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On generalized Pell numbers of order $r \geq 2$

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ABSTRACT. In this paper we investigate the generalized Pell numbers of order $r \geq 2$ through the properties of their related fundamental system of generalized Pell numbers. That is, the generalized Pell number of order $r \geq 2$, are expressed as a linear combination of a fundamental system of generalized Pell numbers. The properties of this fundamental system are examined and results can be established for generalized Pell numbers of order $r \geq 2$. Some identities and combinatorial results are established. Moreover, the analytic study of the fundamental system of generalized Pell is provided. Furthermore, the generalized Pell Cassini identity type is provided.

Keywords: generalized Pell fundamental system, generalized Pell numbers, combinatorial identities, analytic representations, Generalized Cassini identity.

1 INTRODUCTION

The usual sequence of Pell numbers $(P_n)_{n\geq 0}$ is defined by the initial conditions $P_0 = 0$, $P_1 = 1$, and the recurrence relation $P_{n+1} = 2P_n + P_{n-1}$, for $n \geq 1$. In the literature, there are various generalizations of this well known sequence of integers (see, for example, [4, 6, 10, 11], and references therein). The sequence $(P_n)_{n\geq 0}$ and their generalizations are widely studied from both algebraic, analytic, combinatorial and matrix perspective, and it is an interesting subject of several important properties and identities (see, for example, [1, 4, 6, 10, 11]).

In this paper we are concerned with the generalization defined by the following linear difference equation of order $r \geq 2$,

$$P_{n+1} = 2P_n + P_{n-1} + \cdots + P_{n-r+1}, \qquad \text{for} \qquad n \geq r, \tag{1.1}$$

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where the initial conditions $P_0 = \alpha_0, \ldots, P_{r-1} = \alpha_{r-1}$ are adequately chosen. Let consider the set $\mathfrak{P}_r = \{(P_n^{(s)})_{n\geq 0},\ 1 \leq s \leq r\}$ of sequences of generalized Pell numbers $(P_n^{(s)})_{n\geq 0}$ defined as follows,

$$\begin{aligned} P_n^{(s)} &= 2P_{n-1}^{(s)} + \sum_{i=1}^{r-1} P_{n-i-1}^{(s)} \qquad \text{for} \qquad n \geq r, \\ P_{r-s}^{(s)} &= 1 \text{ and } P_n^{(s)} = 0 \text{ for } 0 \leq n \neq r-s \leq r-1. \end{aligned} \tag{1.2}$$

For example, in case $r = 3$, the set $\mathfrak{P}_3 = \{(P_n^{(s)})_{n\geq 0},\ 1 \leq s \leq 3\}$ of sequences of generalized Pell numbers $(P_n^{(s)})_{n\geq 0}$ is given by

$$\begin{cases} P_{n+1}^{(s)} = 2P_n^{(s)} + P_{n-1}^{(s)} + P_{n-2}^{(s)}, & \text{for } n \geq 3, \\ P_{3-s}^{(s)} = 1 \text{ and } P_n^{(s)} = 0 \text{ for } 0 \leq n \neq 3-s \leq 2. \end{cases}$$

The set $\mathfrak{P}_r$ will play a central role in the sequel of this work. Indeed, we explore the family of generalized Pell numbers (1.1), through the properties of the set $\mathfrak{P}_r$. More precisely, we describe explicitly the closed connection between the sequences $(P_n^{(s)})_{n\geq 0}$ $(2 \leq s \leq r-1)$ and the sequence of generalized Pell numbers $(P_n^{(r)})_{n\geq 0}$. This approach permits us to elaborate some combinatorial identities and examined the analytical properties of each sequence of the set $\mathfrak{P}_r$. Finally, the combinatorial and the analytical formula of the generalized Pell Cassini identity are investigate.

The content of this paper is organized as follows. In Section 2, we establish that the set $\mathfrak{P}_r$ is fundamental system of solutions of (1.1), considered as a difference equation. Moreover, for every j $(1 \leq j \leq r)$, we show that $P_n^{(j)}$ can be expressed with the aid of $P_k^{(r)}$ $(n-1 \leq k \leq n-j)$. Sections 3 and 4 are devoted to some results, identities and combinatorial relation, related to the sequences of generalized Pell numbers. In Section 5 we study the analytical properties of the elements of the set $\mathfrak{P}_r$, and derive the analytic aspect of every sequence of generalized Pell numbers (1.1). Section 6 concerns the generalized Pell Cassini identity, where its combinatorial and analytical expressions are considered. Finally, conclusion and perspective are provided in Section 7.

2 GENERALIZED PELL NUMBERS (1.1) AND THE SET $\mathfrak{P}_R$

Generally, for the usual generalized Pell numbers $(P_n)_{n\geq 0}$ of order $r \geq 2$, the initial conditions are given by,

$$P_0 = \cdots = P_{r-2} = 0 \text{ and } P_{r-1} = 1. \tag{2.1}$$

We can show that the sequence $(P_n^{(r)})_{n\geq 0}$ of the Pell fundamental system $\mathfrak{P}_r$, is nothing else but the generalized Pell numbers $(P_n)_{n\geq 0}$ defined by (1.1) and initial conditions (2.1). Let study the closed connection between the sequence $(P_n^{(r)})_{n\geq 0}$, or equivalently the sequence $(P_n)_{n\geq 0}$, and the other sequences $(P_n^{(s)})_{n\geq 0}$ $(1 \leq s \leq r-1)$ of the set $\mathfrak{P}_r$.

First, we establish that we have $P_n^{(1)} = P_{n-1}^{(r)}$, for every $n \geq 1$, and second that each $P_n^{(j)}$ $(2 \leq j \leq r)$ can be expressed in terms of the generalized Pell numbers $P_n^{(r)}$ or $P_n^{(1)}$. That is, we have $P_0^{(1)} = 1$, $P_1^{(1)} = \cdots = P_{r-1}^{(1)} = 0$ and $P_r^{(1)} = 1$. On the other hand, we have $P_0^{(r)} = P_1^{(r)} = \cdots = P_{r-2}^{(r)} = 0$, $P_{r-1}^{(r)} = 1$. Therefore, we have $P_{j+1}^{(1)} = P_j^{(r)} = 0$, for $j = 0, \ldots, r-2$ and $P_r^{(1)} = P_{r-1}^{(r)} = 1$. And an induction process allows us to show that $P_n^{(1)} = P_{n-1}^{(r)}$, for all $n \geq 1$.

For $2 \leq j \leq r-2$, let prove that $P_n^{(j)} = P_n^{(1)} + \cdots + P_{n-j+1}^{(1)}$, for all $n \geq r$. To this aim, we proceed by induction, involving a slight similar process as in the proof of $P_n^{(1)} = P_{n-1}^{(r)}$, for all $n \geq 1$. For $j = 2$, we set $w_n^{(2)} = P_n^{(1)} + P_{n-1}^{(1)}$, for $n \geq 1$ with initial conditions $w_1^{(2)} = 1, = w_s^{(2)} = 0$, for $2 \leq s \leq r-1$, and $w_r^{(2)} = 1$. For $(P_n^{(2)})_{n\geq 0}$, we have $P_1^{(2)} = 1$, $P_s^{(2)} = 0$, for $0 \leq s \leq r-1$, and $P_r^{(2)} = 1$. Hence, we have $P_1^{(2)} = w_1^{(2)} = 1$, $P_s^{(2)} = w_s^{(2)} = 0$, for $0 \leq s \leq r-1$, and $P_r^{(2)} = w_r^{(2)} = 1$. Therefore, an induction process permits to derive that $P_n^{(2)} = w_n^{(2)} = P_n^{(1)} + P_{n-1}^{(1)}$, for every $n \geq 1$.

For $3 \leq j \leq r-2$, suppose that $P_n^{(k)} = P_n^{(1)} + \cdots + P_{n-k+1}^{(1)}$, for every $n \geq k-1$. The sequence $(w_n^{(j+1)})_{n\geq 1}$ defined by $w_n^{(j+1)} = P_n^{(1)} + P_{n-1}^{(j)}$. For $P_{n-1}^{(j)}$, the first r terms are $P_{n-1}^{(j)} = 0$ for $n = 1, \ldots, j-1$, $P_j^{(j)} = 1, P_{j+1}^{(j)} = 1$ for $n = j+1, \ldots, r-1$ and $P_r^{(j)} = 1$. Since $P_0^{(1)} = 0$ and $P_n^{(1)} = 0$, for $n = 1, \ldots, r-1$, by summation $P_n^{(1)} + P_{n-1}^{(j)}$ $(1 \leq n \leq r)$, and comparison with the values of $P_n^{(j+1)}$ $(n = 1, \ldots, r)$, we derive that,

$$\begin{aligned} w_n^{(j+1)} &= P_n^{(j+1)} = 0, \text{ for } n = 1, \ldots, j-1, \text{ and } n = j+1, \ldots, r-1 \\ w_j^{(j+1)} &= P_j^{(j+1)} = 1 \text{ and } w_r^{(j+1)} = P_r^{(j+1)} = 1. \end{aligned}$$

Therefore, we obtain $P_n^{(j+1)} = w_n^{(j+1)} = P_n^{(1)} + P_{n-1}^{(j)} = P_n^{(1)} + \cdots + P_{n-j+1}^{(1)}$, for every $n \geq j-1$. Hence, we get the following result.

Theorem 2.1. *Let* $\mathfrak{P}_r = \{(P_n^{(s)})_{n\geq 0};\ 1 \leq s \leq r\}$ *be the generalized Pell numbers* (1.2)*, associated with the generalized Pell numbers* (1.1)*. Then, for every* $1 \leq j \leq r-1$*, we have*

$$P_{n+1}^{(1)} = P_n^{(r)} \ \textit{for } n \geq 0, \ \textit{or equivalently } P_n^{(1)} = P_{n-1}^{(r)} \ \textit{for } n \geq 1, \tag{2.2}$$

$$P_n^{(j)} = P_n^{(1)} + \cdots + P_{n-j+1}^{(1)} = P_{n-1}^{(r)} + \cdots + P_{n-j}^{(r)}, \ \textit{for every } \ n \geq j. \tag{2.3}$$

We observe that, the first part of Theorem 2.1 is equal to the third part of Lemma 2 of [8]. However, the second part is not common in the literature.

Let consider the case $r = 3$, then Theorem 2.1 implies that for the set $\mathfrak{P}_3 = \{(P_n^{(s)})_{n\geq 0},\ 1 \leq s \leq 3\}$ of the basic sequences of generalized Pell numbers $(P_n^{(s)})_{n\geq 0}$, we have,

$$P_{n+1}^{(1)} = P_n^{(3)} \ \text{ for } n \geq 0, \ \text{or equivalently } P_n^{(1)} = P_{n-1}^{(3)} \ \text{ for } n \geq 1,$$

$$P_n^{(2)} = P_n^{(1)} + P_{n-1}^{(1)} = P_{n-1}^{(3)} + P_{n-2}^{(3)}, \ \text{ for every } \ n \geq 2.$$

The Table 1 describes the list of the first terms of the fundamental system $\mathfrak{P}_3 = \{(P_n^{(s)})_{n\geq 0},\ 1 \leq s \leq 3\}$ of the generalized Pell number of order $r = 3$.

Table 1: List of generalized Pell number of order r=3.

n	0	1	2	3	4	5	6	7	8	9	10	11	...
$P_n^{(1)}$	1	0	0	1	2	5	13	33	84	214	545	1388	...
$P_n^{(2)}$	0	1	0	1	3	7	18	46	117	298	759	1933	...
$P_n^{(3)}$	0	0	1	2	5	13	33	84	214	545	1388	3535	...

And a direct computation shows that the property $P_n^{(2)} = P_n^{(1)} + P_{n-1}^{(1)} = P_{n-1}^{(3)} + P_{n-2}^{(3)}$, for every $n \geq 2$, is verified.

Let $(\widetilde{P}_n)_{n\geq 0}$ be a sequence of generalized Pell numbers defined by the recursive relation (1.1), and whose initial conditions are $P_0 = \alpha_0$, $P_1 = \alpha_1, \ldots, P_{r-1} = \alpha_{r-1}$, and $(w_n)_{n\geq 0}$ be the sequence defined by $w_n = \alpha_0 P_n^{(1)} + \alpha_1 P_n^{(2)} + \cdots + \alpha_{r-1} P_n^{(r)}$, for every $n \geq 0$. We can verify that $w_0 = \alpha_0$, $w_1 = \alpha_1, \ldots, w_{r-1} = \alpha_{r-1}$, and the sequence $(w_n)_{n\geq 0}$ satisfies the recursive relation (1.1). Thus, for every $n \geq 0$, we have $\widetilde{P}_n = w_n$. Moreover, suppose that $\alpha_0 P_n^{(1)} + \alpha_1 P_n^{(2)} + \cdots + \alpha_{r-1} P_n^{(r)} = 0$, for every $n \geq 0$. Then, for $n = j$ $(1 \leq j \leq r)$, we derive that $\alpha_j = 0$. Therefore, the sequences of the set $\mathfrak{P}_r = \{(P_n^{(s)})_{n\geq 0};\ 1 \leq s \leq r\}$ are linearly independent. Hence, we have the following proposition.

Proposition 2.1. *Let $(\widetilde{P}_n)_{n\geq 0}$ be a sequence of generalized Pell numbers defined by the recursive relation* (1.1)*, and whose initial conditions are $P_0 = \alpha_0$, $P_1 = \alpha_1, \ldots, P_{r-1} = \alpha_{r-1}$, then*

$$\widetilde{P}_n = \alpha_0 P_n^{(1)} + \alpha_1 P_n^{(2)} + \cdots + \alpha_{r-1} P_n^{(r)}, \text{ for every } n \geq 0. \tag{2.4}$$

In other terms, the set $\mathfrak{P}_r = \{(P_n^{(s)})_{n\geq 0};\ 1 \leq s \leq r\}$ is a basis of the vector space $\mathscr{E}_{\mathbb{K}}^{(r)}$ (over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) of solutions of Equation (1.1)*.*

Proposition 2.1 shows the main role of the set $\mathfrak{P}_r$, known in the literature as *fundamental system of solutions* of (1.1), when (1.1) is considered as a difference equation. According to Theorem 2.1 the sequence $(P_n^{(r)})_{n\geq 0}$ play a fundamental role. The sequence $(P_n^{(r)})_{n\geq 0}$ (and also $(P_n^{(1)})_{n\geq 0}$), is called in the literature the fundamental solution of Equation (1.1). In the sequel, we can also call it the generalized Pell fundamental sequence of order r and denote $P_n^{(r)} = P_n$.

3 PELL FUNDAMENTAL SYSTEM $\mathfrak{P}_R$ AND SOME GENERALIZED PELL IDENTITIES

Let consider the vector column $P(j,n) = (P_n^{(j)};\ P_{n+1}^{(j)};\ \ldots;\ P_{n+r-1}^{(j)})^t$, for $n \geq r-1$, for every j $1 \leq j \leq r$, and the matrix,

$$\widehat{C}_{\mathfrak{P}}(n) = [P(1,n), \ldots, P(j,n), \ldots, P(r,n)].$$

Since the set $\mathfrak{P}_r$ is a fundamental system of solutions of (1.1), considered as a difference equation, then the matrix $\widehat{C}_{\mathfrak{P}}(n) = (c_{ij}^{(n)})_{1\leq i,j\leq r}$, represents the Pell Casoratian matrix associated with $\mathfrak{P}_r$. The main goal here, is to exhibit the explicit expressions for the entries $c_{ij}^{(n)}$ of the matrix

$\widehat{C}_{\mathfrak{P}}(n)$, and derive some identities. A direct verification shows that the Casoratian matrix can be written under the form,

$$\widehat{C}(n) = J \times \mathbb{M}_n \times J,$$

where $J = (b_{i,j})_{1 \le i,j \le r}$ is the anti-diagonal unit matrix, namely, $b_{i,j} = 1$, for $i + j = r + 1$, and $b_{i,j} = 0$, otherwise and $\mathbb{M}_n = (P^{(j)}_{n+r-i-1})_{1 \le i,j \le r}$. We show that the matrix $\mathbb{M}_n$, can be written under the form $\mathbb{M}_n = \mathbb{A}^n$, where $\mathbb{A}$ is the classical companion matrix

$$\mathbb{A} = \mathbb{A}[2,1,...,1] = \begin{pmatrix} 2 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}$$

(for more details see [2] and references therein). Hence, we get the following property.

Proposition 3.2. *Consider the set* $\mathfrak{P}_r = \{(P^{(s)}_n)_{n \ge 0};\ 1 \le s \le r\}$ *of sequences of generalized Pell numbers* (1.2) *. Then, the associated Casoratian matrix* $\widehat{C}(n)$ *and the powers* $\mathbb{A}^n$ *of the companion matrix* $\mathbb{A}$ *are similar. More precisely, we have the matrix identity,*

$$\widehat{C}(n) = J\mathbb{A}^n J = (c^{(n)}_{ij})_{1 \le i,j \le r}, \tag{3.1}$$

for every $n \ge 0$*, where the entries* $c^{(n)}_{ij}$ *are given by* $c^{(n)}_{ij} = P^{(j)}_{n+i-1}$ *(*$1 \le i,j \le r$*), and* $J = (b_{i,j})_{1 \le i,j \le r}$ *is the anti-diagonal unit matrix.*

Expression (3.1) implies the matrix identity $\widehat{C}(n+m) = \widehat{C}(n).\widehat{C}(m)$, for every n and m. Hence, the entries of the matrix $\widehat{C}(n+m) = (c^{(n+m)}_{ij})_{1 \le i,j \le r}$, are expressed in terms of those of the matrices $\widehat{C}(m) = (c^{(m)}_{ij})_{1 \le i,j \le r}$ and $\widehat{C}(n) = (c^{(n)}_{ij})_{1 \le i,j \le r}$ as follows,

$$c^{(n+m)}_{ij} = \sum_{k=1}^{r} c^{(n)}_{ik} c^{(m)}_{kj} = \sum_{k=1}^{r} c^{(m)}_{ik} c^{(n)}_{kj}, \text{ for every } n, m \ge 0, \tag{3.2}$$

where $1 \le i, j \le r$. In fact, according to Proposition 3.2, Expression (3.2) is equivalent to the identity,

$$P^{(q)}_{m+s+p} = \sum_{d=1}^{r} P^{(d)}_{m+p} P^{(q)}_{s+d-1} = \sum_{d=1}^{r} P^{(d)}_{s+p} P^{(q)}_{m+d-1},$$

for any integer m, $s \ge 0$ and p, q ($1 \le p, q \le r$). Therefore, since $P^{(1)}_{n+1} = P_n$ and $P^{(j)}_n = P^{(r)}_{n-1} + ... + P^{(r)}_{n-j} = P_{n-1} + ... + P_{n-j}$, we have the identity,

$$P^{(r)}_{m+s+p} = P_{m+s+p} = \sum_{d=1}^{r} P^{(d)}_{m+p} P_{s+d-1} = \sum_{d=1}^{r} P^{(d)}_{s+p} P_{m+d-1}.$$

for $q = r$. More generally, for $1 \le q \le r-1$, we have,

$$P^{(q)}_{m+s+p} = \sum_{d=1}^{r-1} \left[\sum_{i=1}^{d} P_{m+p-i} \right] \left[\sum_{j=1}^{q} P_{s+d-j-1} \right] + P_{m+p} \left[\sum_{k=1}^{q} P_{s+r-1-k} \right].$$

Theorem 3.2. *Let $(P_n)_{n\geq 0}$ the generalized Pell fundamental sequence. Then, for every m, $s \geq 0$, q ($1 \leq q \leq r$), we have the following identities,*

$$P_{m+s} = \sum_{d=1}^{r-1}\left[\sum_{j=1}^{d} P_{m-j}\right] P_{s+d-1} + P_m P_{s+r-1}, \tag{3.3}$$

$$\sum_{k=1}^{q} P_{m+s-k} = \sum_{d=1}^{r-1}\left[\sum_{1\leq i\leq d,\, 1\leq j\leq q} P_{m-i}P_{s+d-j-1}\right] + P_m\left[\sum_{k=1}^{q} P_{s+r-1-k}\right]. \tag{3.4}$$

Using Expression (3.3), we obtain the following corollary.

Corollary 3.2. *1) The generalized Pell fundamental sequence $(P_n)_{n\geq 0}$ of order $r = 3$, satisfies the identity,*

$$P_{m+s} = P_{m-1}P_s + (P_{m-1} + P_{m-2})P_{s+1} + P_m P_{s+2},$$

for every $m \geq 3$ and $s \geq 0$.

2) For $r = 4$, the generalized Pell fundamental sequence $(P_n)_{n\geq 0}$ satisfies the identity,

$$P_{m+s} = P_{m-1}P_s + P_{s+1}(P_{m-1} + P_{m-2}) + (P_{m-1} + P_{m-2} + P_{m-3})P_{s+2} + P_m P_{s+3},$$

for every $m \geq 4$ and $s \geq 0$. For example, using values described in Table 1 for $m = 4$ and $s = 7$, we have $P_{11} = P_3P_7 + (P_3 + P_2)P_8 + P_4P_9$.

For a given companion matrix $A = A[a_0, a_1, \ldots, a_{r-1}]$, it was established in [7, Proposition 2.1] that the entries $c_{ij}^{(n)}$ of the powers A^n are expressed in terms of a family of generalized Fibonacci sequences $v_n^{(i)}$, where $v_{n+1}^{(i)} = a_0v_n^{(i)} + a_1v_{n-1}^{(i)} + \ldots + a_{r-1}v_{n-r+1}^{(i)}$, with $v_n^{(i)} = \delta_{n,i}$, for $0 \leq n \leq r-1$ (see also formulas (18)-(19) of [2], page 348). Thus, we can show that Formula (4) of [8] represents a particular case of the preceding studies. Moreover, using the previous formula (3.2), we can recover Theorem 2 of [8].

4 COMBINATORIAL IDENTITIES FOR THE PELL FUNDAMENTAL SYSTEM $\mathfrak{P}_R$

Let a_1, $a_2, \cdots, a_{r-1}$ be real or complex numbers and consider the following combinatorial expression,

$$\rho(n,r) = \sum_{k_1+2k_2+\cdots+rk_r=n-r} \frac{(k_1+\cdots+k_r)!}{k_1!k_2!\ldots k_r!} a_1^{k_1}a_2^{k_2}\ldots a_r^{k_r}, \text{ for every } n \geq r, \tag{4.1}$$

where $\rho(j,r) = 0$, for $0 \leq j \leq r-1$, and $\rho(r,r) = 1$. Since $\frac{(k_1+\cdots+k_r-1)!}{k_1!\ldots k_{j-1}!(k_j-1)k_{j+1}!\ldots k_r!} = k_j\frac{(k_1+\cdots+k_r-1)!}{k_1!k_2!\ldots k_r!}$, we derive that $\rho(n,r)$ satisfies the following linear difference equation $\rho(n+1,r) = a_1\rho(n,r) + a_2\rho(n-1,r) + \cdots + a_r\rho(n-r+1,r)$, for every $n \geq r$. Specially, for $a_1 = 2$, $a_2 = \cdots = a_r = 1$, we get

$$\rho(n+1,r) = \sum_{k_1+2k_2+\cdots+rk_r=n-r+1} \frac{(k_1+\cdots+k_r)!}{k_1!k_2!\ldots k_r!}2^{k_1}, \text{ for every } n \geq r, \tag{4.2}$$

with $\rho(j,r)=0$ for $0\le j\le r-1$ and $\rho(r,r)=1$. Therefore, the sequence $\{\rho(n+1,r)\}_{n\ge 0}$ satisfies the Expression (1.1) and its initial conditions are given by $\rho(0,r)=\cdots=\rho(r-2,r)=0$ and $\rho(r,r)=1$. Therefore, we formulate the following result.

Proposition 4.3. *(Combinatorial expression of generalized Pell numbers) The combinatoric expression of the generalized Pell fundamental sequence $(P_n)_{n\ge 0}$ is given by,*

$$P_n=\rho(n+1,r)=\sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r+1}\frac{(k_0+\cdots+k_{r-1})!}{k_0!k_1!\ldots k_{r-1}!}2^{k_0},\ \textit{for } n\ge r, \tag{4.3}$$

where $P_j=\rho(j,r)=0$, for $0\le j\le r-2$, and $P_{r-1}=\rho(r,r)=1$.

More generally, a direct application of Expressions (2.2)-(2.3) (see Theorem 2.1) and Expression (4.3) (see Proposition 4.3) lead to the combinatorial formulas of the sequences of the Pell fundamental system $\mathfrak{P}_r$.

Proposition 4.4. *Let $\mathfrak{P}_r=\{(P_n^{(s)})_{n\ge 0};\ 1\le s\le r\}$ be the Pell fundamental system, associated with the generalized Pell numbers* (1.1). *The combinatorial expression of each element $(P_n^{(s)})_{n\ge 0}$, where $1\le s\le r$, is given by,*

$$P_n^{(s)} = \sum_{j=1}^{s}\rho(n+s-j,r),\ \ \textit{when } 2\le s\le r, \tag{4.4}$$

$$P_n^{(1)} = P_{n-1}=\rho(n,r),\ \textit{for } n\ge r+1, \tag{4.5}$$

with $n\ge r+s$, where the $\rho(n,r)$ are given as in (4.3)*.*

Proof. Indeed, since $P_n^{(1)}=P_{n-1}$, Expressions (2.2) and (4.3) imply that $P_n^{(1)}=P_{n-1}=\rho(n,r)$, for every $n\ge r+1$. For $2\le j\le r-1$, Formulas (2.3) and (4.3), give immediately Expression (4.4), namely, $P_n^{(s)}=P_{n-1}+\cdots+P_{n-s}=\sum_{j=1}^{s}\rho(n-j+1,r)$. □

By a direct application of Theorem 3.2 and Proposition 4.4, we can obtain some identities involving the combinatorial expression (4.2) of the $\rho(n,r)$. More precisely, by combining Expressions (4.4)-(4.5) and (3.3)-(3.4), we arrive at the identities.

Corollary 4.2. *The combinatorial expressions of the generalized Pell numbers identities* (4.4)-(4.5)*, are given by*

$$\rho(m+s+1,r)=\sum_{d=1}^{r}\left[\sum_{j=1}^{d}\rho(m-j+1,r)\right]\rho(s+d,r), \tag{4.6}$$

$$\sum_{k=1}^{q}\rho(n+s-k+1,r)=\sum_{d=1}^{r}\left[\sum_{1\le i\le d,\ 1\le j\le q}\rho(n-i+1,r)\rho(s+d-j,r)\right]. \tag{4.7}$$

For $r=2$, formulas of Corollary 4.2 show that the combinatorial identities (4.6)-(4.7) take the form,

$$\rho(m+s+1,r)=\rho(m,r)\rho(s+1,r)+\rho(m+1,r)\rho(s,r),$$

for every $n \geq 2$ and $s \geq 0$. Let $r = 3$ then, for every $n \geq 2$ and $s \geq 0$, we have the identity,

$$\rho(m+s+1,r) = \rho(m+1,r)\rho(s+2,r) + (\rho(m,r) + \rho(m-1,r)\rho(s,r) + \rho(m,r)\rho(s+1,r),$$

On the other hand, a direct computation using the identity $\dfrac{(k_0+\cdots+k_{r-1}-1)!}{k_0!\ldots(k_p-1)!\ldots k_{r-1}!} = \dfrac{k_p}{k_0+\cdots+k_{r-1}} \times \dfrac{(k_0+\cdots+k_{r-1})!}{k_0!\ldots k_{r-1}!}$, allows us to obtain,

$$P_n = P_{n+1}^{(1)} = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n} \frac{k_1+\cdots+k_{r-1}}{k_0+\cdots+k_{r-1}} \frac{(k_0+\cdots+k_{r-1})!}{k_0!\ldots k_{r-1}!} 2^{k_0}.$$

This expression can be derived from [2, Proposition 3.1] and [5, Theorem 3.1].

It was established in that the Chen-Louck Theorems [5, Theorem 3.1] can be recovered by a direct computation from Expression (22) of [2]. In [8] the authors recall the Chen-Louck Theorem (see Theorem 5 of [8]), and formulate the combinatorial expression of the generalized Pell numbers $P_n^{(i)}$, by considering other kind of initial conditions, see, for instance [8, Corollary 2].

We conclude this section by observing that Expression (4.1) can be written under the form $\rho(n,r) = H_{n-r+1}^{(r)}(a_0,\ldots,a_{r-1})$, for every $n \geq r$, where the $H_n^{(r)}(x_1, \ldots, x_r)$ are the multivariate Fibonacci polynomials of order r (see [9]). Therefore, according to Expressions (4.3) and (4.5), the fundamental generalized Pell numbers can be written as a multivariate Fibonacci polynomials of order r under the form $P_n = \rho(n+1,r) = H_{n-r+2}^{(r)}(2, 1, \ldots, 1)$, for every $n \geq r$. On the other side, with the aid of Formula (4.4) we can deduce that each element of the set $\mathfrak{P}_r$ takes the form $P_n^{(s)} = \sum_{j=1}^{s} H_{n+s-j-r+2}^{(r)}(2, 1, \ldots, 1)$, for $2 \leq s \leq r$ and every $n \geq r$.

5 ANALYTICAL EXPRESSIONS OF THE GENERALIZED PELL NUMBERS

It well known that, the analytic formula for linear recursive sequences, is related to the roots of the associated (so-called) characteristic polynomial (see, for example, [3, 12], and references therein). Here the roots of the characteristic polynomials of the Pell recursive equation (1.1) given by $P(z) = z^r - 2z^{r-1} - z^{r-2} - \cdots - z - 1$ are simple. Indeed, we observe that for $r = 2$, the simple roots of the characteristic polynomial $P(z) = z^2 - 2z - 1$, are $\lambda_1 = 1 - \sqrt{2}$ and $\lambda_2 = 1 + \sqrt{2}$. For $r = 3$ we obtain the approximating simple roots of the characteristic polynomial $P(z) = z^3 - 2z^2 - z - 1$, are given by $\lambda_1 = -0.2734 + 0.5638i$, $\lambda_2 = -0.2734 - 0.5638i$ and $\lambda_3 = 2.5468$. For the general case $r \geq 4$, we have the following result.

Lemma 5.1. *For $r \geq 4$, the roots of the polynomial $P(z) = z^r - 2z^{r-1} - z^{r-2} - \cdots - z - 1$, are simple.*

Proof. For $r \geq 4$, we have $P(z) = z^r - 2z^{r-1} - z^{r-2} - \cdots - z - 1 = z^r - z^{r-1} - \frac{z^r-1}{z-1}$. Hence, it ensue $P(z) = z^r - z^{r-1} - \frac{z^r-1}{z-1} = \frac{S(z)}{z-1}$, where $S(z) = z^{r+1} - 3z^r + z^{r-1} + 1$. Let $\lambda \in Z(P) = \{z \in \mathbb{C}, P(z) = 0\}$, since $P(1) \neq 0$, we show that $P(\lambda) = 0$ if, and only if, $S(\lambda) = 0$, or equivalently,

$$\lambda^{r+1} - 3\lambda^r + \lambda^{r-1} + 1 = 0. \tag{5.1}$$

Suppose that λ is a root of $P(z)$, with multiplicity ≥ 2. Let $P'(z) = \frac{S(z)-S'(z)(z-1)}{(z-1)^2}$, where $P'(z)$ be the derivative of $P(z)$. Therefore, $P'(\lambda) = 0$ implies $S'(\lambda) = 0$. Thus, get,

$$S'(\lambda) = [(r+1)\lambda^2 - 3r\lambda + (r-1)]\lambda^{r-2} = 0, \tag{5.2}$$

because $S(\lambda) = 0$ and $\lambda \neq 1$. On the other hand, since $S(0) = 1 \neq 0$, we derive that $S'(\lambda) = 0$ is equivalent to the equation,

$$(r+1)\lambda^2 - 3r\lambda + (r-1) = 0, \tag{5.3}$$

whose roots are $\lambda_1 = \dfrac{3r - \sqrt{5r^2+4}}{2(r+1)}$-$\lambda_2 = \dfrac{3r + \sqrt{5r^2+4}}{2(r+1)}$. Taking into account Expressions (5.1) and (5.2), allows us to show that λ satisfies the equation,

$$\lambda^{r+1} - \lambda^{r-1} = \lambda^{r-1}[\lambda^2 - 1] = \lambda^{r-1}[\lambda+1][\lambda-1] = r. \tag{5.4}$$

For the root λ_1, since $\sqrt{5r^2+4} < 3r$, for $r \geq 4$, we show that $\lambda_1 > 0$. Further along, a direct computation implies that $\lambda_1 - 1 < 0$. Therefore, we have, $0 < \lambda_1 < 1$. Using Expression (5.4), we have $r = \lambda^{r-1}[\lambda+1][\lambda-1] < 0$, which is impossible, because $r \geq 4$. Consequently, the root λ_1 is not a root of the polynomial $P'(z)$ or equivalently, λ_1 is not a root of the polynomial $P(z)$ of multiplicity ≥ 2 .

Let consider the root λ_2. For each $r \geq 4$, we have $\lambda_2 \geq \frac{5r}{2(r+1)} = 2 + \frac{r-4}{2(r+1)} > 2$. Once again, using Expression (5.4), we obtain,

$$r = \lambda_2^{r-1}[\lambda_2^2 - 1] > 3\lambda_2^{r-1} > 3 \times 2^{r-1} > 3(r-1) > r.$$

which is impossible. Thus, the root λ_2 is not a root of $P'(z)$, namely, λ_2 is not a root of multiplicity ≥ 2 of the polynomial $P(z)$. Therefore, the roots of the polynomial $P(z)$ are simple. $\square$

In the aim to apply Lemma 5.1 for providing the analytic formula of each sequence of the set $\mathfrak{P}_r$, we are going to use the result of [3, Theorem 2.2], where the combinatorial expression (4.1) of $\rho(n,r)$, is expressed in terms of the roots of the characteristic polynomial $P(z) = z^r - a_1 z^{r-1} - \cdots - a_{r-2}z - a_r$. Indeed, this expression of $\rho(n,r)$ is given by,

Lemma 5.2. *(see [2,3]) Suppose that the roots $\lambda_1, \cdots, \lambda_r$ of $P(z) = z^r - a_1 z^{r-1} - \cdots - a_{r-2}z - a_r$ ($a_r \neq 0$) satisfy $\lambda_i \neq \lambda_j$ for $i \neq j$. Then, we have*

$$\rho(n,r) = \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{P'(\lambda_i)} = \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{\prod\limits_{k \neq i}(\lambda_i - \lambda_k)} \textit{ for every } n \geq r, \tag{5.5}$$

otherwise $\rho(r,r) = 1$, $\rho(i,r) = 0$ for $i \leq r-1$, where $P'(z) = \frac{dP}{dz}(z)$.

Following Propositions 4.3-4.4, the combinatorial expressions of the Pell fundamental system, are given by Expressions (4.3), (4.4) and (4.5), namely, we have,

$$P_n = \rho(n+1,r),\ P_n^{(1)} = P_{n-1} = \rho(n,r) \text{ and } P_n^{(s)} = \sum_{j=1}^{s} \rho(n+s-j,r),$$

for $n \geq r$, $n \geq r+1$ or $r \geq r+s$, respectively, where the $\rho(n,r)$ are given as in (4.1), with $a_1 = 2$, $a_2 = \cdots = a_r = 1$. Using Lemmas 5.1 and 5.2, and Expressions (4.3)-(4.5), we show that the analytical expression of each sequence of $\mathfrak{P}_r$, is as follows.

Theorem 5.3. *Let $\mathfrak{P}_r = \{(P_n^{(s)})_{n\geq 0},\ 1 \leq s \leq r\}$ be the set defined as in* (1.2). *Then, the analytic expression of each sequence $(P_n^{(s)})_{n\geq 0}$ ($1 \leq s \leq r$), is given by,*

$$P_n = \rho(n+1,r) = \sum_{i=1}^{r} \frac{\lambda_i^n}{P'(\lambda_i)} = \sum_{i=1}^{r} \frac{\lambda_i^n}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)}, \textit{for } n \geq r,$$

$$P_n^{(1)} = P_{n-1} = \rho(n,r) = \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{P'(\lambda_i)} = \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)}, \textit{for } n \geq r+1,$$

$$P_n^{(s)} = \sum_{j=1}^{s} \rho(n+s-j,r) = \sum_{j=1}^{s}\sum_{i=1}^{r} \frac{\lambda_i^{n+s-j-1}}{P'(\lambda_i)} = \sum_{j=1}^{s}\sum_{i=1}^{r} \frac{\lambda_i^{n+s-j-1}}{\prod\limits_{k\neq i}(\lambda_i - \lambda_k)}, \textit{for } r \geq r+s,$$

where $\lambda_1,\ldots,\lambda_r$ the simple roots of the polynomial $P(z) = z^r - 2z^{r-1} - z^{r-2} - \cdots - z - 1$.

For $r = 3$, the fundamental system is $\mathfrak{P}_3 = \{(P_n^{(s)})_{n\geq 0},\ 1 \leq s \leq 3\}$. The roots of the polynomial $P(z) = z^3 - 2z^2 - z - 1$ are $\lambda_1 = -0.2734 + 0.5638i$, $\lambda_2 = -0.2734 - 0.5638i$ and $\lambda_3 = 2.5468$. Then, using Theorem 5.3, we obtain,

$$P_n = \frac{\lambda_1^n}{3\lambda_1^2 - 4\lambda_1 - 1} + \frac{\lambda_2^n}{3\lambda_2^2 - 4\lambda_2 - 1} + \frac{\lambda_3^n}{3\lambda_3^2 - 4\lambda_3 - 1}, \text{for } n \geq 3,$$

$$P_n^{(1)} = \frac{\lambda_1^{n-1}}{3\lambda_1^2 - 4\lambda_1 - 1} + \frac{\lambda_2^{n-1}}{3\lambda_2^2 - 4\lambda_2 - 1} + \frac{\lambda_3^{n-1}}{3\lambda_3^2 - 4\lambda_3 - 1}, \text{for } n \geq 4,$$

$$P_n^{(2)} = \frac{\lambda_1^n + \lambda_1^{n-1}}{3\lambda_1^2 - 4\lambda_1 - 1} + \frac{\lambda_2^n \lambda_2^{n-1}}{3\lambda_2^2 - 4\lambda_2 - 1} + \frac{\lambda_3^n + \lambda_3^{n-1}}{3\lambda_3^2 - 4\lambda_3 - 1}, \text{for } n \geq 5.$$

Proposition 2.1, Lemma 5.3 and Theorem 5.3 imply the following general result.

Proposition 5.5. *Let $(\widetilde{P}_n)_{n\geq 0}$ be a sequence of generalized Pell numbers defined by the recursive relation* (1.1)*, and whose initial conditions are $P_0 = \alpha_0$, $P_1 = \alpha_1, \ldots, P_{r-1} = \alpha_{r-1}$. Then, the analytic formula for $(\widetilde{P}_n)_{n\geq 0}$ is given by,*

$$\widetilde{P}_n = \alpha_0 \sum_{i=1}^{r} \frac{\lambda_i^{n-1}}{P'(\lambda_i)} + \alpha_1 \sum_{j=1}^{2}\sum_{i=1}^{r} \frac{\lambda_i^{n+2-j-1}}{P'(\lambda_i)} + \cdots + \alpha_{r-1} \sum_{i=1}^{r} \frac{\lambda_i^n}{P'(\lambda_i)} \tag{5.6}$$

where $\lambda_1,\ldots,\lambda_r$ the simple roots of the polynomial $P(z) = z^r - 2z^{r-1} - z^{r-2} - \cdots - z - 1$. For clarifying, take $r = 3$ and $(\widetilde{P}_n)_{n\geq 0}$ be a sequence of generalized Pell numbers, with initial conditions $P_0 = \alpha_0$, $P_1 = \alpha_1$, and $P_2 = \alpha_2$. Applying Proposition 5.5 we obtain the following analytic formula for $\widetilde{P}_n$.

$$\widetilde{P}_n = \alpha_0 \sum_{i=1}^{3} \frac{\lambda_i^{n-1}}{P'(\lambda_i)} + \alpha_1 \sum_{j=1}^{2} \sum_{i=1}^{3} \frac{\lambda_i^{n+2-j-1}}{P'(\lambda_i)} + \alpha_2 \sum_{i=1}^{3} \frac{\lambda_i^{n}}{P'(\lambda_i)}, \tag{5.7}$$

for every $n \geq 0$, where λ_1,λ_2, λ_3 the simple roots of the polynomial $P(z) = z^3 - 2z^2 - z - 1$.

Result of the previous Theorem 5.3 shows that we have a compact and explicit formula for the family of generalized Pell numbers $\mathfrak{P}_r$. More generally, for a given sequence of general Pell numbers, with arbitrary initial conditions, the compact explicit analytic formula is presented in Proposition 5.5. This formula is obtained without using the usual heavy computation of the determinant. In [8] the authors gave the analytic formulas of the family of generalized Pell $P_n^{(i)}$, only in terms of the determinants, see for instance [8, Theorem 4] and [8, Corollary 1].

6 PELL GENERALIZED CASSINI IDENTITY

We consider here the process of building the type of Cassini identity for the generalized Pell fundamental system $\mathfrak{P}_r$. Following Section 3 the Pell Casoratian matrix is given by

$$\widehat{C}_{\mathfrak{P}}(n) = [P(1,n), \ldots, P(j,n), \ldots, P(r,n)],$$

where $P(j,n)$ is the vector column $P(j,n) = (P_n^{(j)};\ P_{n+1}^{(j)};\ \ldots;\ P_{n+r-1}^{(j)})^t$, for every $n \geq r-1$ and every j, $1 \leq j \leq r$. Hence, the determinant properties imply that the Casoratian of the Pell fundamental system $\mathfrak{P}$, takes the following form,

$$C_{\mathfrak{P}}(n) \quad = \quad det([P(1,n), P(2,n), \ldots,, P(r-2,n), P(r,n-r+2), P(r,n)]),$$

for every $n \geq r$. By iteration of the preceding process and taking into account that $P_{n+1}^{(1)} = P_n$, we show that the generalized Pell Cassini identity, is obtained from the Pell Casoratian of the set $\mathfrak{P}_r$ as follows,

$$det([P(r,n+1), P(r,n), \ldots, P(r,n-j), \ldots, P(r,n-r+2)]) = \varepsilon.C_{\mathfrak{P}}(n),$$

where $\varepsilon = -1$ or $+1$. More precisely, let consider the permutation cycle σ_r defined by $\sigma_r = \tau_{1,2} o \tau_{2,3} o \ldots o \tau_{j,j+1} o \ldots o \tau_{r-1,r}$, where $\tau_{i,j}$ $(i \neq j)$ is the transposition which permutes i and j. Then, a straightforward computation, using $P_n^{(1)} = P_{n-1}$, permit us to obtain $det(\widetilde{\mathscr{C}}_{\mathfrak{P}}(n)) = \varepsilon(\sigma_r) C_{\mathfrak{P}}(n)$, where $\varepsilon(\sigma_r) = (-1)^{r-1}$ is the signature of $\sigma \in \mathscr{S}_r$, the group of permutations of the set $\{1, 2, \ldots, r\}$ and $\widetilde{\mathscr{C}}(n) = (\widetilde{\mathscr{C}}_{i,k}^{(n)})_{1 \leq i,k \leq r}$ is the matrix,

$$\widetilde{\mathscr{C}}_{\mathfrak{P}}(n) = [P(r,n), P(r,n-1), \ldots, P(r,n-j), \ldots, P(r,n-r+1)],$$

called the *Cassini matrix*, whose entries are given by $\widetilde{\mathscr{C}}_{i,k}^{(n)} = P_{n-k+i}$. Summarizing, the generalized Pell Cassini identity is formulated in the following result.

Theorem 6.4. *Let $\mathfrak{P} = \{(P_n^{(s)})_{n \geq 0};\ 1 \leq s \leq r\}$ be the Pell fundamental system, and consider the associated Casoratian $C_{\mathfrak{P}}(n)$. Then, the generalized Pell Cassini Identity, is given by,*

$$\begin{aligned} det([P(r,n), \ldots, P(r,n-r+1)]) \quad &= \quad \sum_{\sigma \in \mathscr{S}_r} \varepsilon(\sigma) P_{n-\sigma(1)+1} \ldots . P_{n-\sigma(r)+r} = \varepsilon(\sigma_r) C_{\mathfrak{P}}(n) \\ &= \quad (-1)^{(n+1)(r-1)}, \end{aligned}$$

where $\mathscr{S}_r$ is the group of permutations of the set $\{1, 2, \ldots, r\}$ and $\varepsilon(\sigma)$ is the signature of $\sigma \in \mathscr{S}_r$. Theorem 6.4 shows that the generalized Pell Cassini identity, is expressed in terms of the fundamental solution $(P_n)_{n\geq 0}$ of (1.1), considered as a linear difference equation. For $r = 3$, the generalized Pell Cassini identity is given by,

$$det([P(3,n),P(3,n-1),P(3,n-2)]) \quad = \quad \varepsilon(\sigma_3).C(n) = (-1)^{2(n+1)} = 1,$$

where $\varepsilon(\sigma_3)$ is the signature of σ_3. And, a direct computation shows that the preceding expression takes the form,

$$P_n(P_n^2 - P_{n+2}P_{n-2} - 2P_{n+1}P_{n-1}) + P_{n+2}P_{n-1}^2 + P_{n-2}P_{n+1}^2 = 1.$$

Moreover, using Expression (4.3), the generalized Cassini identity of order r, takes the following combinatorial form,

$$\sum_{\sigma\in\mathscr{S}_r} \varepsilon(\sigma) \left[\prod_{i=1}^{r} \rho(n-\sigma(i)+i+1,r)\right] = (-1)^{(n+1)(r-1)}, \tag{6.1}$$

where $\mathscr{S}_r$ is the group of permutations of the set $\{1, 2, \ldots r\}$ and $\varepsilon(\sigma)$ is the signature of $\sigma \in \mathscr{S}_r$.

We conclude this section by establishing the analytic formula of the generalized Pell Cassini identity. Indeed, combining the combinatorial identity (6.1), with the analytic formulas of P_n given in Theorem 5.3, we get the following analytic expression of the generalized Pell Cassini identity,

$$det([P(r,n),...,P(r,n-r+1)]) \quad = \quad \sum_{\sigma\in\mathscr{S}_r} \varepsilon(\sigma) \prod_{k=1}^{r} \left[\sum_{i=1}^{r} \frac{\lambda_i^{n-\sigma(k)+k}}{P'(\lambda_i)}\right] = (-1)^{(n+1)(r-1)},$$

where the λ_i are the simple roots of $P(z) = z^r - 2z^{r-1} - z^{r-2} - \cdots - 1$, and $\mathscr{S}_r$ is the permutations group of $\{1, 2, \ldots, r\}$, and $\varepsilon(\sigma)$ is the signature of $\sigma \in \mathscr{S}_r$.

The results of this section allow us to see that the Pell Cassini identity is formulated only in terms of the fundamental solution $(P_n)_{n\geq 0}$.

7 CONCLUSION AND PERSPECTIVES

In this study we have considered another approach for investigating the generalized fundamental Pell system, related to the difference equation (1.1) defining the generalized Pell numbers. Our advance is based on the properties of the fundamental system $\mathfrak{P}_r$. Therefore, some results and various identities about the generalized Pell numbers are established. On the other side, the analytic formula of the sequences that make up the set $\mathfrak{P}_r$ of generalized Pell fundamental system are established without using the usual method of the determinant. Furthermore, the generalized Pell Cassini identity is studied. Moreover, the comparison of literature is considered. It should be emphasized that, in the best of our knowledge, our procedure and results are not common in the literature.

Finally, it is noted that our approach can be used for to examine other type of generalized Pell numbers. Some partial and significant results have been obtained in this direction.

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RESUMO. Neste artigo investigamos os números de Pell generalizados de ordem $r \geq 2$ por meio das propriedades do sistema fundamental de números de Pell generalizados associado. Ou seja, o número de Pell generalizado de ordem $r \geq 2$ é expresso como uma combinação linear de um sistema fundamental de números de Pell generalizados. As propriedades deste sistema fundamental são examinadas e os resultados podem ser estabelecidos para números de Pell generalizados de ordem $r \geq 2$. Algumas identidades e resultados combinatórios são estabelecidos. Além disso, o estudo analítico do sistema fundamental de Pell generalizado e a identidade Pell-Cassini generalizada são fornecidos.

Palavras-chave: sistema fundamental de Pell generalizado, números de Pell generalizados, identidades combinatórias, representações analíticas, identidade de Cassini generalizada.

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