \bar{\alpha}(\varphi_q(t))).
\end{aligned} \tag{4.1}$$

From (D3), we get

$$\begin{aligned}
\Delta\bar{\alpha}(t_k) = \; &\Delta\alpha(t_k) \leq I_k(\alpha(t_k)) - \tfrac{L_k t_k}{T}[\alpha(0) - \alpha(T)] \\
&\leq I_k(\bar{\alpha}(t_k)) + L_k[\bar{\alpha}(t_k) - \alpha(t_k)] - \tfrac{L_k t_k}{T}[\alpha(0) - \alpha(T)] \\
= \; &I_k(\bar{\alpha}(t_k)).
\end{aligned} \tag{4.2}$$

Case 2. $\alpha(0) \leq \alpha(T)$ and $\beta(0) \geq \beta(T)$. In this case, it is trivial that we get (4.1) and (4.2).

Thus, in either case, $\bar{\alpha}$ is a classical lower solution. Similarly, $\bar{\beta}$ is a classical upper solution. Moreover $[\bar{\alpha}, \bar{\beta}] \subseteq [\alpha, \beta]$.

Step 2. For any $\eta \in [\bar{\alpha}, \bar{\beta}]$, we consider

$$\begin{cases}
(u(\theta(t)))' + Mu(t) + \sum_{i=1}^{q} N_i u(\varphi_i(t)) = M\eta(t) + \sum_{i=1}^{q} N_i\eta(\varphi_i(t)) \\
\quad + f(t, \eta(t), \eta(\varphi_1(t)), \ldots, \eta(\varphi_q(t))), \quad t \in J_1, \\
\Delta u(t_k) + L_k u(t_k) = I_k(\eta(t_k)) + L_k\eta(t_k), \quad k = 1, \ldots, p, \\
u(0) = u(T).
\end{cases} \tag{4.3}$$

Then, by Theorem 1, (4.3) has a unique solution $u \in \Omega$.

Define operator A by $u = A\eta$. Then A possesses the following properties:

(E1) $\bar{\alpha} \leq A\bar{\alpha}$, $\bar{\beta} \geq A\bar{\beta}$;

(E2) $A\eta_1 \leq A\eta_2$ for $\eta_1, \eta_2 \in [\bar{\alpha}, \bar{\beta}]$ with $\eta_1 \leq \eta_2$.

Firstly, let $m = \bar{\alpha} - \bar{\alpha}_1$, where $\bar{\alpha}_1 = A\bar{\alpha}$. Then we get

$$\begin{aligned}
(m(\theta(t)))' &= (\bar{\alpha}(\theta(t)))' - (\bar{\alpha}_1(\theta(t)))' \\
&\leq f(t, \bar{\alpha}(t), \bar{\alpha}(\varphi_1(t)), \ldots, \bar{\alpha}(\varphi_q(t))) + M\bar{\alpha}_1(t) + \sum_{i=1}^{q} N_i \bar{\alpha}_1(\varphi_i(t)) \\
&\quad - M\bar{\alpha}(t) - \sum_{i=1}^{q} N_i \bar{\alpha}(\varphi_i(t)) - f(t, \bar{\alpha}(t), \bar{\alpha}(\varphi_1(t)), \ldots, \bar{\alpha}(\varphi_q(t))) \\
&= -M[\bar{\alpha}(t) - \bar{\alpha}_1(t)] - \sum_{i=1}^{q} N_i [\bar{\alpha}_1(\varphi_i(t)) - \bar{\alpha}(\varphi_i(t))] \\
&= -Mm(t) - \sum_{i=1}^{q} N_i m(\varphi_i(t)), \ \ t \in J_1, \\
\Delta m(t_k) &= \Delta\bar{\alpha}(t_k) - \Delta\bar{\alpha}_1(t_k) \\
&\leq I_k(\bar{\alpha}(t_k)) - I_k(\bar{\alpha}(t_k)) - L_k\bar{\alpha}(t_k) + L_k\bar{\alpha}_1(t_k) \\
&= -L_k m(t_k), \ k = 1, \ldots, p, \\
m(0) &= \bar{\alpha}(0) - \bar{\alpha}_1(0) \leq \bar{\alpha}(T) - \bar{\alpha}_1(T) = m(T).
\end{aligned}$$

By Lemma 2, we have $m(t) \leq 0$ on J, i.e., $\bar{\alpha} \leq A\bar{\alpha}$. Similarly, we get $\bar{\beta} \geq A\bar{\beta}$.

Next, set $v_1 = A\eta_1$ and $v_2 = A\eta_2$, where $\eta_1, \eta_2 \in [\bar{\alpha}, \bar{\beta}]$ with $\eta_1 \leq \eta_2$. Let $m = v_1 - v_2$. By (D2), (D3) and (4.3), we get

$$\begin{aligned}
(m(\theta(t)))' &= (v_1(\theta(t)))' - (v_2(\theta(t)))' \\
&= \Big[- Mv_1(t) - \sum_{i=1}^{q} N_i v_1(\varphi_i(t)) + f(t, \eta_1(t), \eta_1(\varphi_1(t)), \ldots, \eta_1(\varphi_q(t))) \\
&\quad + M\eta_1(t) + \sum_{i=1}^{q} N_i \eta_1(\varphi_i(t)) \Big] - \Big[- Mv_2(t) - \sum_{i=1}^{q} N_i v_2(\varphi_i(t)) \\
&\quad + f(t, \eta_2(t), \eta_2(\varphi_1(t)), \ldots, \eta_2(\varphi_q(t))) + M\eta_2(t) + \sum_{i=1}^{q} N_i \eta_2(\varphi_i(t)) \Big] \\
&= -Mm(t) - \sum_{i=1}^{q} N_i m(\varphi_i(t)) + [f(t, \eta_1(t), \eta_1(\varphi_1(t)), \ldots, \eta_1(\varphi_q(t))) \\
&\quad - f(t, \eta_2(t), \eta_2(\varphi_1(t)), \ldots, \eta_2(\varphi_q(t)))] \\
&\quad - \Big[M(\eta_2(t) - \eta_1(t)) + \sum_{i=1}^{q} N_i (\eta_2(\varphi_i(t)) - \eta_1(\varphi_i(t))) \Big]
\end{aligned}$$

$$\begin{aligned}
&\le -Mm(t) - \sum_{i=1}^{q} N_i m(\varphi_i(t)), \quad t \in J_1,\\
\Delta m(t_k) &= \Delta v_1(t_k) - \Delta v_2(t_k)\\
&= [-L_k v_1(t_k) + I_k(\eta_1(t_k)) + L_k \eta_1(t_k)]\\
&\quad - [-L_k v_2(t_k) + I_k(\eta_2(t_k)) + L_k \eta_2(t_k)]\\
&\le -L_k m(t_k), \quad k = 1, \dots, p,\\
m(0) &= m(T).
\end{aligned}$$

By Lemma 2, we get $m(t) \le 0$ on J, i.e., $v_1 \le v_2$ on J. Then $A\eta_1 \le A\eta_2$ for $\eta_1, \eta_2 \in [\bar{\alpha}, \bar{\beta}]$ with $\eta_1 \le \eta_2$.

Step 3. Define the sequence $\{\bar{\alpha}_n(t)\}$, $\{\bar{\beta}_n(t)\}$ by $\bar{\alpha}_{n+1} = A\bar{\alpha}_n$, $\bar{\beta}_{n+1} = A\bar{\beta}_n$, $\bar{\alpha}_0 = \bar{\alpha}$, $\bar{\beta}_0 = \bar{\beta}$. From (E1) and (E2), we get

$$\bar{\alpha}_0 \le \bar{\alpha}_1 \le \ldots \le \bar{\alpha}_n \le \bar{\beta}_n \le \ldots \le \bar{\beta}_1 = \bar{\beta}_0, \quad \forall n \in N.$$

Thus it is immediate to verify that

$$\lim_{n\to\infty} \bar{\alpha}_n(t) = \rho(t) \quad \text{and} \quad \lim_{n\to\infty} \bar{\beta}_n(t) = \psi(t)$$

uniformly hold on J.

We consider the equation

$$\begin{cases}
(\bar{\alpha}_{n+1}(\theta(t)))' + M\bar{\alpha}_{n+1}(t) + \sum_{i=1}^{q} N_i \bar{\alpha}_{n+1}(\varphi_i(t)) = M\bar{\alpha}_n(t)\\
\quad + \sum_{i=1}^{q} N_i \bar{\alpha}_n(\varphi_i(t)) + f(t, \bar{\alpha}_n(t), \bar{\alpha}_n(\varphi_1(t)), \dots, \bar{\alpha}_n(\varphi_q(t))),\ t \in J_1,\\
\Delta\bar{\alpha}_{n+1}(t_k) + L_k \bar{\alpha}_{n+1}(t_k) = I_k(\bar{\alpha}_n(t_k)) + L_k \bar{\alpha}_n(t_k), \quad k = 1, \dots, p,\\
\bar{\alpha}_{n+1}(0) = \bar{\alpha}_{n+1}(T),
\end{cases}$$

and pass to the limit when n tends to ∞. Thus we obtain that ρ is a solution of (1.1). Analogously, ψ is also a solution of (1.1).

Finally, let u be any solution of (1.1) on $[\bar{\alpha}, \bar{\beta}]$. Clearly $\bar{\alpha}_0 \le u$. Assume $\bar{\alpha}_n \le u$. We get that $\bar{\alpha}_{n+1} \le u$ by considering the function $m = u - \bar{\alpha}_{n+1}$ and using Lemma 3 again. Then by passing to the limit, we conclude $\rho \le u$ on J. Similarly, $u \le \psi$ on J. Then $\rho(t)$, $\psi(t)$ are minimal and maximal solutions of (1.1), respectively. Thus, we have completed the proof of Theorem 2.

5 An example

Now we consider the equation

$$\begin{cases} (u(t^2))' = -\frac{1}{24}u^2(t) - \frac{1}{36}\left[u\left(\frac{1}{3}t\right) + u(\sqrt{t}) + u\left(\frac{2}{3}t\right)\right] \\ \qquad\qquad + \frac{1}{24}t, \ \ t \in [0,1], t \neq \zeta_1 = \frac{1}{2}, \\ \Delta u(t_1) = -\frac{1}{2}u(t_1), \quad t_1 = \frac{1}{4}, \\ u(0) = u(1). \end{cases} \tag{5.1}$$

Firstly, it is obvious that $\alpha = 0$ is a classical lower solution for (5.1). Certainly $\alpha = 0$ is a lower solution. Similarly $\beta = 1$ is an upper solution. Moreover $\alpha \leq \beta$ on $J = [0, 1]$. Next,

$$\begin{aligned} &f(t, x_2, y_{12}, y_{22}, y_{32}) - f(t, x_1, y_{11}, y_{21}, y_{31}) \\ &= -\tfrac{1}{24}(x_2^2 - x_1^2) - \tfrac{1}{36}\sum_{i=1}^{3}(y_{i2} - y_{i1}) \\ &= -\tfrac{1}{24}(x_2 + x_1)(x_2 - x_1) - \tfrac{1}{36}\sum_{i=1}^{3}(y_{i2} - y_{i1}) \\ &\geq -\tfrac{1}{12}(x_2 - x_1) - \tfrac{1}{36}\sum_{i=1}^{3}(y_{i2} - y_{i1}), \end{aligned}$$

for $\alpha \leq x_1 \leq x_2 \leq \beta$, $\alpha\left(\frac{1}{3}t\right) \leq y_{11}\left(\frac{1}{3}t\right) \leq y_{12}\left(\frac{1}{3}t\right) \leq \beta\left(\frac{1}{3}t\right)$, $\alpha\left(\sqrt{t}\right) \leq y_{21}(\sqrt{t}) \leq y_{22}(\sqrt{t}) \leq \beta(\sqrt{t})$, $\alpha\left(\frac{2}{3}t\right) \leq y_{31}\left(\frac{2}{3}t\right) \leq y_{32}\left(\frac{2}{3}t\right) \leq \beta\left(\frac{2}{3}t\right)$, where $M = \frac{1}{12}$, $N_1 = N_2 = N_3 = \frac{1}{36}$. Further, $I_1(x) - I_1(y) = -\frac{1}{2}x + \frac{1}{2}y \geq -\frac{1}{2}(x - y)$, for $\alpha\left(\frac{1}{4}\right) \leq y\left(\frac{1}{4}\right) \leq x\left(\frac{1}{4}\right) \leq \beta\left(\frac{1}{4}\right)$, where $0 \leq L_1 = \frac{1}{2} < 1$. Finally,

$$\begin{aligned} \left(M + \sum_{i=1}^{3} N_i\right)\int_0^T \prod_{s<\zeta_k<T}(1 - L_k)ds &= \frac{1}{6}\int_0^1 \prod_{s<\zeta_k<T}(1 - L_k)ds \\ = \frac{1}{6}\left[\int_0^{1/2} \frac{1}{2}ds + \int_{1/2}^{1} ds\right] &= \frac{1}{8} \leq \frac{1}{4} = \prod_{k=1}^{p}(1 - L_k)^2. \end{aligned}$$

Then all the conditions of Theorem 2 are satisfied. Thus (5.1) has minimal and maximal solutions in $[\alpha, \beta]$.

In addition, we consider $\beta_1(t) = \frac{t}{100} + \frac{99}{100}$, $t \in [0, 1]$. Clearly $\beta_1(0) < \beta_1(1)$, then $b(t) = \frac{t}{1200} + \frac{1}{3600}\left(\frac{1}{3}t + \sqrt{t} + \frac{2}{3}t\right) + \frac{t}{50}$ and $b_1 = \frac{1}{400}$. We still take

$M = \frac{1}{12}$, $N_1 = N_2 = N_3 = \frac{1}{36}$. From

$$\begin{aligned}
0 &\geq -\tfrac{1}{24}\Big[\Big(\tfrac{t-1}{100}\Big)^2 + \tfrac{9800}{100^2} + \Big(\tfrac{99}{50} - t\Big)\Big] \\
&= -\tfrac{1}{24}\Big[\Big(\tfrac{t}{100}\Big)^2 - \tfrac{t}{50} \times \tfrac{1}{100} + \tfrac{1}{100^2} + \tfrac{9800}{100^2} - t + 2 \times \tfrac{99}{100}\Big] \\
&= -\tfrac{1}{24}\Big[\Big(\tfrac{t}{100}\Big)^2 + \tfrac{t}{50} \times \tfrac{99}{100} + \Big(\tfrac{99}{100}\Big)^2 - t - \tfrac{t}{50} + 2 \times \tfrac{99}{100}\Big] \\
&= -\tfrac{1}{24}\Big(\tfrac{1}{100}t + \tfrac{99}{100}\Big)^2 - \tfrac{1}{36} \times 3 \times \tfrac{99}{100} + \tfrac{1}{24}t + \tfrac{t}{1200}, \\
\tfrac{t}{50} &\geq -\tfrac{1}{24}\beta_1^2(t) - \tfrac{1}{36}\Big[\Big(\tfrac{1}{100} \times \tfrac{1}{3}t + \tfrac{99}{100}\Big) + \Big(\tfrac{1}{100} \times \sqrt{t} + \tfrac{99}{100}\Big) \\
&\quad + \Big(\tfrac{1}{100} \times \tfrac{2}{3}t + \tfrac{99}{100}\Big)\Big] + \tfrac{1}{24}t + \Big[\tfrac{t}{1200} + \tfrac{1}{3600}\Big(\tfrac{1}{3}t + \sqrt{t} + \tfrac{2}{3}t\Big) + \tfrac{t}{50}\Big],
\end{aligned}$$

we have $(\beta_1(t^2))' \geq -\frac{1}{24}\beta_1^2(t) - \frac{1}{36}\Big[\beta_1\big(\frac{1}{3}t\big) + \beta_1(\sqrt{t}) + \beta_1\big(\frac{2}{3}t\big)\Big] + \frac{1}{24}t + b(t)$. From $0 \geq -\frac{1}{2}\left(\frac{1}{100} \times \frac{1}{4} + \frac{99}{100}\right) + \frac{1}{2} \times \frac{1}{400}$, we get $\Delta\beta_1\big(\frac{1}{4}\big) \geq I_1\big(\beta_1\big(\frac{1}{4}\big)\big) + L_1 b_1$.

These show that $\beta_1(t)$ is an upper solution for (5.1). Moreover $\alpha \leq \beta_1$ on J. Similarly, we get the existence of monotone sequence that approximate the extremal solutions for (5.1) in $[\alpha, \beta_1]$.

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