Tema

Remarks on a Nonlinear Wave Equation in a Noncylindrical Domain[†]

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ABSTRACT. In this paper we investigate the existence of solution for an initial boundary value problem
of the following nonlinear wave equation:

$$u'' - \Delta u + |u|^\rho = f \text{ in } \widehat{Q},$$

where $\widehat{Q}$ represents a non-cylindrical domain of $\mathbb{R}^{n+1}$. The methodology, cf. Lions [3], consists of trans-
forming this problem, by means of a perturbation depending on a parameter $\varepsilon > 0$, into another one defined
in a cylindrical domain Q containing $\widehat{Q}$. By solving the cylindrical problem, we obtain estimates that de-
pend on ε. These ones will enable a passage to the limit, when ε goes to zero, that will guarantee, later, a
solution for the non-cylindrical problem. The nonlinearity $|u_\varepsilon|^\rho$ introduces some obstacles in the process
of obtaining a priori estimates and we overcome this difficulty by employing an argument due to Tartar [8]
plus a contradiction process.

Keywords: nonlinear problem, non-cylindrical domain, hyperbolic equation.

1 INTRODUCTION

Let us consider the general non-cylindrical initial-boundary value problem:

$$\left|
\begin{array}{l}
u'' - \Delta u + \beta \left(\frac{\partial u}{\partial x} \right) + \gamma (u) = f \text{ in } \widehat{Q} \\
u = 0 \text{ on } \widehat{\Sigma} \\
u(x, 0) = u_0(x) , u'(x, 0) = u_1(x) , x \in \Omega_0 .
\end{array}
\right. \tag{1.1}$$

By $\widehat{Q}$ we represent a bounded increasing domain of $\mathbb{R}^n \times (0, \infty)$ and $\widehat{\Sigma}$ its lateral boundary.

This type of question was initially investigated by J.-L. Lions [3] by applying a method, called by
himself the penalty method, which will be described later. He obtained weak solutions for (1.1)

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for the special case $\beta(s) = 0$ and $\gamma(s) = |s|^\rho s$, $\rho > 0$ a real number. Cooper and Bardos [1] extended this result to a larger class of regions by assuming only that there is a smooth mapping $\varphi : \mathbb{R}^n \times (0, T) \longrightarrow \mathbb{R}^n \times (0, T)$ such that $Q^* = \varphi(\widehat{Q})$ is monotone increasing and φ preserves the hyperbolic character of (1.1).

Medeiros [5] generalized the results obtained by Lions [3] in another direction considering the case of $\gamma(s)$ as a real continuous function satisfying the condition $\gamma(s) s \geq 0$, for all $s \in \mathbb{R}$, and $\beta(s) = 0$.

In Nakao-Narazaki [7] the authors worked with general real continuous functions, with restrictions, and obtained existence and decay of the solutions.

In the present work, we shall investigate the existence of weak solutions for (1.1) when $\beta(s) = 0$ and $\gamma(s) = |s|^\rho$.

Let T be a positive real number and let $\{\Omega_t\}_{t \in [0,T]}$ be a family of bounded open sets of $\mathbb{R}^n$, with regular boundary Γ_t. We denote by $\widehat{Q}$ the non-cylindrical domain of $\mathbb{R}^{n+1}$ defined by

$$\widehat{Q} = \bigcup_{0 < t < T} \Omega_t \times \{t\},$$

with regular lateral boundary

$$\widehat{\Sigma} = \bigcup_{0 < t < T} \Gamma_t \times \{t\}.$$

Therefore, we consider the following problem:

$$\left|\begin{array}{ll} u'' - \Delta u + |u|^\rho = f & \text{in} \quad \widehat{Q} \\ u = 0 \quad \text{on} \quad \widehat{\Sigma} \\ u(x, 0) = u_0(x), \quad u'(x, 0) = u_1(x) & \text{in} \quad \Omega_0, \end{array}\right. \tag{1.2}$$

where the derivatives are in the sense of the theory of distributions, Δ represents the usual spatial Laplace operator in $\mathbb{R}^n$ and ρ is a positive real number satisfying certain conditions.

The methodology, cf. Lions [2], consists of transforming (1.2), by means of a perturbation depending on a parameter $\varepsilon > 0$, into a problem defined in a cylindrical domain Q. Then we have to solve the cylindrical problem and get estimates to pass to the limit when $\varepsilon \to 0$.

Let us consider a bounded open set $\Omega \subset \mathbb{R}^n$, with C^2-boundary Γ and such that $\widehat{Q} \subset Q = \Omega \times (0, T)$ (see Fig. 1).

For each $\varepsilon > 0$, we are looking for $u_\varepsilon : Q \longrightarrow \mathbb{R}$ solution of the problem:

$$\left|\begin{array}{ll} u_\varepsilon'' - \Delta u_\varepsilon + |u_\varepsilon|^\rho + \dfrac{1}{\varepsilon} M u_\varepsilon' = \widetilde{f} & \text{in} \quad Q \\ u_\varepsilon = 0 \quad \text{on} \quad \Sigma = \Gamma \times (0, T) \\ u_\varepsilon(x, 0) = \widetilde{u}_0(x), \quad u_\varepsilon'(x, 0) = \widetilde{u}_1(x) & \text{in} \quad \Omega, \end{array}\right. \tag{1.3}$$

where M is defined by

$$M(x, t) = \left\{\begin{array}{ll} 1 & \text{if} \quad (x, t) \in Q - \widehat{Q} \\ 0 & \text{if} \quad (x, t) \in \widehat{Q}, \end{array}\right. \tag{1.4}$$
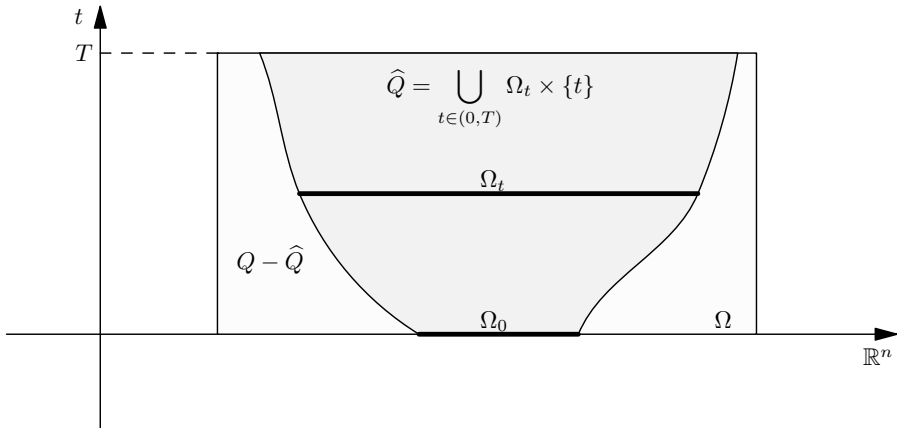
Figure 1: Scheme representing the domains Q and $\widehat{Q}$.

$$\widetilde{f} = \begin{cases} f & \text{in } \widehat{Q} \\ 0 & \text{in } Q - \widehat{Q}, \end{cases} \qquad \widetilde{u}_0 = \begin{cases} u_0 & \text{in } \Omega_0 \\ 0 & \text{in } \Omega - \Omega_0 \end{cases} \quad \text{and} \quad \widetilde{u}_1 = \begin{cases} u_1 & \text{in } \Omega_0 \\ 0 & \text{in } \Omega - \Omega_0. \end{cases}$$

We call attention to the fact that the nonlinearity $|u_\varepsilon|^\rho$ in (1.3) generates some obstacles in the process of obtaining a priori estimates for the problem (1.3), by the energy method, because, at a certain point of our proof, we get a term of the type

$$\int_\Omega |\nabla u_\varepsilon(t)|^2 \, dx + \frac{1}{\rho + 1} \int_\Omega |u_\varepsilon(t)|^\rho \, u_\varepsilon(t) \, dx$$

whose sign cannot be controlled. At this point of the proof we employ an argument due to Tartar [8] plus contradiction process cf. [6].

2   NOTATIONS AND HYPOTHESES

As usual we represent by $L^2(\Omega)$ the Lebesgue space of square integrable functions on Ω. The spaces $L^2(\Omega_t)$ are identified, for all $t \in [0, T]$, with closed subspaces of $L^2(\Omega)$. We denote by $L^p\left(0, T; L^2(\Omega_t)\right)$ and $L^p\left(0, T; H_0^1(\Omega_t)\right)$ the following spaces

$$L^p\left(0, T; L^2(\Omega_t)\right) = \left\{ v \in L^p\left(0, T; L^2(\Omega)\right); \; v(t) \in L^2(\Omega_t) \right\}, 1 \le p \le \infty$$

and

$$L^p\left(0, T; H_0^1(\Omega_t)\right) = \left\{ v \in L^p\left(0, T; H_0^1(\Omega)\right); \; v(t) \in H_0^1(\Omega_t) \right\}, 1 \le p \le \infty.$$

In the following we will denote by $(\cdot, \cdot)$, $|\cdot|$ and $((\cdot, \cdot))$, $\|\cdot\|$ the inner products and norms of the Hilbert spaces $L^2(\Omega)$ and $H_0^1(\Omega)$ respectively.

We will develop our work under the following assumptions:

(H1) *(Geometric condition)* The family $\{\Omega_t\}_{t\in[0,T]}$ is increasing in the following sense: if $t_1 \leq t_2$ then $\Omega_{t_1} \subseteq \Omega_{t_2}$.

(H2) *(Regularity condition)* If $v \in H_0^1(\Omega)$ and $v = 0$ a.e. in $\Omega - \Omega_t$, then $v \in H_0^1(\Omega_t)$;

(H3) *(Immersion condition)* $1 < \rho \leq \frac{n}{n-2}$, for $n \geq 3$, and $\rho > 1$, for $n = 1$ or $n = 2$.

Remark 1. By Sobolev embedding theorem, we have $H^1(\Omega) \hookrightarrow L^q(\Omega)$, with $\frac{1}{q} = \frac{1}{2} - \frac{1}{n}$, that is, $q = \frac{2n}{n-2}$ for $n > 2$. In the case $n = 1$, $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$. In the case $n = 2$, $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$. In the proof of our result, we need the embedding of the space $L^{2\rho}(\Omega)$ into $L^{\rho+1}(\Omega)$. As Ω is bounded and $\rho > 1$, we have, from (H3), that $H^1(\Omega) \hookrightarrow L^{2\rho}(\Omega) \hookrightarrow L^{\rho+1}(\Omega)$.

Remark 2. If the boundary Γ_t of Ω_t is for $t \in [0, t]$ a manifold of class C^2 and $v = 0$ on $\Omega - \Omega_t$ it implies **(H2)** . In fact, Γ is of class C^2. Thus, $\Gamma \cup \Gamma_t$ is of class C^2. Therefore, by the trace theorem

$$\gamma_0 : H^1(\Omega - \Omega_t) \longrightarrow H^{1/2}(\Gamma \cup \Gamma_t),$$

since the boundary of $\Omega - \Omega_t$ is $\Gamma \cup \Gamma_t$, which is continuous. Thus, for each $v \in H^1(\Omega - \Omega_t)$, we have:

$$\|\gamma_0 v\|_{H^{1/2}(\Gamma \cup \Gamma_t)} \leq C \|v\|_{H^1(\Omega - \Omega_t)}.$$

But $v = 0$ on $\Omega - \Omega_t$. Thus,

$$\|\gamma_0 v\|_{H^{1/2}(\Gamma \cup \Gamma_t)} = 0.$$

Thus, $H^{1/2}(\Gamma \cup \Gamma_t)$ is a Hilbert space, what implies $v = 0$ on $\Gamma \cup \Gamma_t$. $\square$

3   MAIN RESULTS

The main result of this work is contained in the following Theorem:

Theorem 3.1. *Given $u_0 \in H_0^1(\Omega_0)$, $u_1 \in L^2(\Omega_0)$ and $f \in L^1(0, \infty; L^2(\Omega_t))$. Set*

$$\gamma(\widetilde{u}_0, \widetilde{u}_1) = \left(|\widetilde{u}_1|^2 + \|\widetilde{u}_0\|^2 + \frac{2}{\rho+1}\int_\Omega |\widetilde{u}_0|^\rho \, \widetilde{u}_0 \, dx + \|\widetilde{f}\|_{L^1(0,\infty;L^2(\Omega))}\right) e^{\|\widetilde{f}\|_{L^1(0,\infty;L^2(\Omega))}},$$

where $\widetilde{u}_0$, $\widetilde{u}_1$ and $\widetilde{f}$ are extensions of u_0, u_1 and f, respectively, and were defined in the previous section. Suppose, in addition to the hypotheses (H1)-(H3), that

$$\|\widetilde{u}_0\| < \left(\frac{1}{2C_0^{\rho+1}}\right)^{\frac{1}{\rho-1}} \tag{H4}$$

and

$$\gamma(\widetilde{u}_0, \widetilde{u}_1) < \frac{1}{2}\left(\frac{1}{2C_0^{\rho+1}}\right)^{\frac{2}{\rho-1}}, \tag{H5}$$

where C_0 is the constant of the embedding of $H_0^1(\Omega)$ into $L^{\rho+1}(\Omega)$. Then, there exists a nonlocal solution for the problem (1.2), satisfying $u \in L^\infty(0, T; H_0^1(\Omega_t))$ and $u' \in L^\infty(0, T; L^2(\Omega_t))$.

Proof.   First we will solve, for each $\varepsilon > 0$, the problem (1.3). Let $\{w_\nu\}_{\nu \in \mathbb{N}}$ be an orthonormal basis of $H_0^1(\Omega)$. For $\varepsilon > 0$ fixed and each $m \in \mathbb{N}$, we consider $u_{\varepsilon m}(x, t) = \sum_{j=1}^{m} g_{jm}(t) w_j(x)$, $x \in \Omega$ and $t \in [0, T_m)$, which is solution of the following approximate problem:

$$
\left|
\begin{aligned}
&(u_{\varepsilon m}''(t), w) + (\nabla u_{\varepsilon m}(t), \nabla w) + (|u_{\varepsilon m}(t)|^\rho, w) \\
&\quad + \frac{1}{\varepsilon}(M(t) u_{\varepsilon m}'(t), w) = (\widetilde{f}(t), w) \\
&u_{\varepsilon m}(0) = \widetilde{u}_{0m} \longrightarrow \widetilde{u}_0 \text{ in } H_0^1(\Omega) \\
&u_{\varepsilon m}'(0) = \widetilde{u}_{1m} \longrightarrow \widetilde{u}_1 \text{ in } L^2(\Omega),
\end{aligned}
\right.
\tag{3.1}
$$

for all $w \in [w_1, w_2, ..., w_m] = \text{span}\{w_1, w_2, ..., w_m\}$.

Remark 3. From (3.1), (H4), (H5) and the Remark 1, there exists $\widetilde{m}$ such that, for all $m \geq \widetilde{m}$, we have

$$
\|\widetilde{u}_{0m}\| < \left(\frac{1}{2C_0^{\rho+1}}\right)^{\frac{1}{\rho-1}} \quad \text{and} \quad \gamma(\widetilde{u}_{0m}, \widetilde{u}_{1m}) < \frac{1}{2}\left(\frac{1}{2C_0^{\rho+1}}\right)^{\frac{2}{\rho-1}}.
\tag{3.2}
$$

Replacing, if necessary, $\widetilde{u}_{0m}$ and $\widetilde{u}_{1m}$ for $m < \widetilde{m}$, we can consider, from now on, that $\widetilde{m} = 1$.

The local existence, for some $T_m > 0$, is a consequence of the results about systems of nonlinear ordinary differential equations.

We need estimates which permit to pass to the limit in the approximate solution $u_{\varepsilon m}(t)$ when m goes to infinity and show that (3.1) has a nonlocal solution.

Estimate 1. Taking $w = 2u_{\varepsilon m}'(t)$ in (3.1) we obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\left|u_{\varepsilon m}'(t)\right|^2 + \|u_{\varepsilon m}(t)\|^2 + \frac{2}{\rho+1}\int_\Omega |u_{\varepsilon m}(x, t)|^\rho u_{\varepsilon m}(x, t)\, dx\right) \\
&\quad + \frac{2}{\varepsilon}\int_\Omega M(x, t)\left|u_{\varepsilon m}'(x, t)\right|^2 dx = 2\left(\widetilde{f}(t), u_{\varepsilon m}'(t)\right).
\end{aligned}
\tag{3.3}
$$

Integrating (3.3) from 0 to $t < T_m$ we have

$$
\begin{aligned}
&\left|u_{\varepsilon m}'(t)\right|^2 + \|u_{\varepsilon m}(t)\|^2 + \frac{2}{\rho+1}\int_\Omega |u_{\varepsilon m}(x, t)|^\rho u_{\varepsilon m}(x, t)\, dx \\
&\quad + \frac{2}{\varepsilon}\int_0^t \int_\Omega M(x, s)\left|u_{\varepsilon m}'(x, s)\right|^2 dx\, ds \\
&\leq |\widetilde{u}_{1m}|^2 + \|\widetilde{u}_{0m}\|^2 + \frac{2}{\rho+1}\int_\Omega |\widetilde{u}_{0m}|^\rho \widetilde{u}_{0m}\, dx \\
&\quad + 2\int_0^t \left|\widetilde{f}(s)\right|\left|u_{\varepsilon m}'(s)\right| ds.
\end{aligned}
\tag{3.4}
$$

The main question at this point of the proof is that we don't know the sign of

$$
J(u) = \frac{1}{2}\|u\|^2 + \frac{2}{\rho+1}\int_\Omega |u|^\rho u\, dx,
$$

for $u = u_{\varepsilon m}(t)$ and $u = \tilde{u}_{0m}$ in the inequality (3.4).

To overcome this difficulty we will do some computation. First, we observe that

$$\left| \int_\Omega |u_{\varepsilon m}(x,t)|^\rho u_{\varepsilon m}(x,t)\,dx \right| \leq \int_\Omega |u_{\varepsilon m}(x,t)|^{\rho+1}\,dx$$
$$= |u_{\varepsilon m}(t)|^{\rho+1}_{L^{\rho+1}(\Omega)} \leq C_0^{\rho+1}\|u_{\varepsilon m}(t)\|^{\rho+1}, \tag{3.5}$$

since the last inequality is a consequence of the immersion $H_0^1(\Omega) \hookrightarrow L^{\rho+1}(\Omega)$, see *(H3)* and *Remark 1*.

From (3.5) we have

$$\int_\Omega |u_{\varepsilon m}(x,t)|^\rho u_{\varepsilon m}(x,t)\,dx \geq -C_0^{\rho+1}\|u_{\varepsilon m}(t)\|^{\rho+1}$$

and thus,

$$J(u) = \frac{1}{2}\|u\|^2 + \frac{2}{\rho+1}\int_\Omega |u|^\rho u\,dx \geq \frac{1}{2}\|u\|^2 - \frac{2}{\rho+1}C_0^{\rho+1}\|u(t)\|^{\rho+1}. \tag{3.6}$$

This functional will be employed for $u = u_{\varepsilon m}(t)$ and $u = \tilde{u}_{0m}$ later.

Therefore, the sign of both sides of (3.4) is related to the sign of the function

$$P(\lambda) = \frac{\lambda^2}{2} - \frac{2}{\rho+1}C_0^{\rho+1}\lambda^{\rho+1},$$

for $\lambda \geq 0$ and $\rho > 1$.

From the definition of $P(\lambda)$, we observe that it is increasing in the open interval

$$\left(0, \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)}\right)$$

and has a maximum value at $\left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)}$. See an example for the graph of $P(\lambda)$, at Figure 2 bellow, when $\rho = 3$ and $C_0 = 0.3$.

As $J(u) = \frac{1}{2}\|u\|^2 + \frac{2}{\rho+1}\int_\Omega |u|^\rho u\,dx \geq P(\|u\|)$, we can conclude that

$$\|u\| \leq \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)} \qquad \text{which implies} \quad J(u) \geq 0. \tag{3.7}$$

Then, from the equation (3.2), we conclude that

$$J(\tilde{u}_{0m}) = \frac{1}{2}\|\tilde{u}_{0m}\|^2 + \frac{2}{\rho+1}\int_\Omega |\tilde{u}_{0m}|^\rho \tilde{u}_{0m}\,dx \geq 0. \tag{3.8}$$
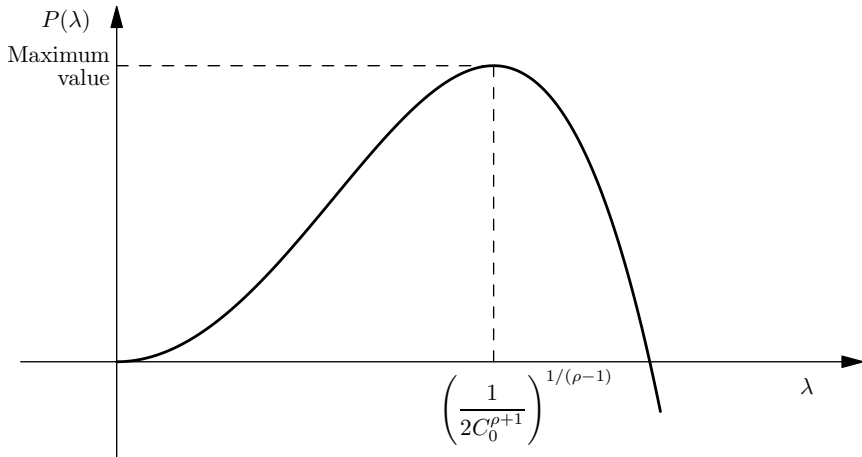
Thus, the right hand side of (3.4) is non negative.

Figure 2: Graph of $P(\lambda)$ for $\rho = 3$ and $C_0 = 0, 3$.

To analyze the left hand side of (3.4) we need the following Lemma:

Lemma 3.1. *From the hypotheses* (H4) *and* (H5)*, it follows that the approximate solution* $u_{\varepsilon m}$ *satisfies*

$$\|u_{\varepsilon m}(t)\| < \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)}, \tag{3.9}$$

for all $t \in [0, T_m)$*,* $m \in \mathbb{N}$ *and* $\varepsilon > 0$ *fixed.*

Proof.   Let us apply a contradiction argument. In fact, suppose there exists $m_0 \in \mathbb{N}$ such that

$$\|u_{\varepsilon m_0}(t)\| \geq \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)} \tag{3.10}$$

for some $0 < t < T_{m_0}$ and $\varepsilon > 0$ fixed. From (3.2) we have

$$0 \leq \|u_{\varepsilon m_0}(0)\| = \|\widetilde{u}_{0m_0}\| < \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)}. \tag{3.11}$$

From (3.11) and the continuity of $\|u_{\varepsilon m_0}(t)\|$, we conclude that there exists $t_0 > 0$ such that

$$0 \leq \|u_{\varepsilon m_0}(t)\| < \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)} \quad \text{for all } t \in (0, t_0).$$

From (3.10), the set

$$\left\{ t > 0; \ \|u_{\varepsilon m_0}(t)\| \geq \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)} \right\}$$

is non-empty, closed and bounded below. Thus, there exists a minimum for this set that we will call t^*. By continuity of $\left\| u_{\varepsilon m_0}(t) \right\|$ we have

$$
\left|
\begin{array}{l}
\left\| u_{\varepsilon m_0}(t) \right\| < \left(\dfrac{1}{2C_0^{\rho+1}} \right)^{1/(\rho-1)} , \ 0 \leq t < t^* \\[4mm]
\left\| u_{\varepsilon m_0}(t^*) \right\| = \left(\dfrac{1}{2C_0^{\rho+1}} \right)^{1/(\rho-1)} .
\end{array}
\right.
\tag{3.12}
$$

Hence, from (3.12) and (3.7), we observe that, for all $t \in [0, t^*]$

$$
J(u_{\varepsilon m_0}(t)) = \frac{1}{2} \left\| u_{\varepsilon m_0}(t) \right\|^2 + \frac{2}{\rho+1} \int_\Omega \left| u_{\varepsilon m_0}(t) \right|^\rho u_{\varepsilon m_0}(t) \, dx \geq 0
\tag{3.13}
$$

So, integrating (3.3) from 0 to $t \leq t^*$, we obtain the inequality bellow

$$
\left| u'_{\varepsilon m_0}(t) \right|^2 + \left\| u_{\varepsilon m_0}(t) \right\|^2 + \frac{2}{\rho+1} \int_\Omega \left| u_{\varepsilon m_0}(t) \right|^\rho u_{\varepsilon m_0}(t) \, dx
$$
$$
\leq \left| \widetilde{u}_{1m_0} \right|^2 + \left\| \widetilde{u}_{0m_0} \right\|^2 + \frac{2}{\rho+1} \int_\Omega \left| \widetilde{u}_{0m_0}(x) \right|^\rho \widetilde{u}_{0m_0}(x) \, dx
\tag{3.14}
$$
$$
+ 2 \int_0^t \left| \widetilde{f}(s) \right| \left| u'_{\varepsilon m_0}(s) \right| ds.
$$

From (3.8) and (3.13), both sides of (3.14) are positive.

For the last term of the right side of (3.14), by Young's inequality, we observe that

$$
2 \int_0^t \left| \widetilde{f}(s) \right| \left| u'_{\varepsilon m}(s) \right| ds \leq \left\| \widetilde{f} \right\|_{L^1(0,\infty:L^2(\Omega))} + \int_0^t \left| \widetilde{f}(s) \right| \left| u'_{\varepsilon m_0}(s) \right|^2 ds.
\tag{3.15}
$$

Therefore, using (3.15) and (3.13) in (3.14) we get, for all $t \in [0, t^*]$,

$$
\left| u'_{\varepsilon m_0}(t) \right|^2 + \frac{1}{2} \left\| u_{\varepsilon m_0}(t) \right\|^2 \leq K_1 + \int_0^t \left| \widetilde{f}(s) \right| \left| u'_{\varepsilon m_0}(s) \right|^2 ds,
\tag{3.16}
$$

where $K_1 = \left| \widetilde{u}_{1m_0} \right|^2 + \left\| \widetilde{u}_{0m_0} \right\|^2 + \dfrac{2}{\rho+1} \int_\Omega \left| \widetilde{u}_{0m_0} \right|^\rho \widetilde{u}_{0m_0} dx + \left\| \widetilde{f} \right\|_{L^1(0,\infty:L^2(\Omega))}$.

Thus, using Gronwall's inequality, we can conclude

$$
K_1 + \int_0^t \left| \widetilde{f}(s) \right| \left| u'_{\varepsilon m_0}(s) \right|^2 ds \leq K_1 \, e^{\|\widetilde{f}\|}, \ \text{for } 0 \leq t \leq t^*.
\tag{3.17}
$$

Applying (3.17) in (3.16) for $t = t^*$, we obtain

$$
\left| u'_{\varepsilon m_0}(t^*) \right|^2 + \frac{1}{2} \left\| u_{\varepsilon m_0}(t^*) \right\|^2 \leq K_1 \, e^{\|\widetilde{f}\|}
$$

and it follows

$$
\frac{1}{2} \left\| u_{\varepsilon m_0}(t^*) \right\|^2 \leq \left(\left| \widetilde{u}_{1m_0} \right|^2 + \left\| \widetilde{u}_{0m_0} \right\|^2 + \frac{2}{\rho+1} \int_\Omega \left| \widetilde{u}_{0m_0} \right|^\rho \widetilde{u}_{0m_0} dx + \left\| \widetilde{f} \right\| \right) e^{\|\widetilde{f}\|},
$$

The right hand side is equal to $\gamma\left(\widetilde{u}_{0m_0}, \widetilde{u}_{1m_0}\right)$ and, combining it with (3.2), allows us to conclude that

$$\left\|u_{\varepsilon m_0}\left(t^*\right)\right\| < \left(\frac{1}{2C_0^{\rho+1}}\right)^{1/(\rho-1)},$$

and this contradicts (3.12). So, the proof of the Lemma 3.1 is finished.   □

From (3.9), we have that for all $t \in [0, T_m)$

$$\frac{1}{2}\left\|u_{\varepsilon m}\left(t\right)\right\|^2 + \frac{2}{\rho+1}\int_\Omega \left|u_{\varepsilon m}\left(t\right)\right|^\rho u_{\varepsilon m}\left(t\right)dx \geq 0,$$

thus, from (3.4), it follows

$$\left|u'_{\varepsilon m}\left(t\right)\right|^2 + \frac{1}{2}\left\|u_{\varepsilon m}\left(t\right)\right\|^2 + \frac{2}{\varepsilon}\int_0^t\int_\Omega M\left(x,s\right)\left|u'_{\varepsilon m}\left(x,s\right)\right|^2 dxds$$
$$\leq \left|\widetilde{u}_{1m}\right|^2 + \left\|\widetilde{u}_{0m}\right\|^2 + \frac{2}{\rho+1}\int_\Omega \left|\widetilde{u}_{0m}\right|^\rho \widetilde{u}_{0m}dx \tag{3.18}$$
$$+ 2\int_0^t \left|\widetilde{f}\left(s\right)\right|\left|u'_{\varepsilon m}\left(s\right)\right|ds.$$

Applying Gronwall's inequality to (3.18) we conclude that there exists a constant $K_2 > 0$, independent of ε, m and t, such that

$$\left|u'_{\varepsilon m}\left(t\right)\right|^2 + \frac{1}{2}\left\|u_{\varepsilon m}\left(t\right)\right\|^2 + \frac{2}{\varepsilon}\int_0^t\int_\Omega M\left(x,s\right)\left|u'_{\varepsilon m}\left(x,s\right)\right|^2 dxds \leq K_2. \tag{3.19}$$

Passage to the limit

From the inequality on (3.19) we obtain a subsequence, still denoted $(u_{\varepsilon m})_{m\in\mathbb{N}}$, such that

$$u_{\varepsilon m} \xrightarrow{\ *\ } u_\varepsilon \text{ in } L^\infty\left(0, T; H_0^1\left(\Omega\right)\right) \tag{3.20}$$

$$u'_{\varepsilon m} \xrightarrow{\ *\ } u'_\varepsilon \text{ in } L^\infty\left(0, T; L^2\left(\Omega\right)\right). \tag{3.21}$$

To take the limit in the nonlinear term, we define

$$W\left(0, T\right) = \left\{u \in L^\infty\left(0, T; H_0^1\left(\Omega\right)\right); \ u' \in L^\infty\left(0, T; L^2\left(\Omega\right)\right)\right\}$$

and observe that, since $H_0^1\left(\Omega\right) \xhookrightarrow{c} L^2\left(\Omega\right)$, from compactness theorem of Lions-Aubin [2],

$$W\left(0, T\right) \xhookrightarrow{c} L^2\left(0, T; L^2\left(\Omega\right)\right).$$

Observe that, from (3.20) and (3.21), we have

$$\left(u_{\varepsilon m}\right)_{m\in\mathbb{N}} \text{ is bounded in } W\left(0, T\right)$$

and, thus, we can extract a subsequence, still denoted by $(u_{\varepsilon m})_{m \in \mathbb{N}}$, such that

$$u_{\varepsilon m} \longrightarrow u_\varepsilon \text{ strongly in } L^2\left(0, T; L^2(\Omega)\right) \equiv L^2(Q).$$

Then, there is a subsequence such that

$$|u_{\varepsilon m}|^\rho \longrightarrow |u_\varepsilon|^\rho \text{ a.e. in } Q. \tag{3.22}$$

On the other hand, by the hypothesis $H3$, we have that $H_0^1(\Omega) \hookrightarrow L^{2\rho}(\Omega)$ and, therefore, by (3.19), we obtain

$$\left| |u_{\varepsilon m}(t)|^\rho \right|^2_{L^2(\Omega)} = |u_{\varepsilon m}(t)|^{2\rho}_{L^{2\rho}(\Omega)} \leq C_1^{2\rho} \|u_{\varepsilon m}(t)\|^{2\rho} < K_3. \tag{3.23}$$

From (3.22) and (3.23), thanks to Lions [2], Lemma 1.3, we have

$$|u_{\varepsilon m}|^\rho \longrightarrow |u_\varepsilon|^\rho \text{ in } L^2\left(0, T; L^2(\Omega)\right). \tag{3.24}$$

From (3.20), (3.21) and (3.24) it follows that, for all $v \in H_0^1(\Omega)$,

$$((u_{\varepsilon m}, v)) \overset{*}{\longrightarrow} ((u_\varepsilon, v)) \text{ in } L^\infty(0, T)$$

$$\left(|u_{\varepsilon m}|^\rho, v\right) \overset{*}{\longrightarrow} \left(|u_\varepsilon|^\rho, v\right) \text{ in } L^\infty(0, T)$$

$$\left(M u'_{\varepsilon m}, v\right) \overset{*}{\longrightarrow} \left(M u'_\varepsilon, v\right) \text{ in} L^\infty(0, T).$$

The convergences obtained above allow us to take the limit in the approximate equation, when m goes to infinity, and get

$$\frac{d}{dt}\left(u'_\varepsilon(t), v\right) + ((u_\varepsilon(t), v)) + \left(|u_\varepsilon(t)|^\rho, v\right) + \frac{1}{\varepsilon}\left(M(t) u'_\varepsilon(t), v\right)$$
$$= \left(\widetilde{f}(t), v\right) \text{ in } \mathcal{D}'(0, T), \text{ for all } v \in H_0^1(\Omega). \tag{3.25}$$

Applying (3.25) to $\theta \in \mathcal{D}(0, T)$ it follows that

$$-\int_0^T \left(u'_\varepsilon(t), v\right) \theta'(t) \, dt - \int_0^T \langle \Delta u_\varepsilon(t), v \rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \theta(t) \, dt$$
$$+ \int_0^T \left(|u_\varepsilon(t)|^\rho, v\right) \theta(t) \, dt + \frac{1}{\varepsilon} \int_0^T \left(M(t) u'_\varepsilon(t), v\right) \theta(t) \, dt$$
$$= \int_0^T \left(\widetilde{f}(t), v\right) \theta(t) \, dt,$$

for all $v \in H_0^1(\Omega)$.

Therefore,

$$\left(-\int_0^T u'_\varepsilon(t) \theta'(t) \, dt, v\right)$$
$$= \left\langle \int_0^T \left(\widetilde{f}(t) + \Delta u_\varepsilon(t) - |u_\varepsilon(t)|^\rho - \frac{1}{\varepsilon} M(t) u'_\varepsilon(t)\right) \theta(t) \, dt, v\right\rangle_{H^{-1}(\Omega) \times H_0^1(\Omega)} \tag{3.26}$$

for all $v \in H_0^1 (\Omega)$, that is,

$$-\int_0^T u_\varepsilon' (t)\, \theta' (t)\, dt = \int_0^T g(t)\, \theta(t)\, dt \; \text{in} H^{-1} (\Omega), \qquad (3.27)$$

where $g(t) = \widetilde{f}(t) + \Delta u_\varepsilon (t) - |u_\varepsilon (t)|^\rho - \frac{1}{\varepsilon} M(t) u_\varepsilon'(t)$.

Thus, $u_\varepsilon'' = g$ in the sense of the distributions.

But,

$$g = \left(\widetilde{f} + \Delta u_\varepsilon - |u_\varepsilon|^\rho - \frac{1}{\varepsilon} M(t) u_\varepsilon' \right) \in L^1 \left(0, T; H^{-1} (\Omega) \right)$$

and therefore,

$$u_\varepsilon'' - \Delta u_\varepsilon + |u_\varepsilon|^\rho + \frac{1}{\varepsilon} M(t) u_\varepsilon' = \widetilde{f} \text{ in } L^1 \left(0, T; H^{-1} (\Omega) \right). \qquad (3.28)$$

We will study the initial conditions on (3.28). From the convergences (3.20) and (3.21) we obtain that $u_{\varepsilon m}(0) \longrightarrow u_\varepsilon(0)$ as $m \to \infty$ in $L^2(\Omega)$. Then, using the approximate problem (3.1), we conclude that

$$u_\varepsilon(0) = \widetilde{u}_0. \qquad (3.29)$$

From the approximate problem (3.1) we have, for all $v \in V_m$ and almost every $t \in [0, T]$,

$$(u_{\varepsilon m}''(t), v) = \left(\widetilde{f} - |u_{\varepsilon m}(t)|^\rho - \frac{1}{\varepsilon} M(t) u_{\varepsilon m}'(t), v \right) - (\nabla u_{\varepsilon m}, \nabla v).$$

As $u_{\varepsilon m}''(t) \in V_m$, we can conclude, for almost every $t \in [0, T]$,

$$\|u_{\varepsilon m}''(t)\|_{H^{-1}(\Omega)} \leq C_1 \left(|\widetilde{f}| + |u_{\varepsilon m}(t)|^\rho_{L^{2\rho}(\Omega)} + \frac{1}{\varepsilon} |M(t) u_{\varepsilon m}'(t)| \right) + \|u_{\varepsilon m}(t)\|, \qquad (3.30)$$

where C_1 is the constant of the embedding of $H_0^1 (\Omega)$ into $L^2(\Omega)$.

From (3.19) and (3.30), we obtain that $\|u_{\varepsilon m}''\|_{L^1(0,T;H^{-1}(\Omega))} \leq K_3$, not depending on m and ε and, then, $u_{\varepsilon m}'' \xrightarrow{\;*\;} u_\varepsilon''$ in $L^1(0, T; H^{-1}(\Omega))$. This convergence, (3.21) and (3.1) imply that

$$u_\varepsilon'(0) = \widetilde{u}_1. \qquad (3.31)$$

Now, to finish the proof of theorem 3.1, we need take the limit when ε goes to zero.

In fact, note that from (3.19), we have the same estimates in ε as those obtained for m. That is,

$$\left| u_\varepsilon'(t) \right|^2 + \|u_\varepsilon(t)\|^2 + \frac{1}{\varepsilon} \int_0^t \int_\Omega M(x, s) \left| u_\varepsilon'(x, s) \right|^2 dx ds \leq K_3. \qquad (3.32)$$

Thus, there exists a subsequence, still denoted by $(u_\varepsilon)_{\varepsilon>0}$, such that

$$u_\varepsilon \xrightarrow{\;*\;} w \text{ in } L^\infty \left(0, T; H_0^1 (\Omega) \right) \qquad (3.33)$$

$$u_\varepsilon' \xrightarrow{\;*\;} w' \text{ in } L^\infty \left(0, T; L^2 (\Omega) \right). \qquad (3.34)$$

Applying, as above, the same arguments of compactness, it follows

$$|u_\varepsilon|^\rho \longrightarrow |w|^\rho \text{ in } L^2\left(0, T; L^2(\Omega)\right). \tag{3.35}$$

Also, by (3.32), we have

$$\int_0^t \int_\Omega M(x, s) \left|u'_\varepsilon(x, s)\right|^2 dx ds \leq \varepsilon K_3. \tag{3.36}$$

The limitation in (3.36), combined with the convergence in (3.34), allows us to conclude that

$$M(x, t) w'(x, t) = 0 \text{ in } Q$$

and thus, see the definition of M in (1.4),

$$w' = 0 \text{ a.e. in } Q - \widehat{Q}. \tag{3.37}$$

As $w(x, 0) = \widetilde{u}_0(x)$, then $w(x, 0) = 0$ in $\Omega - \Omega_0$ and, therefore, by (3.37) and the *geometric condition* (H1),

$$w = 0 \text{ a.e. in } Q - \widehat{Q}. \tag{3.38}$$

By (3.38) and the fact that $w(t) \in H_0^1(\Omega)$, it follows, from the *regularity condition (H2)*, that $w(t) \in H_0^1(\Omega_t)$. That is,

$$w \in L^\infty\left(0, T; H_0^1(\Omega_t)\right).$$

From (3.32), we observe that there is a subsequence of $\frac{1}{\sqrt{\varepsilon}}M(t)u'_\varepsilon(t)$ that converges weakly to some function in $L^2(0, T; L^2(\Omega))$ when ε tends to zero. Then, passing to the limit the equation (3.28), we observe that w could not be a weak solution for the PDE $u'' - \Delta u + |u^\rho| = f$ over all the extended domain Q. But, as M vanishes at $\widehat{Q}$, we can consider the restriction of w to $\widehat{Q}$ and the theorem will be proved.

If we denote by u, the restriction of w to $\widehat{Q}$, we have that

$$u \in L^\infty\left(0, T; H_0^1(\Omega_t)\right)$$

$$u' \in L^\infty\left(0, T; L^2(\Omega_t)\right).$$

On the other hand, from (3.28), by restriction to $\widehat{Q}$, we obtain, in the sense of distributions

$$\widehat{u}''_\varepsilon - \Delta\widehat{u}_\varepsilon + |\widehat{u}_\varepsilon|^\rho = f, \tag{3.39}$$

where $\widehat{u}_\varepsilon$ denote the restriction of u_ε to $\widehat{Q}$.

Finally, from $(3.33) - (3.35)$ we can pass to the limit in (3.39), when ε goes to zero and we can obtain, in the sense of distributions,

$$u'' - \Delta u + |u|^\rho = f.$$

The initial conditions for the problem (1.2) follow from (3.29) and (3.31) and the same kind of arguments used to show these equations.

RESUMO. Nesse artigo investigamos a existência de solução para um problema de valor inicial e de contorno associado à seguinte equação da onda não linear

$$u^{''} - \Delta u + |u|^{\rho} = f \ \text{ em } \ \widehat{Q} \,,$$

onde $\widehat{Q}$ representa um domínio não cilíndrico do $\mathbb{R}^{n+1}$. A metodologia, conforme Lions [3], consiste em transformar o problema original, por meio de uma perturbação dependendo de um parâmetro $\varepsilon > 0$, em um outro definido em um domínio cilíndrico Q que contêm $\widehat{Q}$. Resolvendo o problema no domínio cilíndrico, obtemos estimativas que dependem de ε. Tais estimativas nos permitirão tomar o limite, quando ε tende a zero, garantindo assim a existência de uma solução para o problema não cilíndrico. A não linearidade $|u|^{\rho}$ introduz alguns obstáculos no processo de obtenção das estimativas, tais dificuldades são superadas por meio de um argumento devido a Tartar [8] combinado com um argumento de contradição.

Palavras-chave: problema não linear, domínio não cilíndrico, equação hiperbólica.

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