

## SORM DG – an efficient SORM based on differential geometry

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### Abstract

The first order reliability method (FORM) efficiently performs first order structural reliability analysis, but with limited accuracy. On the other hand, the traditional second order reliability method (SORM) was established to improve the results of FORM, but with a supplementary computational process. Proposed herein, is a new SORM, based on differential geometry making SORM more efficient without hindering accuracy. It can be used to perform the second order structural reliability analysis in engineering. A case in the geotechnical engineering field obtained from literature was solved aiming to demonstrate the ability of the analytical procedure via differential geometry. The advantages of the newly proposed approach whereby the reliability method by differential geometry (SORM DG) over the traditional SORM are discussed. The results show that the SORM DG optimizes the outcomes from FORM and achieves the accuracy of the traditional SORM, more efficiently.

**Keywords:** SORM DG, SORM, failure probability, main curvatures, differential geometry.

### 1. Introduction

Many variables involved in engineering designs are random, since their parameters have uncertainties. The presence of uncertainty is generally treated using an overall safety factor, according to the traditional approach of the permissible stress method. This safety factor is selected based on past experience or practical overall rules and it does not reflect the uncertainty of the individual underpinning parameters nor its correlation structure. These limitations can be overcome through reliability-based designs, in which the safety of a structure is depicted by a reliability index rather than the safety factor. The reliability index is able to explain the uncertainties and the parametric correlations and provides a

way to assess the failure probability of the structure and its components.

Due to its greater simplicity and efficiency, FORM, which performs a linear approximation by a hyperplane to the limit state surface (LSS), has been largely used in reliability analyses. However, the inherent linearization to FORM adds errors in several cases and so SORM has been employed as an alternative (Zeng *et al.*, 2016).

According to Chan and Low (2012), SORM is well established in the structural mechanics field, but its applications in geotechnical engineering problems in recent years suggest a relevant interest and, thus, there is much space for further research in this field

with the main challenge being to calculate the main curvatures of the LSS that involves a lot of mathematical complexity and computational effort.

Brzakala and Pula (1996) and Bauer and Pula (2000) estimated the failure probability of foundations using SORM and a polynomial response surface method (RSM)-based SORM, respectively; Cho (2009) combined a response surface based on an artificial neural network and SORM to calculate the reliability of slopes; Lü and Low (2011), Lü *et al.* (2011) and Lü *et al.* (2012) employed several RSM and SORM calculations to examine tunnel supports; Chan and Low (2012) introduced SORM to a reliability analysis of foundations employing

a practical procedure to estimate the main curvatures of LSS with reasonable accuracy; Zeng and Jimenez (2014) and Zeng *et al.* (2015) applied a quasi-Newton approximation-based SORM to assess the reliability of geotechnical problems, and Zeng *et al.* (2017) presented an extension of a quasi-Newton approximation-based SORM for reliability analysis of the series systems for geotechnical engineering problems. However, the above-mentioned methods have to assess the main curvatures of LSS or build the response surface function.

The new approach, called SORM

## 2. Review of the traditional SORM

The traditional SORM evaluates second order failure probability using the coordinates calculated by FORM of the design point ( $\mathbf{V}^*$ ) approximating a hyperparaboloid to LSS at  $\mathbf{V}^*$  point. In addition, it also requires the calculation of the Hessian matrix at the  $\mathbf{V}^*$

DG, which aims to improve the results of FORM more efficiently than the traditional SORM, with no prejudice to the accuracy of the results, can be considered a useful contribution to engineering. In order for SORM DG to perform the second order structural reliability analysis, it is necessary that FORM provide the coordinates of the point of maximum local density of probability (design point) and that the main curvatures of LSS are calculated at this point, by differential geometry, and provided to SORM DG. Herein, then is presented an analytical procedure for the calculation of these

curvatures in the multidimensional space (space  $\mathbb{R}^n$ ), where  $n$  is the number of random variables of the LSS function involved in the analysis.

For many engineering problems, the LSS function is computationally intensive to be evaluated and the analytical derivatives do not exist (Du, 2005). In this case, the derivatives are calculated by the finite difference method. Due to this fact, for this study, all the partial derivatives required for solution of the assessments of the first and second order probabilities of failure were calculated by the finite difference method.

point, so that the main curvatures of the hyperparaboloid approximated to LSS at the  $\mathbf{V}^*$  point, are considered as equal to the main curvatures of the LSS in that point.

In the following items of this Section, the conventional methods used to

solve FORM and the Hessian matrix are discussed. The reliability analysis of SORM, based on these methods, is also discussed, in addition to its efficiency. More details can be found in Madsen *et al.* (2006), Ditlevsen and Madsen (2007) and Melchers and Beck (2018).

### 2.1 The first order reliability method (FORM)

FORM and traditional SORM are considered very efficient methods to perform structural reliability analysis (Zhao and Ono, 1999). In FORM, the first order failure probability is obtained

by transformation of original random variables ( $\mathbf{U}$ ) of the  $U$  space (original space) into reduced random variables  $\mathbf{V}$  (standard normal and statistically independent variables) of the  $V$  space,

being LSS represented in the  $U$  space as the  $G(\mathbf{U})=0$  function, and in the  $V$  space as the  $g(\mathbf{V})=0$  function. Otherwise, it is equivalent to solving the following constrained optimization problem:

$$\mathbf{V}^* = \min \|\mathbf{V}\| \text{ subject to } g(\mathbf{V}) = 0 \quad (1)$$

where  $\|\cdot\|$  is the norm of a vector. Then, the first order reliability index  $\beta_f$  may be calculated as

$$\beta_f = \|\mathbf{V}^*\| \quad (2)$$

and the failure probability can be approximated by

$$P_f = \Phi(-\beta_f) \quad (3)$$

where  $\Phi(\cdot)$  is the cumulative probability function for the standardized normal distribution.

In order to solve Equation 1, FORM with the iHLRF – an improved HLRF (Hasofer and Lind, 1974; Rackwitz and Fiessler, 1978) algorithm is used herein – FORM (iHLRF), which according to Zeng *et al.* (2016) is the first order method more often used in solving engineering cases.

According to Zeng *et al.* (2016), to measure the computer efficiency, the number of deterministic evaluations of the

function (NFE) of the LSS is used, needed in each analysis, due to the fact that the computer effort required for other parts of the algorithm is frequently insignificant when compared with the NFE, specifically as numerical methods are involved, such as finite differences. Hence, NFE may be used as an overall indicator of computer efficiency in real engineering problems, i.e. greater NFE implies less efficiency and vice versa.

In the case of FORM (iHLRF), when the coordinates of the gradient of the  $G(\mathbf{U})=0$  function are calculated by the finite difference method, there are needed  $n+1$  evaluations of the  $G(\mathbf{U})=0$  function, for each iteration ( $i$ ) in the search algorithm (iHLRF) of the design point at the original space ( $\mathbf{U}^*$ ), i.e. the number of deterministic evaluations of the function of LSS for FORM (iHLRF) is:

$$NFE_{FORM} = i(n+1) \quad (4)$$

### 2.2 Calculation of the Hessian matrix

Before calculating the second order failure probability, the variables

$\mathbf{V}$  of the space  $V$  must be converted, by rotation, into the standard normal

space  $Y$  performing an orthogonal transformation.

$$\mathbf{Y} = \mathbf{R} \mathbf{V} \quad (5)$$

where,  $\mathbf{R}$  is an orthogonal rotation matrix, with dimension  $n \times n$ , whose last column contains the coordinates of the

unit normal vector  $\mathbf{\alpha}^*$  ( $\mathbf{\alpha}^* = -\mathbf{V}^*/\beta_F$ ) of the LSS, and can be obtained by an orthogonalization process, such as

that of Gram-Schmidt. After rotating the coordinates, the matrix  $\mathbf{H}_R$  is obtained:

$$\mathbf{H}_{Rij} = \frac{(\mathbf{RHR}^T)_{ij}}{\|\nabla g(\mathbf{V}^*)\|}, i, j = 1, \dots, n-1 \quad (6)$$

where  $\mathbf{H}$  is the Hessian matrix,  $\mathbf{R}^T$  is an orthogonal rotation transposed matrix and  $\|\nabla g(\mathbf{V}^*)\|$  is the norm of the

gradient of the LSS at the design point (evaluated in the last iteration of the iHLRF algorithm). The eigenvalues of

the matrix  $\mathbf{H}_R$  are the main curvatures  $\kappa_j$  ( $j=1, 2, \dots, n-1$ ) of LSS.

### 2.3 Appraisal of the second order failure probability

In possession of the coordinates of  $\mathbf{V}^*$ , as well as the  $\kappa_j$  values, the appraisal of the second order failure probability of traditional SORM can be attained, for ex-

ample, according to Chan and Low (2012) and Zeng *et al.* (2016), by the average values provided by formulas, such as those proposed by Tvedt (1983), Breitung (1984),

Hohenbichler and Rackwitz (1988), Köyliüoğlu and Nielsen (1994), Cai and Elishakoff (1994), Hong P3 (1999), Hong P4 (1999), Zhao and Ono (1999), namely:

#### Tvedt (1983)

$$Pf_T = A_1 + A_2 + A_3 \quad (7)$$

$$A_1 = \Phi(-\beta_F) \prod_{j=1}^{n-1} (1 + \beta_F \kappa_j)^{-1/2} \quad (8)$$

$$A_2 = [\beta_F \Phi(-\beta_F) - \varphi(\beta_F)] \left\{ \prod_{j=1}^{n-1} (1 + \beta_F \kappa_j)^{-1/2} - \prod_{j=1}^{n-1} (1 + (\beta_F + 1) \kappa_j)^{-1/2} \right\} \quad (9)$$

$$A_3 = (\beta_F + 1) [\beta_F \Phi(-\beta_F) - \varphi(-\beta_F)] \left\{ \prod_{j=1}^{n-1} (1 + \beta_F \kappa_j)^{-1/2} - \text{Re} \left[ \prod_{j=1}^{n-1} (1 + (\beta_F + i) \kappa_j)^{-1/2} \right] \right\} \quad (10)$$

#### Breitung (1984)

$$Pf_B = \Phi(-\beta_F) \prod_{j=1}^{n-1} (1 + \beta_F \kappa_j)^{-1/2} \quad (11)$$

#### Hohenbichler and Rackwitz (1988)

$$Pf_{HR} = \Phi(-\beta_F) \prod_{j=1}^{n-1} (1 + \psi(\beta_F) \kappa_j)^{-1/2} \quad (12)$$

#### Cai and Elishakoff (1994)

$$Pf_C = \Phi(-\beta_F) - \varphi(\beta_F) (D_1 + D_2 + D_3) \quad (13)$$

$$D_1 = \sum_{j=1}^{n-1} \lambda_j \quad (14)$$

$$D_2 = -\frac{1}{2} \beta_F \left( 3 \sum_{j=1}^{n-1} \lambda_j^2 + \sum_{j \neq l}^{n-1} \lambda_j \lambda_l \right) \quad (15)$$

$$D_3 = \frac{1}{6} (\beta_F^2 - 1) \left( 15 \sum_{j=1}^{n-1} \lambda_j^3 + 9 \sum_{j \neq l}^{n-1} \lambda_j^2 \lambda_l + \sum_{j \neq l \neq m}^{n-1} \lambda_j \lambda_l \lambda_m \right) \quad (16)$$

**Köylüoğlu and Nielsen (1994)**

$$Pf_K = \Phi(-\beta_F) - \varphi(-\beta_F) \left( \prod_{j=m}^{n-1} [1 - 0.5 \xi(\beta_F) \kappa_j^-]^{-1/2} \left\{ 1 - \prod_{j=1}^{m-1} [1 + \psi(\beta_F) \kappa_j^+]^{-1/2} \right\} \right) + \Phi(\beta_F) \left( \prod_{j=1}^{m-1} [1 + 0.5 \psi(\beta_F) \kappa_j^+]^{-1/2} \left\{ 1 - \prod_{j=m}^{n-1} [1 - \xi(\beta_F) \kappa_j^-]^{-1/2} \right\} \right) \quad (17)$$

**Hong P<sub>3</sub> (1999)**

$$Pf_{HP_3} = C_1 P_0 \quad (18)$$

$$P_0 = \Phi(-\beta_F) \prod_{j=1}^{n-1} [1 + 2 \psi(\beta_F) \lambda_j]^{-1/2} \quad (19)$$

$$C_1 = \frac{1}{n-1} \sum_{j=1}^{n-1} \frac{\Phi(-\beta_F - \{[(n-1) \lambda_j] / [1 + 2 \psi(\beta_F) \lambda_j]\})}{\Phi(-\beta_F)} \exp \left[ \frac{(n-1) \psi(\beta_F) \lambda_j}{1 + 2 \psi(\beta_F) \lambda_j} \right] \quad (20)$$

**Hong P<sub>4</sub> (1999)**

$$Pf_{HP_4} = C_2 P_0 \quad (21)$$

$$P_0 = \Phi(-\beta_F) \prod_{j=1}^{n-1} [1 + 2 \psi(\beta_F) \lambda_j]^{-1/2} \quad (22)$$

$$C_2 = 1 - \frac{n-1}{3} + \frac{1}{3} \sum_{j=1}^{n-1} \frac{\Phi(-\beta_F - \{3 \lambda_j / [1 + 2 \psi(\beta_F) \lambda_j]\})}{\Phi(-\beta_F)} \exp \left[ \frac{3 \psi(\beta_F) \lambda_j}{1 + 2 \psi(\beta_F) \lambda_j} \right] \quad (23)$$

**Zhao and Ono (1999)**

$$R = \frac{n-1}{K_s} \quad (24)$$

$$\beta_s = \left( 1 + \frac{2.5 K_s}{2n-5R + \frac{25(23-5\beta_F)}{R^2}} \right) \beta_F + 0.5 K_s \left( 1 + \frac{K_s}{40} \right); K_s < 0 \quad (25)$$

$$Pf_{z0} = \Phi(-\beta_s) \quad (26)$$

The meanings of the symbols necessary for understanding Equations 7 to 26 are listed below:

- $\beta_F$  = First order reliability index.
- $\Phi(\cdot)$  = Probability cumulative function for the standardized normal distribution.
- $\varphi(\cdot)$  = Probability density function for the standardized normal distribution.
- Re[.] = Real part.

- $i = \sqrt{-1}$  = Imaginary unit.
- $n$  = Number of random variables of the LSS function involved in the analysis.
- $\kappa_j$  = Main curvatures of the LSS at the design point, with  $j=1, 2, \dots, n-1$ .
- $\kappa_j^+$  = Positive main curvatures of the LSS at the design point, with  $j=1, 2, \dots, m-1$ .
- $\kappa_j^-$  = Negative main curvatures

- of the LSS at the design point, with  $j= m, \dots, n-1$ .
- $K_s$  = The sum of the main curvatures.
- $R$  = The average main curvature radius.
- $\beta_s$  = The empirical second-order reliability index.

$$\xi(\beta_F) = \frac{\varphi(\beta_F)}{\Phi(\beta_F)}, \quad \psi(\beta_F) = \frac{\varphi(\beta_F)}{\Phi(-\beta_F)} \quad \text{and} \quad \lambda_j = \frac{1}{2} \kappa_j, \quad \text{with } j=1, 2, \dots, n-1.$$

In order for the failure probability value at the design point to be calculated by SORM, it is necessary that the condition  $\beta_F \kappa_j > -1$  be met.

## 2.4 Efficiency of the traditional SORM

In practice, with regard to engineering problems, when the analytical solutions for the partial derivatives are not available, the Hessian matrix is calculated

by finite differences. Therefore, the number of additional analyses of the function of the LSS for calculation of all second order derivatives by finite differences is

$$\text{NFE}_{\text{SORM}} = \frac{n(n+1)}{2} \quad (27)$$

It could significantly enlarge the computational cost, especially when the

LSS function has a high number of random variables and is time-consuming to

$n(n+1)$ . However, as the Hessian matrix is symmetrical, the number of deterministic evaluations of the LSS function for the traditional SORM is:

be evaluated, when using laborious methods such as the finite difference method.

## 3. Procedure to obtain the main curvatures by differential geometry (Ferreira, 2015)

In this Section, the procedure is established, by differential geometry, to calculate the main curvatures of the LSS at the  $p$  point, in order to perform the esti-

mate of the second order failure probability. According to the proposed method, the estimate of the second order failure probability will be calculated, using the average

of the results of the eight formulas shown by Equations 7, 11, 12, 13, 17, 18, 21 and 26 and compared with the result calculated by traditional SORM, provided by literature.

### 3.1 Surfaces in $\mathbb{R}^3$

To get a better understanding, in this Subsection, the main curvatures at the  $p$

point of a surface in  $\mathbb{R}^3$  are calculated by differential geometry and in the Subsection

3.2 the generalization of this process for the space  $\mathbb{R}^n$  is performed.

#### 3.1.1 Parameterization of a regular surface in $\mathbb{R}^3$

The graph of an equation of the  $F(x,y,z)=0$  form, where  $F$  is a differentiable function and its partial derivatives do not cancel each other, simultaneously, at any  $p$  point, such that  $F(p)=0$ , is an example of a regular surface in  $\mathbb{R}^3$ . It is verified that the graph of a differentiable function  $f: \mathbb{R}^2 \rightarrow \mathbb{R}$  is also an example of regular surface. More generally, a subset  $S$  of  $\mathbb{R}^3$  is named regular surface if, for each point

$p \in S$ , there is an open vicinity  $V \subset \mathbb{R}^3$  of  $p$ , an open  $U \subset \mathbb{R}^2$  and a bijection  $\varphi: U \rightarrow V \cap S$ , being  $\varphi$  of way  $\varphi(u,v) = \{x(u,v), y(u,v), z(u,v)\}$ , with the properties described as follows (Rodrigues, 2001):

a)  $\varphi$  is Class  $C^\infty$ , i.e.  $\varphi$  has continuous partial derivatives of all orders at the  $p$  point;

b)  $\varphi$  is a homeomorphism (i.e., its inverse is continuous); and

c) for any point  $q \in U$  the Jacobian matrix of  $\varphi$  has rank two. The referred matrix has rank two, which means that the image of the linear transformation obtained has dimension two, i.e. eliminating a line, conveniently chosen, the resulting  $2 \times 2$  matrix has a determinant different from zero. The Jacobian matrix, in this case, has dimensions  $3 \times 2$ , represented by:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{bmatrix} \quad (28)$$

In these conditions, it is said that  $\varphi$  is a parameterization for  $S$ , as illustrated in Figure 1:

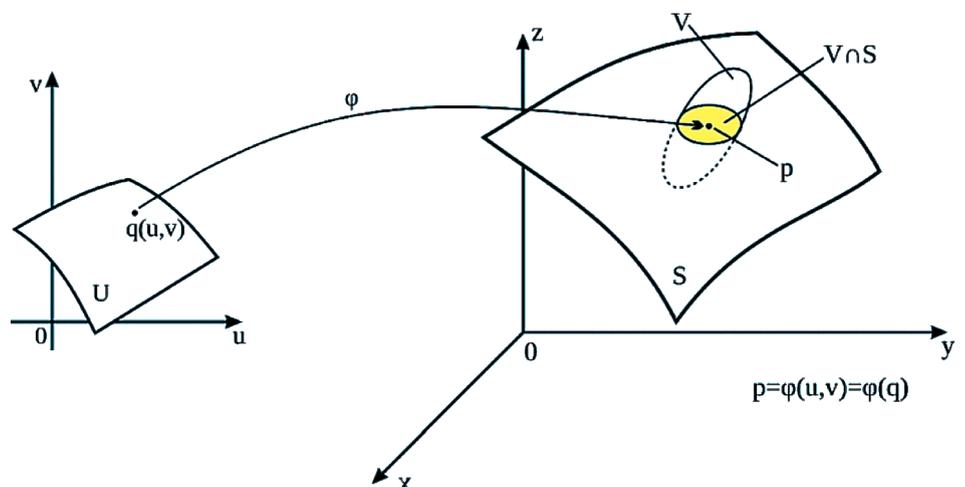


Figure 1  
Parameterization of a regular surface.  
Source: Adapted from Carmo (2006)

A regular surface  $S \subset \mathbb{R}^3$  is orientable, if and only if there is a differentiable field

### 3.1.2 Curvatures of a surface in $\mathbb{R}^3$

Being  $S$  an orientable surface, the Gauss application is the field of normal vectors  $\mathbf{N}: S \rightarrow S^2$ , where  $S^2 \subset \mathbb{R}^3$  is the sphere of radius 1 and center at origin.

where  $\Phi_u$  and  $\Phi_v$  are partial derivatives of parameterization  $\varphi(u,v)$ , i.e, they are tangent vectors that generate the plane  $T_pS$ .

The vectors  $\mathbf{N}$  and  $\Phi_u$  are orthogonal, as well as  $\mathbf{N}$  and  $\Phi_v$ . Derivat-

### 3.1.3 Normal curvature

Being  $\alpha:(a,b) \rightarrow S$  a parameterized curve by arch length. The normal curvature of  $\alpha$  in  $\alpha(s)$  is the component of  $\alpha''(s)$  according to the nor-

$\mathbf{N}: S \rightarrow \mathbb{R}^3$  of normal vectors in  $S$ , according to Carmo (2006).

$\mathbf{N}$  is a differentiable application and its derivative  $-DN_p: T_pS \rightarrow T_pS$  is an endomorphism (i.e. a linear transformation  $T:U \rightarrow V$ , being  $U=V$ ), where  $T_pS$  is the

$$- \mathbf{N}_u = - DN_{\varphi(u,v)}(\Phi_u) \tag{29}$$

$$- \mathbf{N}_v = - DN_{\varphi(u,v)}(\Phi_v) \tag{30}$$

ing the scale products  $\langle \Phi_u, \mathbf{N} \rangle = 0$  and  $\langle \Phi_v, \mathbf{N} \rangle = 0$ , it was concluded that  $-DN_p$  is a self-adjunct linear application of  $T_pS$  in  $T_pS$ . Thus, according to Araújo (1998), the eigenvalues  $\kappa_1(p)$  and  $\kappa_2(p)$

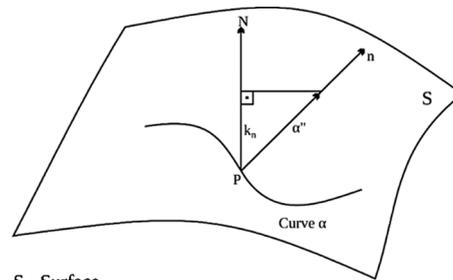
space (plane) tangent to  $S$  surface at the  $p = \varphi(u,v)$  point. From the definition of derivative (rule of the chain), highlighted by Araújo (1998), one has to:

of the linear operator  $(-DN_p)$  are named main curvatures of  $S$  at the  $p$  point and the orthogonal directions defined in  $T_pS$  by eigenvalues  $\kappa_1(p)$  and  $\kappa_2(p)$  are named main directions.

mal to  $S$  at this point and is given by  $k_n(\alpha, s) = \langle \alpha''(s), \mathbf{N} \circ \alpha(s) \rangle$  (inner product between  $\alpha''(s)$  and  $\mathbf{N}$ , with  $\mathbf{N}$  applied in the point  $\alpha(s)$ ) and illustrated as Figure

2. If the curve was not parameterized by arch length, the formula of the normal curvature, according to Rodrigues (2001), becomes:

$$k_n(\alpha, t) = \frac{1}{\|\alpha'(t)\|^2} \langle \alpha''(t), \mathbf{N} \circ \alpha(t) \rangle \tag{31}$$



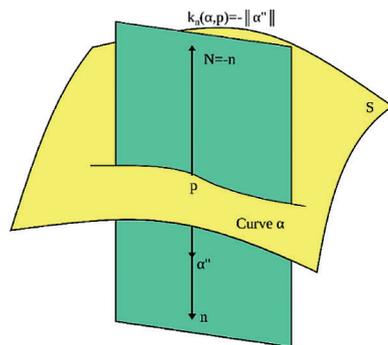
S - Surface  
N - Vector normal to the surface at p  
n - Vector normal to the curve at p  
 $k_n$  - Normal curvature of the curve at p

Figure 2  
Normal curvature at point p.

According to Rodrigues (2001), the maximum and minimum values of

the normal curvatures of the normal sections at  $p$  are the main curvatures

of the surface in point  $p$ , as illustrated in Figure 3:



S - Surface  
N - Vector normal to the surface at p  
n - Vector normal to the curve at p  
 $k_n$  - Normal curvature of the curve  $\alpha$

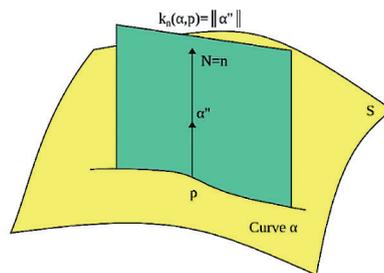


Figure 3  
Normal curvatures when  $\alpha$  is a normal section in point p.

## 3.2 Surfaces in $\mathbb{R}^n$

Most of the literature on differential geometry shows the coefficients of the 1<sup>st</sup> and 2<sup>nd</sup> fundamental form to sim-

plify the calculation of the curvatures in a surface in  $\mathbb{R}^3$  and also to obtain other information, such as a surface area.

Herein, as the interest is the generalization of the surface idea (hypersurface), such simplification using these coef-

ficients could not be done, due to the complexity of mathematical operations involved in calculating the curvatures when  $n > 3$ , hence in Subsection 3.1 it

was opted to use the linear operator  $-DN_p$  (it also could be performed by the option for linear operator  $DN_p$ , but this issue is explained in Section 4).

To simplify calculation of normal vector to surface, it is described as the graph of a differentiable function  $f: \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ .

### 3.2.1 Parameterization of the surface $g(\mathbf{V})=0$

A parameterization for the surface in this vicinity can be given by:

$$\varphi_{(p)}(V_1, V_2, \dots, V_{n-1}) = [V_1, V_2, \dots, V_{n-1}, f(V_1, V_2, \dots, V_{n-1})] \quad (32)$$

The function  $f(\mathbf{V})$ , with  $\mathbf{V}=(V_1, V_2, \dots, V_{n-1}) \in \mathbb{R}^{n-1}$ , is obtained by explicitness for any vari-

able of the vector  $\mathbf{V}$  of the function  $g(\mathbf{V})=0$ , where  $\mathbf{V}=(V_1, V_2, \dots, V_{n-1}, V_n) \in \mathbb{R}^n$ . Considering,

for example, the explicitness of the last variable of  $g(\mathbf{V})=0$ ; it has  $V_n=f(V_1, V_2, \dots, V_{n-1})$ .

### 3.2.2 Obtaining the vectors tangent to the surface $g(\mathbf{V})=0$

The vectors tangent, which correspond to partial derivatives of Equation 32, are calculated in point  $p$  according to:

$$\boldsymbol{\varphi}_{V_1}(p) = (1, 0, \dots, 0, f_{V_1}(p)) = \left(1, 0, \dots, 0, \frac{\partial f(p)}{\partial V_1}\right) \quad (33)$$

$$\boldsymbol{\varphi}_{V_i}(p) = (0, \dots, 1, \dots, 0, f_{V_i}(p)) = \left(0, \dots, 1, \dots, 0, \frac{\partial f(p)}{\partial V_i}\right); 1 < i < n-1 \quad (34)$$

$$\boldsymbol{\varphi}_{V_{n-1}}(p) = (0, 0, \dots, 0, 1, f_{V_{n-1}}(p)) = \left(0, 0, \dots, 1, \frac{\partial f(p)}{\partial V_{n-1}}\right) \quad (35)$$

### 3.2.3 Obtaining the normal vector to the surface $g(\mathbf{V})=0$ and its partial derivatives

The normal vector at point  $p$  is calculated by extending the equation shown in Carmo (2006) for this vector, i.e.

$$\mathbf{N}(p) = \frac{(-f_{V_1}(p), -f_{V_2}(p), \dots, -f_{V_{n-1}}(p), 1)}{\sqrt{(f_{V_1}(p))^2 + (f_{V_2}(p))^2 + \dots + (f_{V_{n-1}}(p))^2 + 1}} \quad (36)$$

The partial derivatives of the normal vector are obtained by

$$\mathbf{N}_{V_j}(p) = \frac{\partial \mathbf{N}(p)}{\partial V_j}, j = 1, 2, \dots, n-1 \quad (37)$$

### 3.2.4 Obtaining the main curvatures of the surface $g(\mathbf{V})=0$

Once performed the calculation of the normal vector and its partial derivatives  $\mathbf{N}_{V_j}$ , they can be written as a linear

combination of the vectors,  $\boldsymbol{\varphi}_{V_1}, \dots, \boldsymbol{\varphi}_{V_{n-1}}$ , of the tangent plane, obtaining the matrix ( $\mathbf{M}$ ) of the linear operator  $-DN_p$ , whose ei-

genvalues are the main curvatures. Extending the equation shown by Araújo (1998) for the referred linear operator, it has:

$$(-DN_p)(\boldsymbol{\varphi}_{V_j}) = -\mathbf{N}_{V_j}, j = 1, 2, \dots, n-1 \quad (38)$$

thus:

$$\begin{aligned} -\mathbf{N}_{V_1} &= (-N_{1,1}) \boldsymbol{\varphi}_{V_1} + \dots + (-N_{1,n-1}) \boldsymbol{\varphi}_{V_{n-1}} \\ &\vdots \\ -\mathbf{N}_{V_{n-1}} &= (-N_{n-1,1}) \boldsymbol{\varphi}_{V_1} + \dots + (-N_{n-1,n-1}) \boldsymbol{\varphi}_{V_{n-1}} \end{aligned} \quad (39)$$

$$\mathbf{M} = \begin{bmatrix} -N_{1,1} & \cdot & \cdot & -N_{1,n-1} \\ -N_{2,1} & -N_{2,2} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ -N_{n-1,1} & -N_{n-1,2} & \cdot & -N_{n-1,n-1} \end{bmatrix} \quad (40)$$

## 4. SORM DG (Ferreira, 2015)

SORM DG is proposed as an alternative for performing a second order structural reliability analysis because it is more efficient than the traditional

SORM, without affecting accuracy, as shown in Subsection 4.2. For this purpose, it needs coordinates of  $\mathbf{V}^*$ , in addition to calculation of the Hessian matrix

( $\mathbf{H}$ ) and the coordinates of the normal vector ( $\mathbf{N}^*$ ) of the LSS in the referred point. It is from the vector  $\mathbf{N}^*$ , calculated by Equation 36 in the point  $p=\mathbf{V}^*$ , which

the main curvatures of LSS, provided to SORM DG, are calculated via differential geometry, as the procedure established in Subsection 3.2. In addition, the vector  $\mathbf{N}^*$  should be the same orientation of the vector  $\boldsymbol{\alpha}^*$ , since it also

is the unit normal vector of the LSS at the design point ( $\mathbf{V}^*$ ). As the calculation methods are distinct, it is required to do this differentiation, as to orientation, for a correct use of the main curvatures in Equations 7 to 26, by the inner product

among the referred vectors. If the inner product between  $\mathbf{N}^*$  and  $\boldsymbol{\alpha}^*$  is negative, they will have different orientations and the use of linear operator  $-DN_p$  must be kept, otherwise it must be replaced by linear operator  $DN_p$ .

#### 4.1 Algorithm for application of SORM DG

a) FORM provides the design point ( $\mathbf{V}^*$ ), transforms the variables of vector  $\mathbf{U}$  of the function  $G(\mathbf{U})=0$ , when required, into equivalent normal variables (see Ditlevsen, 1981) and eliminates the correlations of these variables (see Kiureghian and Liu, 1986), transforming them into standard normal and statistically independent variables;

b) In order to eliminate the effects of the correlation in variables of function  $g(\mathbf{V})=0$ , it must be done by the orthogonal transformation (see Appendix B, Section B.3 and B.4, of Melchers and Beck, 2018) of the vector  $\mathbf{U}$  (which was reported in item “a”) into vector  $\mathbf{V}$ , whose variables are standard normal and statistically independent. After the orthogonal transformation, replacing the values of  $\mathbf{U}$  in

function  $G(\mathbf{U})=0$ , it becomes the function  $g(\mathbf{V})=0$ ;

c) To obtain the function  $f(\mathbf{V})$ , with  $\mathbf{V}=(V_1, V_2, \dots, V_{n-1}) \in \mathbb{R}^{n-1}$ , it can explicit any variables of the vector  $\mathbf{V}$  of the function  $g(\mathbf{V})=0$ , where  $\mathbf{V}=(V_1, V_2, \dots, V_{n-1}, V_n) \in \mathbb{R}^n$ . Considering, for example, the explicitness of last variable of  $g(\mathbf{V})=0$  it has  $V_n=f(V_1, V_2, \dots, V_{n-1})$ ;

d) Generate in the vicinity of  $p=\mathbf{V}^*$ , from the function  $V_n=f(V_1, V_2, \dots, V_{n-1})$ , the parameterization according to the Equation 32;

e) Calculate the partial derivatives of the parameterization to obtain the vectors tangent to surface, at point  $p=\mathbf{V}^*$ , using the Equations 33, 34 and 35;

f) Obtain the normal vector to the surface ( $\mathbf{N}^*$ ), at point  $p=\mathbf{V}^*$ , according to

Equation 36;

g) Calculate the partial derivatives of the normal vector to the surface, at point  $p=\mathbf{V}^*$ , by Equation 37;

h) Obtain by Equation 40 the eigenvalues of the matrix ( $\mathbf{M}$ ) of the linear operator, considering that they are the main curvatures of the surface  $g(\mathbf{V})=0$  at the point  $p=\mathbf{V}^*$ ;

i) Perform the inner product between  $\mathbf{N}^*$  and  $\boldsymbol{\alpha}^*$ , to assure that the orientation of  $\mathbf{N}^*$  is the same the  $\boldsymbol{\alpha}^*$ , in order for the main curvatures of the surface, calculated in previous item, can be applied with correct orientation in the Equations 7 to 26;

j) Perform the second order structural reliability analysis by calculation of the average of the results provided by Equations 7, 11, 12, 13, 17, 18, 21 and 26.

#### 4.2 Efficiency of the SORM DG

Analyzing the algorithm shown in Subsection 4.1, it is noted that for item “e”, it is required to obtain the first order partial derivatives of  $f(\mathbf{V})$ , while for item

“g”, the second order partial derivatives of  $f(\mathbf{V})$ , i.e, the Hessian matrix ( $\mathbf{H}$ ) must be calculated. As the values of the first order partial derivatives of  $f(\mathbf{V})$  were

$$NFE_{SORM\ DG} = \frac{n(n-1)}{2} \quad (41)$$

because, as mentioned in item “c” of the Subsection 4.1, the function  $f(\mathbf{V})$  is obtained by the explicitness

of one of the variables  $\mathbf{V}$  of the function  $g(\mathbf{V})=0$ , thus showing one variable less than the function  $g(\mathbf{V})=0$ .

obtained with values of the second order partial derivatives of  $f(\mathbf{V})$  during the calculation of the Hessian matrix, it is said that

Therefore, replacing  $n-1$  in the place of  $n$  in Equation 27 it is obtained the Equation 41.

### 5. Results

In the example below, the reliability analysis is performed by SORM DG of a

geotechnical engineering case obtained from literature, aiming to show the ability

of the analytical procedure by differential geometry established in Subsection 3.2.

#### 5.1 Example – Case of bearing capacity of a shallow footing

The example, also proposed by Chan and Low (2012) and Zeng *et al.* (2016),

envisages the bearing capacity of a shallow footing resting on a homogeneous silty sand;

see Figure 4. The LSS function, due to the exceedance of its bearing capacity is given as

$$G(\mathbf{U}) = q_{ult} - q = 0 \quad (42)$$

and  $q_{ult}$  is the vertical bearing resistance computed with the polynomial bearing capacity equation; and  $q$  is the vertical

applied pressure (corrected to explain the eccentricity of the loads). Five random variables — cohesion,  $c'$ ; friction angle,  $\phi'$ ; unit

weight,  $\gamma$ ; horizontal load,  $P_H$ ; and vertical load,  $P_V$  — are considered, and they are all presumed to be normally distributed.

$$q_{ult} = c' N_c s_c i_c + \gamma D N_q s_q i_q + 0.5 \gamma B' N_\gamma s_\gamma i_\gamma \quad (43)$$

where  $N_q = e^{(\pi \tan \phi') \tan^2 [45^\circ + (\phi'/2)]}$ ,  $N_c = (N_q - 1) \cot \phi'$ ,  $N_\gamma = 2 (N_q - 1) \tan \phi'$  are the conventional dimensionless bearing resistance factors that depend (strongly

non-linearly) on the soil's friction angle and  $s_q = 1 + (B'/L') \sin \phi'$ ,  $s_c = (s_q N_q - 1) / (N_q - 1)$  and  $s_\gamma = 1 - 0.3 (B'/L')$  are dimensionless factors that introduce corrections to

explain the form of the footing ( $B$  and  $L$  are the width and length of the footing) and the eccentricity of the loads. Where  $B' = B - 2 e_B$ ,  $L' = L$  and  $e_B = h (P_H / P_V)$ .

Similarly,  $i_q = \{1 - [P_H / (P_V + B' L' c' \cot \phi')]\}^m$ ,  $i_c = i_q - [(1 - i_q) / N_c \tan \phi']$  and  $i_\gamma = \{1 - [P_H / (P_V + B' L' c' \cot \phi')]\}^{m+1}$  are dimensionless correction factors responsible for

the inclination of the resultant load, where  $c'$  is the cohesion of the soil and  $m = [2 + (B'/L)] / [1 + (B'/L)]$ . Employing this approach, the bearing capacity

failure would be theoretically exceeded when the vertical applied pressure,  $q$ , which can be computed as

$$q = P_V / B' \tag{44}$$

becomes higher than the value of  $q_{ult}$  computed employing the Equation 43.

### 5.1.1 Obtaining the $g(\mathbf{V})$ function

As written on item “b” from Subsection 4.1, the orthogonal transformation ac-

complished from vector  $\mathbf{U} = [c' \ \phi' \ \gamma \ P_H \ P_V]$ , being the vector  $\mathbf{V}$  obtained with standard

normal and statistically independent variables, whose coordinates are:

$$\mathbf{V} = \begin{bmatrix} c' = 4.5 V_1 + 15 \\ \phi' = -0.043633231299858 V_1 + 0.075574973509759 V_2 + 0.436332312998582 \\ \gamma = 1.1547005388379252 V_2 + 1.632993161855452 V_3 + 20 \\ P_H = 40 V_4 + 400 \\ P_V = 40 V_4 + 69.282032302755098 V_5 + 800 \end{bmatrix}$$

Replacing these values, respectively, in place of  $c', \phi', \gamma, P_H, P_V$ , in  $G(\mathbf{U})=0$

function (Equation 42), it becomes  $g(\mathbf{V})=0$  function.

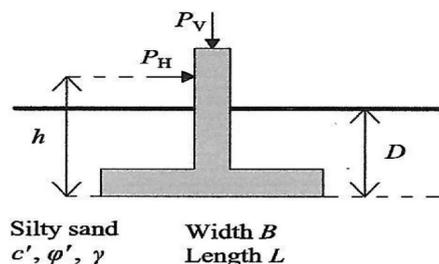
### 5.1.2 Obtaining the $f(\mathbf{V})$ function

The  $f(\mathbf{V})$  function, where  $\mathbf{V}=(V_1, V_2, \dots, V_{n-1})$ , as shown in the item “c” in Subsection 4.1, is obtained by explicitness of any variables of the vector  $\mathbf{V}$  in  $g(\mathbf{V})=0$  function, where  $\mathbf{V}=(V_1, V_2, \dots, V_{n-1}, V_n)$ . Then, the  $f(\mathbf{V})$  function was obtained from the explicitness of the variable  $V_3$ , as recommended by item “c” of the Subsection 4.1,

$V_3 = f(V_1, V_2, V_4, V_5) = f(\mathbf{V})$ . Therefore,  $f(\mathbf{V})$  is the new LSS function, now with one variable less, in the case of  $V_3$  variable. Thus, the reliability analysis of this example, performed by SORM DG, from the  $f(\mathbf{V})$  function, is more efficient than that performed by traditional SORM and without affecting the accuracy, as shown in Table 4. In Table

1, shown are the features (moments and probability distributions) of the random variables, in addition of the coordinates of the design point. Table 2 provides the main curvatures at the design point calculated by differential geometry, and in Table 3, the estimate of the second order failure probability at design point by SORM DG is presented.

Figure 4 Description of the example of the shallow footing and of the deterministic parameters involved. Source: Adapted from Zeng et al. (2016)



#### Deterministic parameters

Width	$B$ (m)	5
Length	$L$ (m)	25
Depth	$D$ (m)	1.8
Load position	$h$ (m)	2.5

Correlation matrix ( $\Omega$ )

	$c'$	$\phi'$	$\gamma$	$P_H$	$P_V$
$c'$ (KPa)	1	-0.5	0	0	0
$\phi'$ ( $^\circ$ )	-0.5	1	0.5	0	0
$\gamma$ (KN/m <sup>3</sup> )	0	0.5	1	0	0
$P_H$ (KN/m)	0	0	0	1	0.5
$P_V$ (KN/m)	0	0	0	0.5	1

$$\mathbf{L} = [\text{Cholesky factorization } (\Omega)]^T \tag{45}$$

Lower Cholesky matrix ( $\mathbf{L}$ )

1	0	0	0	0
-0.5	0.866	0	0	0
0	0.577	0.816	0	0
0	0	0	1	0
0	0	0	0.5	0.866

Distribution	Variable	$\mu_i$	$\sigma_i$	$U_i^*$	$V_i^{/*} = (U_i^* - \mu_i) / \sigma_i$	$V^* = L^{-1} V^{/*}$
Normal	$c'$ (KPa)	15	4.5	14.915	-0.019	-0.019
Normal	$\phi'$ (°)	25	5	18.490	-1.302	-1.514
Normal	$\gamma$ (KN/m <sup>3</sup> )	20	2	17.934	-1.033	-0.195
Normal	$P_H$ (KN/m)	400	40	422.600	0.565	0.565
Normal	$P_V$ (KN/m)	800	80	808.400	0.105	-0.205

Table 1  
Random variables and coordinates of the design point.

$i = 1, 2, \dots, n$ ;  $\mu_i$  = Mean;  $\sigma_i$  = Standard deviation;  $U_i^*$  = Coordinates of the vector  $U^*$  at in original space;  $V_i^{/*}$  = Coordinates of the vector  $V^{/*}$  in reduced space;  $V^*$  = Design point for correlated  $U$ .

$K_j$
-0.0886
-0.0574
-0.0039
0.0090

Table 2  
Main curvatures by differential geometry.

SORM DG	$Pf_T$	$Pf_B$	$Pf_C$	$Pf_K$	$Pf_{HR}$	$Pf_{HP3}$	$Pf_{HP4}$	$Pf_{ZO}$	Average <sup>a</sup>
$Pf(x10^{-2})^b$	5.93	5.70	5.86	5.71	5.91	5.87	5.88	5.86	<b>5.84</b>
$\beta^c$	1.561	1.580	1.567	1.579	1.563	1.566	1.565	1.567	<b>1.568</b>

Table 3  
Estimation the second order failure probability by SORM DG – Average of eight formulas.

<sup>a</sup> To assess the second order failure probability, the average of eight formulas (Equations 7, 11, 12, 13, 17, 18, 21 and 26) based on the same information (i.e.,  $\beta_i$  and  $K_j$ ) is used in this study, <sup>b</sup> Failure probability, <sup>c</sup> Reliability index.

n = 5 variables	NFE <sup>a</sup>	$\beta^g$	$Pf(x10^{-2})^h$	$\Delta$ (%) <sup>b</sup>
FORM (iHLRF)	42 <sup>c</sup>	1.641	5.04	-18.31
Traditional SORM	42 <sup>c</sup> + 15 <sup>d</sup> = 57 <sup>e</sup>	1.569 <sup>e</sup>	5.83 <sup>e</sup>	-5.51 <sup>e</sup>
<b>SORM DG</b>	42 <sup>c</sup> + 10 <sup>d</sup> = 52	<b>1.568</b> <sup>f</sup>	<b>5.84</b> <sup>f</sup>	- <b>5.35</b> <sup>f</sup>
MCS	5x10 <sup>5</sup> <sup>e</sup>	1.541 <sup>e</sup>	6.17 <sup>e</sup>	reference

<sup>a</sup> NFE = Number of deterministic function evaluations, <sup>b</sup>  $\Delta$  = Relative error in relation to Monte Carlo Simulation (MCS), computed based on results of  $Pf$ , <sup>c</sup> NFE needed by FORM, <sup>d</sup> NFE demanded for computing Hessian matrix, <sup>e</sup> Results given by Zeng et al. (2016), <sup>f</sup> Results (average of eight formulas), <sup>g</sup> Reliability index, <sup>h</sup> Failure probability.

Table 4  
Computed reliability results.

All calculations for this example were performed using MATLAB software (Lee, 2018), which are summarized in Table 4. Analyzing this table, it is noted that the proposed method (SORM DG) was the one providing

the failure probability value closest to the reference value, calculated by MCS, significantly improving the result provided by FORM, but with slightly higher accuracy (almost equivalent) than traditional SORM. In addition,

$$NFE_{SORMDG} = NFE_{SORM} - n \tag{46}$$

it was more efficient than traditional SORM, which required an NFE=15 to calculate the Hessian matrix, while SORM DG needed an NFE=10 for this purpose. Analyzing the Equations 27 and 41 concluded that:

(46)

efficiency then the traditional SORM, as indicated in Equation 46.

Whenever the function  $g(\mathbf{V})=0$  of the LSS shows at least one random variable involved in the reliability analysis,

which can be explicit, the SORM DG will calculate the failure probability, at the design point, with higher

## 6. Conclusions

The example of geotechnical engineering shown herein, whose LSS function correlates variables and features a high degree of non-linearity, was analyzed to test the proposed approach (SORM DG), which was able to perform the second order structural reliability analysis, optimizing the results calculated by FORM and with higher efficiency than the traditional

SORM, without affecting accuracy. It demonstrates the contribution of analytical procedure via differential geometry for the calculation of the main curvatures at a p point, established for a hypersurface in Subsection 3.2.

FORM is very efficient, but limited in terms of accuracy. The MCS is very accurate, but suffers from a lack of efficiency, so whenever it is feasible

to use FORM/SORM, there will be increased efficiency in relation to the MCS. In this sense, the FORM/SORM DG contributes to further increased efficiency in relation to MCS. For limit state surfaces with many random variables involved in the analysis, NFE is often relevant; thus a search for efficiency in structural reliability analysis is required.

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