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Group of Isometries of Niederreiter-Rosenbloom-Tsfasman Block Space

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ABSTRACT. Let $P = (\{1,2,\ldots,n\},\leq)$ be a poset that is an union of disjoint chains of the same length and $V = \mathbb{F}_q^N$ be the space of N-tuples over the finite field $\mathbb{F}_q$. Let $V_i = \mathbb{F}_q^{k_i}$, with $1 \leq i \leq n$, be a family of finite-dimensional linear spaces such that $k_1 + k_2 + \ldots + k_n = N$ and let $V = V_1 \times V_2 \times \ldots \times V_n$ endow with the poset block metric $d_{(P,\pi)}$ induced by the poset P and the partition $\pi = (k_1,k_2,\ldots,k_n)$, encompassing both Niederreiter-Rosenbloom-Tsfasman metric and error-block metric. In this paper, we give a complete description of group of isometries of the metric space $(V,d_{(P,\pi)})$, also called the Niederreiter-Rosenbloom-Tsfasman block space. In particular, we reobtain the group of isometries of the Niederreiter-Rosenbloom-Tsfasman space and obtain the group of isometries of the error-block metric space.

Keywords: error-block metric, poset metric, Niederreiter-Rosenbloom-Tsfasman metric, ordered Hamming metric, isometries, automorphisms.

1 INTRODUCTION

One of the main classical problem of the coding theory is to find sets with q^k elements in $\mathbb{F}_q^N$, the space of N-tuples over the finite field $\mathbb{F}_q$, with the largest minimum distance possible. There are many possible metrics that can be defined in $\mathbb{F}_q^N$, but the most common ones are the Hamming and Lee metrics.

In 1987 Harald Niederreiter generalized the classical problem of coding theory (see [8]): given positive integers s and $m_1,\ldots,m_s$, to find sets C of vectors $c_{ij} \in \mathbb{F}_q^N$, for $1 \leq i \leq s$ and $1 \leq j \leq m_i$, with the largest minimum sum $\sum_{i=1}^{s} d_i$, where the minimum is extended over all integers $d_1,\ldots,d_s$ with $0 \leq d_i \leq m_i$ for $1 \leq i \leq s$ and $\sum_{i=1}^{s} d_i \geq 1$ for which the subset $\{c_{i,j} : 1 \leq i \leq s \text{ and } 1 \leq j \leq d_i\}$ is linearly dependent in $\mathbb{F}_q^N$. The classical problem corresponds to the special case where $s > m$ and $m_i = 1$ for all $1 \leq i \leq s$.

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Brualdi, Graves and Lawrence (see [2]) also provided in 1995 a wider situation for the Niederreiter's problem: using partially ordered sets (*posets*) and defining the concept of poset codes, they started to study codes with a poset metric. Later Feng, Xu and Hickernell ([4], 2006) introduced the block metric, by partitioning the set of coordinate positions of $\mathbb{F}_q^N$ into families of blocks. Both kinds of metrics are generalizations of the Hamming metric, in the sense that the latter is attained when considering the trivial order (in the poset case) or one-dimensional blocks (in the block metric case). In 2008, Alves, Panek and Firer (see [1]) combined the poset and block structure, obtaining a further generalization, the poset block metrics. As a unified reading we cite the book of Firer *et al.* [5].

A particular instance of poset block codes and spaces, with one-dimensional blocks, are the spaces introduced by Niederreiter in 1991 (see [8]) and Rosenbloom and Tsfasman in 1997 (see [12]), where the posets taken into consideration have a finite number of disjoint chains of equal size. This spaces are of special interest since there are several rather disparate applications, as noted by Rosenbloom and Tsfasman (see [12]) and Park e Barg (see [11]).

In [7], [3] and [10] the groups of linear isometries of poset metrics were determined for the Rosenbloom-Tsfasman space, crown space and arbitrary poset-space respectively. In [9] we describe the full isometry group (which includes non-linear isometries) of a poset metric that is a product of Rosenbloom-Tsfasman spaces and in [6] the author studied the full isometry group to any poset metric. The full description of the group of linear isometries of a poset block space were determined by Alves, Panek and Firer in [1].

In this work, we describe the group of isometries (not necessarily linear ones) of the poset block space whose underlying poset is a finite union of disjoint chains of same length. We call this space the *Niederreiter-Rosenbloom-Tsfasman block space* (or *NRT block space*, for short).

2 POSET BLOCK METRIC SPACE

Let $[n] := \{1,2,\ldots,n\}$ be a finite set with n elements and let $\leq$ be a partial order on $[n]$. We call the pair $P := ([n],\leq)$ a *poset* and say that k is *smaller than* j if $k \leq j$ and $k \neq j$. An *ideal* in $([n],\leq)$ is a subset $I \subseteq [n]$ that contains every element that is smaller than some of its elements, i.e., if $j \in I$ and $k \leq j$, then $k \in I$. Given a subset $X \subseteq [n]$, we denote by $\langle X \rangle$ the smallest ideal containing X, called the *ideal generated by* X. An order on the finite set $[n]$ is called a *linear order* or a *chain* if any two elements are comparable, that is, given $i, j \in [n]$ we have that either $i \leq j$ or $j \leq i$. In this case, n is said to be *the length* of the chain and the set can be labeled in such a way that $i_1 < i_2 < \ldots < i_n$. For the simplicity of the notation, in this situation we will always assume that the order P is defined as $1 < 2 < \ldots < n$.

Let q be a power of a prime, $\mathbb{F}_q$ be the finite field of q elements and $V := \mathbb{F}_q^N$ the N-dimensional vector space of N-tuples over $\mathbb{F}_q$. Let $\pi = (k_1,k_2,\ldots,k_n)$ be a partition of N, that is,

$$N = k_1 + k_2 + \ldots + k_n,$$

with $k_i > 0$ an integer. For each integer k_i, let $V_i := \mathbb{F}_q^{k_i}$ be the k_i-dimensional vector space over the finite field $\mathbb{F}_q$ and define

$$V = V_1 \times V_2 \times \ldots \times V_n,$$

called the *π-direct product decomposition* of V. A vector $v \in V$ can be uniquely decomposed as

$$v = (v_1, v_2, \ldots, v_n),$$

with $v_i \in V_i$ for each $1 \leq i \leq n$. We will call this the *π-direct product decomposition* of v. Given a poset $P = ([n], \leq)$, we define the *poset block weight* $\omega_{(P,\pi)}(v)$ (or simply the (P,π)*-weight*) of a vector $v = (v_1, v_2, \ldots, v_n)$ to be

$$\omega_{(P,\pi)}(v) :=\mid \langle supp(v) \rangle \mid$$

where $supp(v) := \{i \in [n] : v_i \neq 0\}$ is the *π-support* of the vector v and $|X|$ is the cardinality of the set X. The block structure is said to be *trivial* when $k_i = 1$, for all $1 \leq i \leq n$. The (P,π)-weight induces a metric $d_{(P,\pi)}$ on V, that we call the *poset block metric* (or simply (P,π)*-metric*):

$$d_{(P,\pi)}(u, v) := \omega_{(P,\pi)}(u - v).$$

The pair $(V, d_{(P,\pi)})$ is a metric space and where no ambiguity may rise, we say it is a *poset block space*, or simply a (P,π)*-space*.

An *isometry* of $(V, d_{(P,\pi)})$ is a bijection $T : V \to V$ that preserves distance, that is,

$$d_{(P,\pi)}(T(u), T(v)) = d_{(P,\pi)}(u, v),$$

for all $u, v \in V$. The set $Isom(V, d_{(P,\pi)})$ of all isometries of $(V, d_{(P,\pi)})$ is a group with the natural operation of composition of functions, and we call it the *isometry group* of $(V, d_{(P,\pi)})$. An *automorphism* is a linear isometry.

In [9] the group of isometries of a product of Niederreiter-Rosenbloom-Tsfasman spaces is characterized. In [6] is studied a subgroup of the full isometry group for any given poset. In this work, we will describe the full isometry group of an important class of poset block spaces, namely, those induced by posets that are an union of disjoint chains of the same length. This class includes the block metric spaces over chains and the Niederreiter-Rosembloom-Tsfasman spaces with trivial block structures.

We remark that the initial idea is the same as in [9]. The main differences are that we follow a more coordinate free approach an that the dimensions of the blocks pose a new restraint. We first study the isometry group of NRT block space induced by one simple chain (Theorem 1), analogous to those of [9]. In this work, we prove some results on isometries, also anologous to those of [9], plus a result on preservation of block dimensions (Lemma 4), and conclude that $Isom(V, d_{(P,\pi)})$ is the semi-direct product of the direct product of the isometry groups induced by each chain and the automorphism group of the permutations of chains that preserves the block dimensions (Theorem 6).

3 ISOMETRIES OF LINEAR ORDERED BLOCK SPACE

Let $P = ([n], \leq)$ be the linear order $1 < 2 < \ldots < n$, let $\pi = (k_1, k_2, \ldots, k_n)$ be a partition of N and let

$$V = V_1 \times V_2 \times \ldots \times V_n,$$

where $V_i = \mathbb{F}_q^{k_i}$, for $i = 1, 2, \ldots, n$, be the π-direct product decomposition of the vector space $V = \mathbb{F}_q^N$ endow with the poset block metric $d_{(P,\pi)}$. In this section we will describe the full isometry group of the poset block space $(V, d_{(P,\pi)})$. This description will be used in the next section to describe the isometry group of the NRT block space. In this section, $P = ([n], \leq)$ will be the linear order $1 < 2 < \ldots < n$.

We note that, given $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n)$ in the total ordered block space V,

$$d_{(P,\pi)}(u, v) = \max\{i : u_i \neq v_i\}.$$

For each $i \in \{1, 2, \ldots, n\}$, let

$$F_i : V_i \times V_{i+1} \times \ldots \times V_n \to V_i$$

be a map that is a bijection with respect to the first block space V_i, that is, given $(v_{i+1}, \ldots, v_n) \in V_{i+1} \times \ldots \times V_n$, the map $\widetilde{F}_{v_{i+1},\ldots,v_n} : V_i \to V_i$ defined by

$$\widetilde{F}_{v_{i+1},\ldots,v_n}(v_i) = F_i(v_i, v_{i+1}, \ldots, v_n)$$

is a bijection. Let $S_{q,\pi,i}$ be the set of such maps F_i. Given $F_i \in S_{q,\pi,i}$, with $1 \leq i \leq n$, we define a map $T_{(F_1,F_2,\ldots,F_n)} : V \to V$ by

$$T_{(F_1,F_2,\ldots,F_n)}(v_1, \ldots, v_n) := (F_1(v_1, \ldots, v_n), F_2(v_2, \ldots, v_n), \ldots, F_n(v_n)).$$

Theorem 1. *Let $P = ([n], \leq)$ be the linear order $1 < 2 < \ldots < n$ and let $V = V_1 \times V_2 \times \ldots \times V_n$ be the π-direct product decomposition of $V = \mathbb{F}_q^N$ endowed with the poset block metric induced by the poset P and the partition π. Then, the group $Isom(V, d_{(P,\pi)})$ of isometries of $(V, d_{(P,\pi)})$ is the set of all maps $T_{(F_1,F_2,\ldots,F_n)} : V \to V$.*

Proof. Given $u = (u_1, \ldots, u_n)$ and $v = (v_1, \ldots, v_n) \in V$, let $l = d_{(P,\pi)}(u, v) = \max\{i : u_i \neq v_i\}$. Since each $F_i : V_i \times V_{i+1} \times \ldots \times V_n \to V_i$ is a bijection in relation to the first block space V_i, it follows that

$$F_l(u_l, u_{l+1}, \ldots, u_n) \neq F_l(v_l, v_{l+1}, \ldots, v_n)$$

and

$$F_t(u_t, u_{t+1}, \ldots, u_n) = F_t(v_t, v_{t+1}, \ldots, v_n)$$

for any $t > l$. It follows that

$$d_{(P,\pi)}\left(T_{(F_1,\ldots,F_n)}(u), T_{(F_1,\ldots,F_n)}(v)\right) = \max\{i : F_i(u_i, \ldots, u_n) \neq F_i(v_i, \ldots, v_n)\} = l$$

and hence $T_{(F_1,F_2,\ldots,F_n)}$ is distance preserving. Since V is a finite metric space, it follows that $T_{(F_1,F_2,\ldots,F_n)}$ is also a bijection.

Now let T be an isometry of V. Let us write

$$T(v_1,v_2,\ldots,v_n) = (T_1(v_1,v_2,\ldots,v_n),\ldots,T_n(v_1,v_2,\ldots,v_n)).$$

We prove first that $T_j(v_1,v_2,\ldots,v_n) = F_j\left(v_j,v_{j+1},\ldots,v_n\right)$, that is, T_j does not depend on the first $j-1$ coordinates. In other words, we want to prove that

$$T_j\left(v_1,\ldots,v_{j-1},v_j,\ldots,v_n\right) = T_j\left(u_1,\ldots,u_{j-1},v_j,\ldots,v_n\right)$$

regardless of the values of the first $j-1$ coordinates. Since

$$\begin{aligned} d_{(P,\pi)}\left(\left(u_1,\ldots,u_{j-1},v_j,\ldots,v_n\right),\left(v_1,\ldots,v_{j-1},v_j,\ldots,v_n\right)\right) &= \max_i \{i : v_i \neq u_i\} \\ &\leq j-1 \end{aligned}$$

and since T is an isometry, it follows that

$$\begin{aligned} &d_{(P,\pi)}\left(T\left(v_1,\ldots,v_{j-1},v_j,\ldots,v_n\right),T\left(u_1,\ldots,u_{j-1},v_j,\ldots,v_n\right)\right) = \\ &= d_{(P,\pi)}\left(\left(u_1,\ldots,u_{j-1},v_j,\ldots,v_n\right),\left(v_1,\ldots,v_{j-1},v_j,\ldots,v_n\right)\right) \leq j-1, \end{aligned}$$

and so,

$$T_j\left(v_1,\ldots,v_{j-1},v_j,\ldots,v_n\right) = T_j\left(u_1,\ldots,u_{j-1},v_j,\ldots,v_n\right),$$

for any $(v_1,\ldots,v_{j-1}),(u_1,\ldots,u_{j-1}) \in V_1\times\ldots\times V_{j-1}$ and $(v_j,\ldots,v_n) \in V_j\times\ldots\times V_n$. Thus,

$$T(v_1,v_2,\ldots,v_n) = (F_1(v_1,v_2,\ldots,v_n),F_2(v_2,\ldots,v_n),\ldots,F_n(v_n))$$

and the first statement is proved. Now, we need to prove that each $\widetilde{F}_{v_{i+1},\ldots,v_n}$ is a bijection, what is equivalent to prove those maps are injective. If $\widetilde{F}_{v_{i+1},\ldots,v_n}$ is not injective, then there are $v_i \neq u_i$ in V_i such that

$$\widetilde{F}_{v_{i+1},\ldots,v_n}(v_i) = \widetilde{F}_{v_{i+1},\ldots,v_n}(u_i).$$

Considering i minimal with this property, it follows that

$$\begin{aligned} i &= d_{(P,\pi)}((v_1,\ldots,v_i,\ldots,v_n),(v_1,\ldots,u_i,\ldots,v_n)) \\ &= d_{(P,\pi)}(T(v_1,\ldots,v_i,\ldots,v_n),T(v_1,\ldots,u_i,\ldots,v_n)) < i \end{aligned}$$

contradicting the assumption that T is an isometry of $(V,d_{(P,\pi)})$. $\square$

Let S_m be the symmetric group of permutations of a set with m elements and $V = V_1\times V_2\times\ldots\times V_n$ be the π-direct product decomposition of $V = \mathbb{F}_q^N$ with $\pi = (k_1,k_2,\ldots,k_n)$. Since V has q^N elements we can identify the group $S_{q,\pi,1}$ of functions $F : V_1\times V_2\times\ldots\times V_n \to V_1$ such that $\widetilde{F}_{v_2,\ldots,v_n}$ is a permutation of $V_1 = \mathbb{F}_q^{k_1}$, with operation

$$(F\cdot G)(v) := F(G(v_1,v_2,\ldots,v_n),v_2,\ldots,v_n),$$

with $F,G \in S_{q,\pi,1}$ and $v = (v_1, v_2, \ldots, v_n) \in V$, with the direct product[1] $\left(S_{q^{k_1}}\right)^{q^{N-k_1}}$. With this notations, it follows the following result.

Theorem 2. *Let $P = ([n], \leq)$ be the linear order $1 < 2 < \ldots < n$ and let $V = V_1 \times V_2 \times \ldots \times V_n$ be the π-direct product decomposition of $V = \mathbb{F}_q^N$ endowed with the poset block metric induced by the poset P and the partition π. If $\pi = (k_1, k_2, \ldots, k_n)$, then the group of isometries $Isom(V, d_{(P,\pi)})$ has a semi-direct product[2] structure given by*

$$(S_{q^{k_1}})^{q^{N-k_1}} \rtimes \left(\ldots \left((S_{q^{k_{n-1}}})^{q^{N-k_1-k_2-\ldots-k_{n-1}}} \rtimes (S_{q^{k_n}})^{q^{N-k_1-k_2-\ldots-k_{n-1}-k_n}}\right)\ldots\right).$$

Proof. Let $G_{(\widehat{P},\widehat{\pi})}$ be the isometry group $Isom(\widehat{V}, d_{(\widehat{P},\widehat{\pi})})$ of

$$\widehat{V} = \widehat{V}_1 \times \widehat{V}_2 \times \ldots \times \widehat{V}_{n-1},$$

where $\widehat{V}_i = V_{i+1}$ for each $i = 1, 2, \ldots, n-1$, $\widehat{P} = ([n-1], \leq)$ is the linear order $1 < 2 < \ldots < n-1$ and $\widehat{\pi} = (k_2, k_3, \ldots, k_n)$. Let

$$H = \{T \in Isom(V, d_{(P,\pi)}) : T = (F_1, \mathrm{Pr}_{V_2}, \ldots, \mathrm{Pr}_{V_n}) \text{ with } F_1 \in S_{q,\pi,1}\}$$

and

$$K = \{T \in Isom(V, d_{(P,\pi)}) : T = (\mathrm{Pr}_{V_1}, F_2, \ldots, F_n) \text{ with } F_i \in S_{q,\pi,i}\},$$

where each $\mathrm{Pr}_{V_i} : V_i \times V_{i+1} \times \ldots \times V_n \to V_i$ is the projection map given by $\mathrm{Pr}_{V_i}(v_i, v_{i+1}, \ldots, v_n) = v_i$. We claim that $Isom(V, d_{(P,\pi)})$ is a semi-direct product of H by K. Clearly, $Isom(V, d_{(P,\pi)}) = HK$, because each isometry of $(V, d_{(P,\pi)})$ is a composition $T_1 \circ T_2$ with $T_1 \in H$ and $T_2 \in K$. Let $L \in H \cap K$. Since $L \in H$, $L(x_1, x_2, \ldots, x_{n+1}) = (x_1', x_2, \ldots, x_{n+1})$ and, since L is also in K, it follows that $x_1' = x_1$. Hence, $L = id_V$ and the groups H and K intersect trivially. Now, we prove that H is a normal subgroup of $Isom(V, d_{(P,\pi)})$. In fact, since $Isom(V, d_{(P,\pi)}) = HK$, it suffices to check that $THT^{-1} \subset H$ for each $T \in K$. Let $L \in H$ and $T \in K$. Then $L(x_1, \ldots, x_n) = (F_1(x_1, \ldots, x_n), x_2, \ldots, x_n)$ and $T(x_1, \ldots, x_n) = (x_1, \widetilde{T}(x_2, \ldots, x_n))$ for some $F_1 \in S_{q,\pi,1}$ and $\widetilde{T} \in Isom(\widehat{V}, d_{(\widehat{P},\widehat{\pi})})$. If $(x_1, \ldots, x_n) \in V$, then

$$\begin{aligned}
\left(T \circ L \circ T^{-1}\right)(x_1, \ldots, x_n) &= (T \circ L)\left(x_1, \widetilde{T}^{-1}(x_2, \ldots, x_n)\right) \\
&= T\left(F_1\left(x_1, \widetilde{T}^{-1}(x_2, \ldots, x_n)\right), \widetilde{T}^{-1}(x_2, \ldots, x_n)\right) \\
&= \left(F_1\left(x_1, \widetilde{T}^{-1}(x_2, \ldots, x_n)\right), \widetilde{T} \circ \widetilde{T}^{-1}(x_2, \ldots, x_n)\right) \\
&= \left(F_1\left(x_1, \widetilde{T}^{-1}(x_2, \ldots, x_n)\right), x_2, \ldots, x_n\right).
\end{aligned}$$

[1] If $H_1, \ldots, H_l$ are groups, then their *direct product*, denoted by $H_1 \times \ldots \times H_l$, is the group with elements $(h_1, \ldots, h_l)$, $h_i \in H_i$ for each $1 \leq i \leq l$, and with operation $(h_1, \ldots, h_l)(h_1', \ldots, h_l') = (h_1 h_1', \ldots, h_l h_l')$.

[2] Let G be a group with identity 1_G and let N_1 and Q_1 be subgroups of G. We recall that the group G is a *semi-direct product* of N by Q (see [13], p. 167), denoted by $G = N \rtimes Q$, if $N \cong N_1$, $Q \cong Q_1$, $N_1 \cap Q_1 = \{1_G\}$, $N_1 Q_1 = G$ and N_1 is a normal subgroup of G.

Since F_1 is a bijection with respect to the first block space V_1, it follows that $TLT^{-1} \in H$. This shows that H is a normal subgroup of $Isom(V, d_{(P,\pi)})$ and that

$$Isom(V, d_{(P,\pi)}) = H \rtimes K.$$

In order to simplify notation, we will denote the elements of $\left(S_{q^{k_1}}\right)^{q^{N-k_1}}$ by (π_X), where

$$(\pi_X) := (\pi_X)_{X \in \mathbb{F}_q^{N-k_1}}.$$

The group $G_{(\widehat{P},\widehat{\pi})}$ acts on $V = V_1 \times \widehat{V}$ by

$$T(x_1, \ldots, x_n) = (x_1, T(x_2, \ldots, x_n))$$

and $(S_{q^{k_1}})^{q^{N-k_1}}$ acts by

$$(\pi_X)(x_1, \ldots, x_n) = \left(\pi_{(x_2,\ldots,x_n)}(x_1), x_2, \ldots, x_n\right).$$

Both groups act as groups of isometries and both act faithfully. Therefore these actions establish isomorphisms of these groups with subgroups H and K: $H \cong (S_{q^{k_1}})^{q^{N-k_1}}$ and $K \cong G_{(\widehat{P},\widehat{\pi})}$. Using the aforementioned isomorphisms involving H and K, it follows that

$$Isom(V, d_{(P,\pi)}) = (S_{q^{k_1}})^{q^{N-k_1}} \rtimes G_{(\widehat{P},\widehat{\pi})},$$

which concludes the proof. □

Corollary 3. *Let $P = ([n], \leq)$ be the linear order $1 < 2 < \ldots < n$ and let $V = V_1 \times V_2 \times \ldots \times V_n$ be the π-direct product decomposition of $V = \mathbb{F}_q^N$ endowed with the poset block metric induced by the poset P and the partition π. If*

$$H = \{T \in Isom(V, d_{(P,\pi)}) : T = (F_1, \mathrm{Pr}_{V_2}, \ldots, \mathrm{Pr}_{V_n}) \text{ with } F_1 \in S_{q,\pi,1}\}$$

and

$$K = \{T \in Isom(V, d_{(P,\pi)}) : T = (\mathrm{Pr}_{V_1}, F_2, \ldots, F_n) \text{ with } F_i \in S_{q,\pi,i}\},$$

then[3]

$$Isom(V, d_{(P,\pi)}) \cong H \rtimes_\theta K$$

with $\theta : K \to Aut(H)$ given by

$$\theta_T(L)(x_1, x_2, \ldots, x_n) = \left(F_1\left(x_1, \widetilde{T}^{-1}(x_2, \ldots, x_n)\right), x_2, \ldots, x_n\right)$$

for all $L = (F_1, \mathrm{Pr}_{V_2}, \ldots, \mathrm{Pr}_{V_n}) \in H$, $T = (\mathrm{Pr}_{V_1}, F_2, \ldots, F_n) \in K$ and $(x_1, \ldots, x_n) \in V$ with $\widetilde{T} := (F_2, \ldots, F_n)$.

[3]Given groups Q and N and a homomorphism $\theta : Q \to Aut(N)$, then $N \times Q$ equipped with the operation $(a,x)(b,y) := (a\theta_x(b), xy)$ is a semi-direct product of N by Q (see [13], Theorem 7.22), denoted by $N \rtimes_\theta Q$. If $G = N \rtimes Q$ and $\theta_x(a) = xax^{-1}$, for all $x \in Q$ and $a \in N$, then $G \cong N \rtimes_\theta Q$ (see [13], Theorem 7.23).

Corollary 4. *Let $P = ([n], \leq)$ be the linear order $1 < 2 < \ldots < n$ and let $V = V_1 \times V_2 \times \ldots \times V_n$ be the π-direct product decomposition of $V = \mathbb{F}_q^N$ endowed with the poset block metric induced by the poset P and the partition $\pi = (k_1, k_2, \ldots, k_n)$. Then*

$$|Isom(V, d_{(P,\pi)})| = (q^{k_1}!)^{q^{N-k_1}} \cdot (q^{k_2}!)^{q^{N-k_1-k_2}} \cdot \ldots \cdot (q^{k_{n-1}}!)^q \cdot (q^{k_n}!).$$

Now, if the partition $\pi = (1, 1, \ldots, 1)$, it follows the following result (see [9], Corollary 3.1):

Corollary 5. *Let $P = ([n], \leq)$ be the linear order $1 < 2 < \ldots < n$ and let $V = \mathbb{F}_q^n$ be the vector space endowed with the poset metric induced by the poset P. Then the group of isometries $Isom(V, d_P)$ is a semi-direct product*

$$(S_q)^{q^{n-1}} \rtimes (\ldots((S_q)^q \rtimes S_q)\ldots).$$

In particular,

$$|Isom(V, d_P)| = (q!)^{\frac{q^n-1}{q-1}}.$$

4 ISOMETRIES OF NRT BLOCK SPACE

In this section, we consider an order $P = ([m \cdot n], \leq)$, that is, the union of m disjoint chains $P_1, P_2, \ldots, P_m$ of order n. We identify the elements of $[m \cdot n]$ with the set of ordered pairs of integers (i, j), with $1 \leq i \leq m,\ 1 \leq j \leq n$, where $(i, j) \leq (k, l)$ *iff* $i = k$ and $j \leq_{\mathbb{N}} l$, where $\leq_{\mathbb{N}}$ is just the usual order on $\mathbb{N}$. We denote $P_i = \{(i, j) : 1 \leq j \leq n\}$. Each P_i is a chain and those are the connected components of $([m \cdot n], \leq)$.

Let $\pi = (k_{11}, \ldots, k_{1n}, \ldots, k_{m1}, \ldots, k_{mn})$ be a partition of $N = mn$ and for each $1 \leq i \leq m$ let $\pi_i = (k_{i1}, \ldots, k_{in})$. Let

$$V = U_1 \times U_2 \times \ldots \times U_m, \tag{4.1}$$

where

$$U_i := V_{i1} \times V_{i2} \times \ldots \times V_{in}$$

and $V_{ij} = \mathbb{F}_q^{k_{ij}}$, for all $1 \leq i \leq m,\ 1 \leq j \leq n$. The space V with the poset metric induced by the order $P = ([m \cdot n], \leq)$ is called the (m, n, π)-*NRT block space*. Note that if $n = 1$, then $P = ([m \cdot 1], \leq)$ induces just the error-block metric on V, and in particular, if $\pi = (1, 1, \ldots, 1)$, then $P = ([m \cdot 1], \leq)$ induces just the Hamming metric on $\mathbb{F}_q^m$. Hence the induced metric from the poset $P = ([m \cdot n], \leq)$ can be viewed as a generalization of the error-block metric.

Let $V = U_1 \times U_2 \times \ldots \times U_m$ as in (4.1), called the *canonical decomposition of* V. Given the canonical decompositions $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_m)$ with $u_i, v_i \in U_i$, we have that

$$d_{(P,\pi)}(u, v) = \sum_{i=1}^{m} d_{(P_i,\pi_i)}(u_i, v_i),$$

where $d_{(P_i,\pi_i)}$, the restriction of $d_{(P,\pi)}$ to U_i, is a linear poset block metric. We note that the restriction of $d_{(P,\pi)}$ to each U_i turns it into a poset space defined by a linear order, that is, each U_i is isometric to $(U_i, d_{([n],\pi_i)})$ with the metric $d_{([n],\pi_i)}$ determined by the chain $1 < 2 < \ldots < n$. Let $G_{i,\pi_i,n}$ be the group of isometries of $(U_i, d_{([n],\pi_i)})$. The direct product $\prod_{i=1}^{m} G_{i,\pi_i,n}$ acts on V in the following manner: given $T = (T_1, \ldots, T_m) \in \prod_{i=1}^{m} G_{i,\pi_i,n}$ and $v = (v_1, \ldots, v_m) \in V$,

$$T(v) := (T_1(v_1), \ldots, T_m(v_m)).$$

Lemma 1. *Let $(V, d_{(P,\pi)})$ be the (m,n,π)-NRT block space over $\mathbb{F}_q$ and let $G_{i,\pi_i,n}$ be the group of isometries of $\left(U_i, d_{([n],\pi_i)}\right)$. Given $T_i \in G_{i,\pi_i,n}$, with $1 \le i \le m$, the map $T = (T_1, \ldots, T_m)$ defined by*

$$T(v) := (T_1(v_1), \ldots, T_m(v_m))$$

is an isometry of $(V, d_{(P,\pi)})$.

Proof. Given $u, v \in V$, consider the canonical decompositions $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_m)$ with $u_i, v_i \in U_i$. Then,

$$\begin{aligned} d_{(P,\pi)}(T(u), T(v)) &= \sum_{i=1}^{m} d_{(P_i,\pi_i)}(T_i(u_i), T_i(v_i)) \\ &= \sum_{i=1}^{m} d_{(P_i,\pi_i)}(u_i, v_i) \\ &= d_{(P,\pi)}(u, v), \end{aligned}$$

which concludes the proof. □

Let S_m be the permutation group of $\{1, 2, \ldots, m\}$. We will call a permutation $\sigma \in S_m$ *admissible* if $\sigma(i) = j$ implies that $k_{il} = k_{jl}$, for all $1 \le l \le n$. Cleary, the set S_π of all admissible permutations is a subgroup of S_m.

Let us consider the canonical decomposition $v = (v_1, \ldots, v_m)$ of a vector v in the (m,n,π)-NRT block space V. The group S_π acts on V as a group of isometries: given $\sigma \in S_\pi$ and $v = (v_1, \ldots, v_m) \in V$, we define

$$T_\sigma(v) = (v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(m)}).$$

Lemma 2. *Let $(V, d_{(P,\pi)})$ be the (m,n,π)-NRT block space V and let $\sigma \in S_\pi$. Then T_σ is an isometry of $(V, d_{(P,\pi)})$.*

Proof. Given $u, v \in V$, we consider their canonical decompositions $u = (u_1, \ldots, u_m)$ and $v = (v_1, \ldots, v_m)$ with $u_i, v_i \in U_i$. Then,

$$\begin{aligned} d_{(P,\pi)}\left(T_\sigma(u),T_\sigma(v)\right) &= \sum_{i=1}^{m} d_{(P_i,\pi_i)}\left(u_{\sigma(i)},v_{\sigma(i)}\right) \\ &= \sum_{i=1}^{m} d_{(P_i,\pi_i)}\left(u_i,v_i\right) \\ &= d_{(P,\pi)}\left(u,v\right), \end{aligned}$$

which concludes the proof. □

The Lemmas 1 and 2 assure that the groups $\prod_{i=1}^{m} G_{i,\pi_i,n}$ and S_π are both isometry groups of the (m,n,π)-NRT block space V, and so is the group $G_{(m,n,\pi)}$ generated by both of them. We identify $\prod_{i=1}^{m} G_{i,\pi_i,n}$ and S_π with their images in $G_{(m,n,\pi)}$ and make an abuse of notation, denoting the images in $G_{(m,n,\pi)}$ by the same symbols. With this notation, analogous calculations as those of Theorem 2 show that

$$\left(\prod_{i=1}^{m} G_{i,\pi_i,n}\right) \cap S_\pi = \{id_V\}$$

and

$$\sigma \circ \left(\prod_{i=1}^{m} G_{i,\pi_i,n}\right) \circ \sigma^{-1} = \prod_{i=1}^{m} G_{i,\pi_i,n},$$

for every $\sigma \in S_\pi$. Since $\prod_{i=1}^{m} G_{i,\pi_i,n}$ is normal in $G_{(m,n,\pi)}$ and $G_{(m,n,\pi)}$ is generated by $\prod_{i=1}^{m} G_{i,\pi_i,n}$ and S_π, it follows that

$$G_{(m,n,\pi)} = \left(\prod_{i=1}^{m} G_{i,\pi_i,n}\right) \cdot S_\pi,$$

and therefore, it follows the following proposition:

Proposition 3. *The group $G_{(m,n,\pi)}$ has the structure of a semi-direct product given by*

$$\left(\prod_{i=1}^{m} G_{i,\pi_i,n}\right) \rtimes S_\pi.$$

We need two more lemmas in order to prove that every isometry of the (m,n,π)-NRT block space V is in $G_{(m,n,\pi)}$, i.e., that $G_{(m,n,\pi)}$ is the group of isometries of V. We will identify the block space U_i of V with the subspace of V of vectors $v=(v_1,\dots,v_m)$ such that $v_j=0$ for $j\neq i$.

Lemma 4. *Let $(V,d_{(P,\pi)})$ be the (m,n,π)-NRT block space and let $V=U_1\times U_2\times\cdots\times U_m$ be the canonical decomposition of V. If*

$$\pi = (k_{11},\dots,k_{1n},\dots,k_{m1},\dots,k_{mn})$$

and $T:V\to V$ is an isometry such that $T(0)=0$, then for each index $1\le i\le m$ there is another index $1\le j\le m$ such that

$$T(U_i)=U_j$$

and

$$k_{il} = \dim(V_{il}) = \dim(V_{jl}) = k_{jl},$$

for all $1 \leq l \leq n$.

Proof. In the following we denote the subspace $V_{i1} \times V_{i2} \times \ldots \times V_{ik}$ by U_{ik}. We begin by showing that for each index $1 \leq i \leq m$ there is another index $1 \leq j \leq m$ such that $T(U_{i1}) = U_{j1}$ and $k_{i1} = k_{j1}$. Let $v_i \in U_{i1}$, with $v_i \neq 0$. Since

$$d_{(P,\pi)}(T(v_i),0) = d_{(P,\pi)}(v_i,0) = 1,$$

it follows that $v_j = T(v_i)$ is a vector of (P,π)-weight 1. Thus $T(v_i) \in U_{j1}$ for some index $1 \leq j \leq m$. If $v'_i \in U_{i1}$, with $v'_i \neq v_i$ and $v'_i \neq 0$, then $T(v'_i) = v_k$ for some $v_k \in U_{k1}$ with $v_k \neq 0$, but also

$$d_{(P,\pi)}(T(v_i),T(v'_i)) = d_{(P,\pi)}(v_i,v'_i) = 1.$$

If $k \neq j$, then $d_{(P,\pi)}(T(v_i),T(v'_i)) = d_{(P,\pi)}(v_j,v_k) = 2$. Hence $k = j$ and $T(U_{i1}) \subseteq U_{j1}$. Now apply the same reasoning to T^{-1}. If $v_i \in U_{i1}$, with $v_i \neq 0$, and $T(v_i) = v_j$ with $v_j \in U_{j1}$, then $T^{-1}(v_j) \in U_{i1}$ and therefore $T^{-1}(U_{j1}) \subseteq U_{i1}$. So that $U_{j1} \subseteq T(U_{i1})$. Therefore $T(U_{i1}) = U_{j1}$. Since T is bijective, it follows that $k_{i1} = k_{j1}$. For induction on k, suppose that for each s there exists an index l such that

$$T(U_{sk}) = U_{lk}$$

and $k_{sj} = k_{lj}$ for all $1 \leq j \leq k$ and for all $1 \leq k \leq n$. We note that $U_{sn} = U_s$. Without loss of generality, let us consider $s = 1$, $P_1 = \{(1,1),\ldots,(1,n)\}$. Let P_l be the chain that begins at $(l,1)$ such that $T(U_{11}) = U_{l1}$ and suppose that $U_{1(k-1)}$ is taken by T onto $U_{l(k-1)}$ with $k_{1j} = k_{lj}$ for all $1 \leq j \leq k-1$. Let $v = (v_{11},\ldots,v_{1k})$, $v_{1i} \in V_{1i}$ with $v_{1k} \neq 0$, and let $T(v) = (u_1,\ldots,u_m)$, $u_i \in U_i$. Since $T(0) = 0$, it follows that

$$\omega_{(P,\pi)}(v) = \omega_{(P,\pi)}(T(v)) = \omega_{(P,\pi)}(u_1) + \ldots + \omega_{(P,\pi)}(u_m).$$

We will use this to show that $T(v) = u_l$. First suppose that $u_l = 0$. In this case, $\omega_{(P,\pi)}(v) = \sum_{j \neq l} \omega_{(P,\pi)}(u_j)$ and therefore, if $u_{11} \in U_{11}$, with $u_{11} \neq 0$ and $T(u_{11}) = u_{l1}$, then

$$k = d_{(P,\pi)}(u_{11},v) = d_{(P,\pi)}(T(u_{11}),T(v)) = \sum_{j \neq l} \omega_{(P,\pi)}(u_j) + \omega_{(P,\pi)}(u_{l1}) = k+1,$$

a contradiction. Hence $u_l \neq 0$. Let $u_l = (u_{l1},\ldots,u_{lt})$, $u_{li} \in V_{li}$, and suppose now there is another summand $u_i \neq 0$. Then $k = \sum_j \omega_{(P,\pi)}(u_j) > \omega_{(P,\pi)}(u_l)$ and therefore $t < k$. By the induction hypothesis, it follows that $T^{-1}(u_l)$ is a vector in $V_{1(k-1)}$ with $\omega_{(P,\pi)}(T^{-1}(u_l)) < k$. Hence

$$k = d_{(P,\pi)}(T^{-1}(u_l),v) = d_{(P,\pi)}(u_l,T(v)) = \sum_{j \neq l} \omega_{(P,\pi)}(u_j) < k,$$

again a contradiction. Hence, $T(v) \in U_{lk}$. From the induction hypothesis and from the fact that T is a weight-preserving bijection, it follows that

$$T(v_{11},\ldots,v_{1k}) = (u_{l1},\ldots,u_{lk}),$$

where $v_{1k} \neq 0$ implies $u_{lk} \neq 0$. Therefore, $T(U_{1k}) = U_{lk}$. Since $k_{1j} = k_{lj}$ for all $1 \leq j \leq k-1$ and T is a bijection, it follows that $k_{1k} = k_{lk}$. Hence $T(U_1) = U_l$ with $k_{1j} = k_{lj}$ for all $1 \leq j \leq n$. $\square$

We recall that we defined an action of the group S_π of the admissible permutations of S_m on the canonical decomposition $U_1 \times U_2 \times \cdots \times U_m$ of V by

$$T_\sigma(v) := (v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(m)})$$

and that we defined an action of $\prod_{i=1}^m G_{i,\pi_i,n}$ on V by

$$(g_1, g_2, \ldots, g_m)(v_1, v_2, \ldots, v_m) = (g_1(v_1), \ldots, g_m(v_m)).$$

Lemma 5. *Let $(V, d_{(P,\pi)})$ be the (m,n,π)-NRT block space. Each isometry of V that preserves the origin is a product $T_\sigma \circ g$, with σ in S_π and g in $\prod_{i=1}^m G_{i,\pi_i,n}$.*

Proof. Let T be an isometry of V, with $T(0) = 0$. By the Lemma 4, for each $1 \leq i \leq m$ there is a $\sigma(i)$ such that $T(U_i) = U_{\sigma(i)}$ with $k_{il} = k_{\sigma(i)l}$ for all $1 \leq l \leq n$. Since T is a bijection, it follows that the map $i \mapsto \sigma(i)$ is an admissible permutation of the set $\{1, \ldots, m\}$. We define $T_\sigma : V \to V$ by

$$T_\sigma(v) := (v_{\sigma(1)}, v_{\sigma(2)}, \ldots, v_{\sigma(m)}).$$

Thus, $T = T_\sigma^{-1}(T_\sigma T) = T_{\sigma^{-1}}(T_\sigma T)$, where $\sigma \in S_\pi$. Let $g = T_\sigma T$. Since $g(U_i) = (T_\sigma T)(U_i) = T_\sigma(U_{\sigma(i)}) = U_{\sigma^{-1}\sigma(i)} = U_i$, we have that $g|_{U_i}$ is an isometry of U_i. Defining $g_i := g|_{U_i}$ it follows that $g = (g_1, \ldots, g_m)$, and hence, $g \in \prod_{i=1}^m G_{i,\pi_i,n}$. □

Theorem 6. *Let $(V, d_{(P,\pi)})$ be the (m,n,π)-NRT block space. The group of isometries of V is isomorphic to*

$$\left(\prod_{i=1}^{m} G_{i,\pi_i,n} \right) \rtimes S_\pi.$$

Proof. Let $G_{(m,n,\pi)}$ be the group of isometries of V generated by the action of $\prod_{i=1}^m G_{i,\pi_i,n}$ and S_π. Let T be an isometry of V and let $v = T(0)$. The translation $S_{-v}(u) := u - v$ is clearly an isometry of V and $(S_{-v} \circ T)(0) = S_{-v}(v) = 0$ is an isometry that fixes the origin. Hence, by the previous lemma, it follows that $S_{-v} \circ T \in G_{(m,n,\pi)}$. Consider the canonical decomposition of v on the chain spaces, $v = (v_1, \ldots, v_m)$, $v_i \in U_i$. Since the restriction S_{v_i} of $S_v = (S_{v_1}, \ldots, S_{v_m})$ to U_i, with $S_{v_i}(u_i) = u_i + v_i$ for each i, is the translation by v_i, it follows that is an isometry of U_i. Thus, $S_v \in \prod_{i=1}^m G_{i,\pi_i,n} \subset G_{(m,n,\pi)}$ and hence, that $T = S_v \circ (S_{-v} \circ T)$ is in $G_{(m,n,\pi)}$. Thus $G_{(m,n,\pi)}$ is the isometry group of V. By Proposition 3, it follows that $G_{(m,n,\pi)}$ is isomorphic to $(\prod_{i=1}^m G_{i,\pi_i,n}) \rtimes S_\pi$. □

If $n = 1$ (P is an antichain) and $\pi = (k_1, k_2, \ldots, k_m)$, where

$$k_1 = \ldots = k_{m_1} = l_1, \ldots, k_{m_1+\ldots+m_{l-1}+1} = \ldots = k_{m_1+\ldots+m_l} = l_r$$

with $l_1 > \ldots > l_r$ and $m_1, \ldots, m_l$ positive integers such that $m_1 + \ldots + m_l = m$, it follows that $G_{i,(k_i),1} = S_{q^{k_i}}$, for $1 \leq i \leq m$, and $S_\pi = S_{m_1} \times \ldots \times S_{m_l}$ (S_π only permutes those blocks with same dimensions). Therefore it follows the following result.

Corollary 7. *If P is an antichain, then*

$$Isom(V, d_{(P,\pi)}) = \left(\prod_{i=1}^{m} S_{q^{k_i}}\right) \rtimes \left(\prod_{i=1}^{l} S_{m_i}\right).$$

When $n = 1$ and $\pi = (1, 1, \ldots, 1)$, the (P, π)-weight is the usual Hamming weight on $\mathbb{F}_q^m$. In this case, each $G_{i,(1),1}$ in Corollary 7 is equal to S_q and every permutation in S_m is also admissible. Thus, we reobtain the isometry groups of Hamming space:

Corollary 8. *Let d_H be the Hamming metric over $\mathbb{F}_q^m$. The isometry group of $(\mathbb{F}_q^m, d_H)$ is isomorphic to $S_q^m \rtimes S_m$.*

If $\pi = (1, 1, \ldots, 1)$, then every permutation in S_m is admissible. Hence, it follows the following result (see [9], Theorem 4.1):

Corollary 9. *Let $V = \mathbb{F}_q^{mn}$ be the vector space endowed with the poset metric d_P induced by the poset $P = ([mn], \leq)$ which is union of chains $P_1, \ldots, P_m$ of length n. Then*

$$Isom(V, d_P) = (G_n)^m \rtimes S_m,$$

where $G_n := (S_q)^{q^{n-1}} \rtimes (\ldots((S_q)^q \rtimes S_q)\ldots)$. In particular,

$$|Isom(V, d_P)| = (q!)^{m \cdot \frac{q^n - 1}{q-1} + m} \cdot m!.$$

5 AUTOMORPHISMS

The group of automorphisms of $\left(V, d_{(P,\pi)}\right)$ is easily deduced from the Lemma 5 and Theorem 6. Let $T = T_\sigma \circ g$ be a isometry. Since T_σ is linear, it follows that the linearity of T is a matter of whether g is linear or not. Now, if $g = (g_1, g_2, \ldots, g_m)$ is linear, then each component g_i must also be linear. Since each g_i is an isometry, it follows that g_i is in the group $Aut\left(U_i, d_{([n],\pi_i)}\right)$ of linear isometries of $(U_i, d_{([n],\pi_i)})$. Therefore $g \in \prod_{i=1}^{m} Aut\left(U_i, d_{([n],\pi_i)}\right)$. On the other hand, any element of this group is a linear isometry. Hence, it follows the following result:

Theorem 1. *The automorphism group $Aut\left(V, d_{(P,\pi)}\right)$ of $\left(V, d_{(P,\pi)}\right)$ is isomorphic to*

$$\left(\prod_{i=1}^{m} Aut\left(U_i, d_{([n],\pi_i)}\right)\right) \rtimes S_\pi.$$

Corollary 2. *Let $n = 1$ and $\pi = (k_1, k_2, \ldots, k_m)$ be a partition of N. If*

$$k_1 = \ldots = k_{m_1} = l_1, \ldots, k_{m_1 + \ldots + m_{l-1} + 1} = \ldots = k_n = l_r$$

with $l_1 > l_2 > \ldots > l_r$, then

$$\left|Aut\left(\mathbb{F}_q^N, d_{(P,\pi)}\right)\right| = \left(\prod_{i=1}^{m}\left(q^{k_i}-1\right)\left(q^{k_i}-q\right)\ldots\left(q^{k_i}-q^{k_i-1}\right)\right)\cdot\left(\prod_{j=1}^{l} m_j!\right).$$

Proof. Note initially that there is a bijection from $Aut\left(U_i\right)$ and the family of all ordered bases of U_i. Let $\left(e_1, e_2, \ldots, e_{k_i}\right)$ be an ordered basis of U_i. If $T \in Aut\left(V_i\right)$, then $\left(T\left(e_1\right), T\left(e_2\right), \ldots, T\left(e_{k_i}\right)\right)$ is an ordered basis of U_i. If $\left(v_1, v_2, \ldots, v_{k_i}\right)$ is an ordered basis of U_i, then there exists a unique automorphism T with $T\left(e_j\right) = v_j$ for all $j \in \{1, 2, \ldots, k_i\}$. Since the number of ordered basis of U_i is equal to

$$\left(q^{k_i}-1\right)\left(q^{k_i}-q\right)\ldots\left(q^{k_i}-q^{k_i-1}\right)$$

follows that $\left|Aut\left(U_i\right)\right| = \left(q^{k_i}-1\right)\left(q^{k_i}-q\right)\ldots\left(q^{k_i}-q^{k_i-1}\right)$. Since

$$Aut\left(U_i, d_{([n],\pi_i)}\right) = Aut(U_i)$$

for each i, from Theorem 1

$$\left|Aut\left(\mathbb{F}_q^N, d_{(P,\pi)}\right)\right| = \left(\prod_{i=1}^{m}\left|Aut\left(U_i\right)\right|\right)\cdot\left|S_\pi\right|.$$

Since $\left|S_\pi\right| = \prod_{j=1}^{l} m_j!$, it follows the result. $\square$

Restricting to the Hamming case again, it follows that

$$Aut\left(U_i, d_{([n],\pi_i)}\right) = Aut\left(U_i\right) = Aut\left(\mathbb{F}_q\right) \simeq \mathbb{F}_q^*$$

and $S_\pi = S_m$, and therefore, it follows the following corollary:

Corollary 3. *The automorphism group of $\left(\mathbb{F}_q^m, d_H\right)$ is $\left(\mathbb{F}_q^*\right)^m \rtimes S_m$.*

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RESUMO. Seja $P = (\{1,2,\ldots,n\}, \leq)$ um conjunto parcialmente ordenado dado por uma união disjunta de cadeias de mesmo comprimento e $V = \mathbb{F}_q^N$ o espaço vetorial das N-uplas sobre o corpo finito $\mathbb{F}_q$. Seja $V = V_1 \times V_2 \times \ldots \times V_n$ um produto direto de V, em blocos de subespaços $V_i = \mathbb{F}_q^{k_i}$ com $k_1 + k_2 + \ldots + k_n = N$, munido com a métrica de blocos ordenados $d_{(P,\pi)}$ induzida pela ordem P e pela partição $\pi = (k_1, k_2, \ldots, k_n)$. Neste trabalho descrevemos o grupo de isometrias do espaço métrico $(V, d_{(P,\pi)})$.

Palavras-chave: métrica de bloco, métrica de ordem, métrica de Niederreiter-Rosenbloom-Tsfasman, isometrias, automorfismos.

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