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Hybrid Quaternions of Leonardo

M. C. dos S. MANGUEIRA[1*], F. R. V. ALVES[2] and P. M. M. C. CATARINO[3]

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ABSTRACT. In this article, we intend to investigate the Leonardo sequence presenting the hybrid Leonardo quaternions. To explore Hybrid Quaternions of Leonardo, the *priori*, sequence of Leonardo, quaternions and hybrid numbers were presented. Soon after, its recurrence, characteristic equation, its relation with the Fibonacci quaternions, generating function, Binet's formula, as well as its extension to non-positive integer indices were developed. Finally, identities involving Leonardo's hybrid quaternions are presented.

Keywords: Leonardo sequence, hybrid Leonardo quaternions, hybrid numbers.

1 INTRODUCTION

Sequence of Leonardo has been discussed in works on pure mathematics. Initially this sequence was addressed by [5] which is presented as a recurrent sequence of integers that is related to the Fibonacci and Lucas sequences that can be applied in almost every field of science. Historically, little is known about this sequence, however [2] believes that this sequence may have been defined by Leonardo of Pisa (1170-1250). Studies around this sequence can be found in the works of [3, 24, 25].

Sequence of Leonardo corresponds to the following recurrence relationship:

$$Le_n = Le_{n-1} + Le_{n-2} + 1, n \geq 2, \tag{1.1}$$

being $Le_0 = Le_1 = 1$ its initial conditions. For $n+1$ we can rewrite this recurrence relationship as $Le_{n+1} = Le_n + Le_{n-1} + 1$. Subtracting $Le_n - Le_{n+1}$ we obtain another equivalent recurrence relation for this sequence. Watch:

$$Le_n - Le_{n+1} = Le_{n-1} + Le_{n-2} + 1 - Le_n - Le_{n-1} - 1$$

*Corresponding author: Milena Carolina Santos Mangueira – E-mail: milenacarolina24@gmail.com
[1]Federal Institute of Education, Science, Technology of Ceará - IFCE, Department of Mathematics, Treze of Maio, 2081, 60040-531, Fortaleza, CE, Brazil – E-mail: milenacarolina24@gmail.com https://orcid.org/0000-0003-2690-7033
[2]Federal Institute of Education, Science, Technology of Ceará - IFCE, Department of Mathematics, Treze of Maio, 2081, 60040-531, Fortaleza, CE, Brazil – E-mail: fregis@ifce.edu.br https://orcid.org/0000-0003-3710-1561
[3]University of Trás-os-Montes, Department of Mathematics, Portugal – E-mail: pcatarino23@gmail.com https://orcid.org/0000-0001-6917-5093

$$Le_{n+1} \quad = \quad 2Le_n - Le_{n-2} \tag{1.2}$$

The Leonardo sequence is related to the Fibonacci sequence, for $n \geq 0$ and $n \in \mathbf{N}$, is expressed by:

$$Le_n = 2F_{n+1} - 1.$$

In [5] we can find some properties involving this sequence, including the Binet formula and the generator function. Thus, progressing the studies of this sequence, we will present the hybrid quaternions of the Leonardo sequence.

Quaternions were developed by Willian Rowan Hamilton (1805-1865). In the paper [19], the author says that the quaternions arise from the attempt to generalize complex numbers in the form $z = a + bi$ in three dimensions. In [9], the author state that quatenions are hypercomplex numbers, they are studied in abstract algebra and *priori*, there are two quaternionic structures: the quaternions over $\mathbf{R}$, having real components, and the biquaternions over the $\mathbf{C}$, having complex components.

The quaternions are presented as formal sums of scalars with usual vectors of three-dimensional space, in four dimensions. Thus, a quaternion is described by:

$$q = a + bi + cj + dk$$

where a, b, c and d are real numbers and i, j, k the orthogonal part at the base $\mathbf{R}^3$. And still, in [13] the authors presents the quaternionic product as $i^2 = j^2 = k^2 = -1, ij = k = -ij, jk = i = -kj$ and $ki = j = -ik$.

On the other hand, there is the set of hybrid numbers, denoted by $\mathbf{K}$, presented by [20] where he studied three number systems together, namely: the complex, hyperbolic and dual numbers being combined with each other. A hybrid number is defined as:

$$\mathbf{K} = \{z = a + bi + c\varepsilon + dh : a,b,c,d \in \mathbf{R}, i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i\}$$

We can perform mathematical operations with hybrid numbers, such as: addition, subtraction, multiplication by scalar and the product between two hybrid numbers. And yet, we have the conjugate of a hybrid number $z = a + bi + c\varepsilon + dh$, denoted by $\bar{z}$, is defined as

$$\bar{z} = a - bi - c\varepsilon - dh$$

and the real number

$$C(z) = z\bar{z} = \bar{z}z = a^2 + (b-c)^2 - c^2 - d^2 = a^2 + b^2 - 2bc - d^2$$

is called the hybrid number character, where the root of that real number will be the hybrid number norm z, so we have to: $\|z\| = \sqrt{|C(z)|}$.

We can associate quaternions and hybrid numbers with linear recursive sequences, so we find work on the Padovan and Perrin quaternions in [10], Pell-Padovan in [23] and Fibonacci and Fibonacci Complexes in [9, 11, 12, 14] and hybrid sequence numbers in [4, 7, 16, 17, 18, 21, 22].

In the paper On the horadam hybrid quaternions, [8], the authors intended to present the hybrid Leonardo quaternions and some definitions, Binet's formula, generating function, some properties and its extension to non-positive integer index will be enunciated.

2 MAIN RESULTS

In this section, we will define Leonardo's hybrid quaternion numbers and we find some results.

At first we will define the hybrid numbers and Leonardo's quaternions. It is noteworthy that hybrid numbers of Leonardo were defined by Alp and Kocer (2021) [1].

Definition 2.1. *Leonardo's hybrid number, denoted by HLe_{n+1}, is defined by:*

$$HLe_n = Le_n + Le_{n+1}i + Le_{n+2}\varepsilon + Le_{n+3}h.$$

Definition 2.2. *The Recurrence Relationship for Leonardo's Hybrids, $n \geq 2$, is defined by:*

$$HLe_n = HLe_{n-1} + HLe_{n-2} + (1 + i + \varepsilon + h), \tag{2.1}$$

with $HLe_0 = 1 + i + 3\varepsilon + 5h$ and $HLe_1 = 1 + 3i + 5\varepsilon + 9h$ its initial terms.

Definition 2.3. *Leonardo's quaternion number, denoted by QLe_n, it is given by:*

$$QLe_n = Le_n + Le_{n+1}i + Le_{n+2}j + Le_{n+3}k.$$

Definition 2.4. *The recurrence relation for Leonardo's quaternions, $n \geq 2$, is defined by:*

$$QLe_n = QLe_{n-1} + QLe_{n-2} + (1 + i + j + k), \tag{2.2}$$

with $QLe_0 = 1 + i + 3j + 5k$ and $QLe_1 = 1 + 3i + 5j + 9k$ its initial terms.

Table 1: First terms of Leonardo's hybrids and quaternions

n	HLe_n	QLe_n
0	$1+i+3\varepsilon+5h$	$1+i+3j+5k$
1	$1+3i+5\varepsilon+9h$	$1+3i+5j+9k$
2	$3+5i+9\varepsilon+15h$	$3+5i+9j+15k$
3	$5+9i+15\varepsilon+25h$	$5+9i+15j+25k$
4	$9+15i+25\varepsilon+41h$	$9+15i+25j+41k$
5	$15+25i+41\varepsilon+67h$	$15+25i+41j+67k$
6	$25+41i+67\varepsilon+109h$	$25+41i+67j+109k$
$\vdots$	$\vdots$	$\vdots$

Now, from what was seen above, we will approach Leonardo's hybrid quaternions.

Definition 2.5. *Leonardo's hybrid quaternion number, denoted by $\tilde{Le}_n$ is defined as:*

$$\tilde{Le}_n = HLe_n + HLe_{n+1}i + HLe_{n+2}j + HLe_{n+3}k.$$

where i, j, k are the units of the quaternions and HLe_n it's the n-th Leonardo hybrid number. Thus, Leonardo's hybrid quaternions can be rewritten by:

$$\begin{aligned}
\tilde{Le}_n &= (Le_n + Le_{n+1}i + Le_{n+2}\varepsilon + Le_{n+3}h) + \\
&\quad (Le_{n+1} + Le_{n+2}i + Le_{n+3}\varepsilon + Le_{n+4}h)i + \\
&\quad (Le_{n+2} + Le_{n+3}i + Le_{n+4}\varepsilon + Le_{n+5}h)j + \\
&\quad (Le_{n+3} + Le_{n+4}i + Le_{n+5}\varepsilon + Le_{n+6}h)k \\
&= \widehat{HLe}_n + \widehat{HLe}_{n+1}i + \widehat{HLe}_{n+2}\varepsilon + \widehat{HLe}_{n+3}h
\end{aligned}$$

where i, ε and h are the imaginary units of the hybrid numbers and $\widehat{HLe}_n = Le_n + Le_{n+1}i + Le_{n+2}j + Le_{n+3}k$.

Definition 2.6. *The recurrence relationship for Leonardo's hybrid quaternions, $n \geq 2$, is defined by:*

$$\tilde{Le}_{n+1} = 2\tilde{Le}_n - \tilde{Le}_{n-2}, \tag{2.3}$$

with the following initial terms: $\tilde{Le}_0 = HLe_0 + HLe_1 i + HLe_2 j + HLe_3 k$, $\tilde{Le}_1 = HLe_1 + HLe_2 i + HLe_3 j + HLe_4 k$ and $\tilde{Le}_2 = HLe_2 + HLe_3 i + HLe_4 j + HLe_5 k$. And yet, extending these non-positive integer indices, we have:

Definition 2.7. *The recurrence relation for the hybrid quaternions of non-positive Leonardo indices, $n \geq 0$, is defined by:*

$$\tilde{Le}_{-n} = 2\tilde{Le}_{-n+2} - \tilde{Le}_{-n+3}.$$

The work of [2] reports that the Leonardo sequence is similar to the Fibonacci sequence, differing from the Leonardo recurrence, which presents the sum of unit 1 in its recurrence. Knowing this information, another recurrence for the hybrid Leonardo quaternions can be presented, as follows.

Theorem 2.1. *Leonardo's hybrid quaternion, $\tilde{Le}_n$, satisfies the following recurrence:*

$$\tilde{Le}_{n+1} = \tilde{Le}_n + \tilde{Le}_{n-1} + [\Theta + \Theta i + \Theta j + \Theta k], \tag{2.4}$$

where $\Theta = 1 + i + \varepsilon + h$. ***Proof.***

$$\begin{aligned}
\tilde{Le}_n + \tilde{Le}_{n-1} + [\Theta + \Theta i + \Theta j + \Theta k] &= HLe_n + HLe_{n+1} i + HLe_{n+2} j + HLe_{n+3} k \\
&+ HLe_{n-1} + HLe_n i + HLe_{n+1} j + HLe_{n+2} k \\
&+ [\Theta + \Theta i + \Theta j + \Theta k] \\
&= (HLe_n + HLe_{n-1} + \Theta) + (HLe_{n+1} + HLe_n + \Theta) i \\
&+ (HLe_{n+2} + HLe_{n+1} + \Theta) j + (HLe_{n+3} + HLe_{n+2} + \Theta) k \\
&= HLe_{n+1} + HLe_{n+2} i + HLe_{n+3} j + HLe_{n+4} k \\
&= \tilde{Le}_{n+1}
\end{aligned}$$

□

We can present a relationship between the hybrid quaternions of Leonardo and Fibonacci.

Theorem 2.2. *For $n \geq 0$, we have the relationship between the hybrid Leonardo quaternions with the hybrid Fibonacci quaternions, defined by:*

$$\tilde{Le}_n = 2\tilde{F}_{n+1} - [\Theta + \Theta i + \Theta j + \Theta k], \tag{2.5}$$

where $\Theta = 1 + i + \varepsilon + h$. **Proof.** Hybrid Fibonacci quaternions were presented in the work of [8]. With that, let's prove this Theorem by induction.
For $n = 0$

$$\begin{aligned}
\tilde{Le}_0 &= 2\tilde{F}_1 - [\Theta + \Theta i + \Theta j + \Theta k] \\
&= 2(HF_1 + HF_2 i + HF_3 j + HF_4 k) - [\Theta + \Theta i + \Theta j + \Theta k] \\
&= (2HF_1 - \Theta) + (2HF_2 - \Theta) i + (2HF_3 - \Theta) j + (2HF_4 - \Theta) k \\
&= (1 + i + 3\varepsilon + 5h) + (1 + 3i + 5\varepsilon + 9h) i + (2 + 5i + 9\varepsilon + 15h) j + \\
&\quad (5 + 9i + 15\varepsilon + 25h) k \\
&= HLe_0 + HLe_1 i + HLe_2 j + HLe_3 k \\
&= \tilde{Le}_0
\end{aligned}$$

Assuming equality holds for all $1 < t \leq n$, we have $\tilde{Le}_t = 2\tilde{F}_{t+1} - [\Theta + \Theta i + \Theta j + \Theta k]$. Now let's prove that it is valid for $t+1$, using the recurrence 2.4 and the induction hypothesis, we get:

$$\begin{aligned}
\tilde{Le}_{t+1} &= \tilde{Le}_t + \tilde{Le}_{t-1} + [\Theta + \Theta i + \Theta j + \Theta k] \\
&= 2\tilde{F}_{t+1} - [\Theta + \Theta i + \Theta j + \Theta k] + 2\tilde{F}_t - [\Theta + \Theta i + \Theta j + \Theta k] + \\
&\quad [\Theta + \Theta i + \Theta j + \Theta k] \\
&= 2(\tilde{F}_{t+1} + \tilde{F}_t) - [\Theta + \Theta i + \Theta j + \Theta k] \\
&= 2\tilde{F}_{t+2} - [\Theta + \Theta i + \Theta j + \Theta k]
\end{aligned}$$

Thus the proposition is true. □

According to the recurrence relationship, $\tilde{Le}_{n+1} = 2\tilde{Le}_n - \tilde{Le}_{n-2}$, one can present its characteristic equation, through the relation

$$\frac{\tilde{Le}_{n+1}}{\tilde{Le}_n} = 2 - \frac{\tilde{Le}_{n-2}}{\tilde{Le}_n}.$$

Using the reasoning performed by T. Koshy in [15], we conjecture that the sequence of quotients $\frac{\tilde{Le}_{n+1}}{\tilde{Le}_n}$ converges to one positive number. So, one has to $\frac{\tilde{Le}_{n+1}}{\tilde{Le}_n} = 2 - \frac{\tilde{Le}_{n-2}}{\tilde{Le}_n} \cdot \frac{\tilde{Le}_{n-1}}{\tilde{Le}_{n-1}} \Rightarrow \frac{\tilde{Le}_{n+1}}{\tilde{Le}_n} = 2 - \frac{1}{\frac{\tilde{Le}_{n-1}}{\tilde{Le}_{n-2}} \cdot \frac{\tilde{Le}_n}{\tilde{Le}_{n-1}}}$. Denoting $x_n = \frac{\tilde{Le}_{n+1}}{\tilde{Le}_n}$, we have: $x_{n-1} = \frac{\tilde{Le}_n}{\tilde{Le}_{n-1}}$ and $x_{n-2} = \frac{\tilde{Le}_{n-1}}{\tilde{Le}_{n-2}}$. Determining the equation $x_n = 2 + \frac{1}{z_{n-1} \cdot z_{n-2}}$. One may note that the sequence is monotonous bounded, so passing the limit and making n tend to infinity in this last expression, we have:

$$\begin{aligned}
\lim_{n\to\infty} x_n &= 2 - \frac{1}{\lim_{n\to\infty} x_{n-1} \cdot \lim_{n\to\infty} x_{n-2}} \\
x &= 2 - \frac{1}{x^2} \\
x^3 - 2x^2 + 1 &= 0,
\end{aligned}$$

where the previous equation having three roots, two of which are equal to the roots of the characteristic equation of the Fibonacci sequence and one is equal to 1.

Definition 2.8. *Leonardo's hybrid quaternion conjugate can be defined in three different types for $\tilde{Le}_n = \widehat{HLe}_n + \widehat{HLe}_{n+1} i + \widehat{HLe}_{n+2} \varepsilon + \widehat{HLe}_{n+3} h$:*

- *Quaternion conjugate, $\overline{\tilde{Le}_n}$: $\overline{\tilde{Le}_n} = \overline{\widehat{HLe}_n} + \overline{\widehat{HLe}_{n+1}} i + \overline{\widehat{HLe}_{n+2}} \varepsilon + \overline{\widehat{HLe}_{n+3}} h$;*

- *Hybrid conjugate, $(\tilde{Le}_n)^C$: $(\tilde{Le}_n)^C = \widehat{HLe}_n - \widehat{HLe}_{n+1} i - \widehat{HLe}_{n+2} \varepsilon - \widehat{HLe}_{n+3} h$;*

- *Total conjugate, $(\tilde{Le}_n)^T$: $(\tilde{Le}_n)^T = \overline{(\tilde{Le}_n)^C} = \overline{\widehat{HLe}_n} - \overline{\widehat{HLe}_{n+1}} i - \overline{\widehat{HLe}_{n+2}} \varepsilon - \overline{\widehat{HLe}_{n+3}} h$.*

Next, we will provide the generating function for $\tilde{Le}_n$, its Binet formula and some identities found involving these sequences.

Theorem 2.3. *The generating function for hybrid quaternions of Leonardo numbers, denoted by $G_{\tilde{Le}_n}(t)$, is:*

$$G_{\tilde{Le}_n}(t) = \frac{(\tilde{Le}_0 + \tilde{Le}_1 t)(1-2t) + \tilde{Le}_2 t^2}{1-2t+t^3}$$

Proof. *Let us write a formal sum, where each portion of this sum has as a coefficient one element of hybrid quaternions of Leonardo number sequence.*

$$G_{\tilde{Le}_n}(t) = \sum_{n=0}^{\infty} \tilde{Le}_n t^n$$

Making algebraic manipulations due to the recurrence relation we can write this series as:

$$\begin{aligned}
G_{\tilde{Le}_n}(t) &= \tilde{Le}_0 + \tilde{Le}_1 t + \tilde{Le}_2 t^2 + \sum_{n=3}^{\infty} \tilde{Le}_n t^n \\
&= \tilde{Le}_0 + \tilde{Le}_1 t + \tilde{Le}_2 t^2 + \sum_{n=3}^{\infty} (2\tilde{Le}_{n-1} - \tilde{Le}_{n-3}) t^n \\
&= \tilde{Le}_0 + \tilde{Le}_1 t + \tilde{Le}_2 t^2 + 2t \sum_{n=3}^{\infty} \tilde{Le}_{n-1} t^{n-1} - t^3 \sum_{n=3}^{\infty} \tilde{Le}_{n-3} t^{n-3} \\
&= \tilde{Le}_0 + \tilde{Le}_1 t + \tilde{Le}_2 t^2 + 2t \sum_{n=0}^{\infty} (\tilde{Le}_n t^n - \tilde{Le}_0 - \tilde{Le}_1 t) - t^3 \sum_{n=0}^{\infty} \tilde{Le}_n t^n \\
&= \tilde{Le}_0 + \tilde{Le}_1 t + \tilde{Le}_2 t^2 - 2\tilde{Le}_0 t - 2\tilde{Le}_1 t^2 + 2t \sum_{n=0}^{\infty} \tilde{Le}_n t^n - t^3 \sum_{n=0}^{\infty} \tilde{Le}_n t^n \\
&= \tilde{Le}_0 + \tilde{Le}_1 t + \tilde{Le}_2 t^2 - 2\tilde{Le}_0 t - 2\tilde{Le}_1 t^2 + 2t G_{\tilde{Le}_n} - t^3 G_{\tilde{Le}_n}
\end{aligned}$$

So we have:

$$\begin{aligned}
G_{\tilde{Le}_n}(t) - 2t G_{\tilde{Le}_n} + t^3 G_{\tilde{Le}_n} &= \tilde{Le}_0 + \tilde{Le}_1 t + \tilde{Le}_2 t^2 - 2\tilde{Le}_0 t - 2\tilde{Le}_1 t^2 \\
G_{\tilde{Le}_n}(t)(1-2t+t^3) &= \tilde{Le}_0(1-2t) + \tilde{Le}_1 t(1-2t) + \tilde{Le}_2 t^2 \\
G_{\tilde{Le}_n}(t) &= \frac{(\tilde{Le}_0 + \tilde{Le}_1 t)(1-2t) + \tilde{Le}_2 t^2}{1-2t+t^3}
\end{aligned}$$

□

Now we will explore the existence of the Binet formula, this formula calculates the n-th term of the sequence, without depending on recurrence, where it is necessary to use the roots of the characteristic equation.

Theorem 2.4. *Binet formula of hybrid quaternions of Leonardo, with $n \in \mathbf{Z}$, is given by:*

$$\tilde{Le}_n = Ax_1^n + Bx_2^n + Cx_3^n,$$

on what $x_1 = \frac{1+\sqrt{5}}{2}$, $x_2 = \frac{1-\sqrt{5}}{2}$, $x_3 = 1$ *are the roots of the characteristic polynomial* $x^3 - 2x^2 + 1 = 0$ *and*

$$\begin{aligned} A &= \frac{(\tilde{Le}_2) + (-x_2 - x_3)(\tilde{Le}_1) + x_2x_3(\tilde{Le}_0)}{x_1^2 - x_1x_2 - x_1x_3 + x_2x_3}, \\ B &= \frac{(\tilde{Le}_2) + (-x_1 - x_3)(\tilde{Le}_1) + x_1x_3(\tilde{Le}_0)}{x_2^2 - x_2x_3 - x_1x_2 + x_1x_3}, \\ C &= \frac{(\tilde{Le}_2) + (-x_1 - x_2)(\tilde{Le}_1) + x_1x_2(\tilde{Le}_0)}{x_3^2 + x_1x_2 - x_1x_3 - x_2x_3}. \end{aligned}$$

Proof.

Through Binet formula $\tilde{Le}_n = Ax_1^n + B\beta x_2^n + Cx_3^n$ *and the initial values* $\tilde{Le}_0 = HLe_0 + HLe_1 i + HLe_2 j + HLe_3 k$, $\tilde{Le}_1 = HLe_1 + HLe_2 i + HLe_3 j + HLe_4 k$ *and* $\tilde{Le}_2 = HLe_2 + HLe_3 i + HLe_4 j + HLe_5 k$, *it is possible to obtain the following system of equations:*

$$\begin{cases} A + B + C & = \tilde{Le}_0 \\ Ax_1 + Bx_2 + Cx_3 & = \tilde{Le}_1 \\ Ax_1^2 + Bx_2^2 + Cx_3^2 & = \tilde{Le}_2 \end{cases}$$

Solving the system, you have to:

$$\begin{aligned} A &= \frac{(\tilde{Le}_2) + (-x_2 - x_3)(\tilde{Le}_1) + x_2x_3(\tilde{Le}_0)}{x_1^2 - x_1x_2 - x_1x_3 + x_2x_3}, \\ B &= \frac{(\tilde{Le}_2) + (-x_1 - x_3)(\tilde{Le}_1) + x_1x_3(\tilde{Le}_0)}{x_2^2 - x_2x_3 - x_1x_2 + x_1x_3}, \\ C &= \frac{(\tilde{Le}_2) + (-x_1 - x_2)(\tilde{Le}_1) + x_1x_2(\tilde{Le}_0)}{x_3^2 + x_1x_2 - x_1x_3 - x_2x_3}. \end{aligned}$$

□

3 IDENTITIES

In this section, we will present identities of hybrid quaternions of Leonardo and identities that relate hybrid quaternions of Leonardo to hybrid quaternions of Fibonacci.

Identity 1. *The sum of the n first numbers of hybrid quaternions of Leonardo is given by:*

$$\sum_{j=0}^{n} \tilde{Le}_j = \tilde{Le}_{n+2} - [n + 1 + (\Theta + \Theta i + \Theta j + \Theta k)], \tag{3.1}$$

being $\Theta = 1+i+\varepsilon+h$. ***Proof.*** *Using the Theorem 2.2 and the work of [6] gives us the identity $\tilde{F}_{n+3} = \tilde{F}_{n+2} + \tilde{F}_{n+1}$. So we have:*

$$
\begin{aligned}
\sum_{j=0}^{n} \tilde{Le}_j &= \sum_{j=0}^{n} [2\tilde{F}_{j+1} - (\Theta + \Theta i + \Theta j + \Theta k)] \\
&= 2\sum_{j=0}^{n} \tilde{F}_{j+1} - [n + (\Theta + \Theta i + \Theta j + \Theta k)] \\
&= 2\sum_{j=0}^{n+1} \tilde{F}_{j} - [n + (\Theta + \Theta i + \Theta j + \Theta k)] \\
&= 2(\tilde{F}_{n+2} + \tilde{F}_{n+1} - 1) - [n + (\Theta + \Theta i + \Theta j + \Theta k)] \\
&= (2\tilde{F}_{n+3} - 1) - [n + 1 + (\Theta + \Theta i + \Theta j + \Theta k)] \\
&= \tilde{Le}_{n+2} - [n + 1 + (\Theta + \Theta i + \Theta j + \Theta k)]
\end{aligned}
$$

□

Identity 2. *The sum of the hybrid Leonardo quaternions of $2n$ indices can be described by:*

$$\sum_{j=0}^{n} \tilde{Le}_{2j} = \tilde{Le}_{2n+1} - n$$

Proof. *Using the Theorem 2.2, we have:*

$$
\begin{aligned}
\sum_{j=0}^{n} \tilde{Le}_{2j} &= \sum_{j=0}^{n} [2\tilde{F}_{2j+1} - (\Theta + \Theta i + \Theta j + \Theta k)] \\
&= 2\sum_{j=1}^{n} \tilde{F}_{2j-1} - [n + (\Theta + \Theta i + \Theta j + \Theta k)] \\
&= 2\tilde{F}_{2n+2} - [n + (\Theta + \Theta i + \Theta j + \Theta k)] \\
&= [2\tilde{F}_{(2n+1)+1} - (\Theta + \Theta i + \Theta j + \Theta k)] - n \\
&= \tilde{Le}_{2n+1} - n
\end{aligned}
$$

□

Identity 3. *The sum of hybrid Leonardo quaternions of indices $2n+1$ can be described by:*

$$\sum_{j=0}^{n} \tilde{Le}_{2j+1} = QLe_{2n+2} - (n+2)$$

Proof. *Using the Theorem 2.2, we have:*

$$\begin{aligned}
\sum_{j=0}^{n} \tilde{Le}_{2j+1} &= \sum_{j=0}^{n} [2\tilde{F}_{2j+2} - (\Theta + \Theta i + \Theta j + \Theta k)] \\
&= 2\sum_{j=0}^{n+1} \tilde{F}_{2j} - [n + (\Theta + \Theta i + \Theta j + \Theta k)] \\
&= 2\tilde{F}_{2n+3} - [n + (\Theta + \Theta i + \Theta j + \Theta k)] \\
&= [2\tilde{F}_{(2n+2)+1} - (\Theta + \Theta i + \Theta j + \Theta k)] - (n+2) \\
&= \tilde{Le}_{2n+1} - (n+2)
\end{aligned}$$

□

Identity 4. *For $n \geq 0$, we have:*

$$\sum_{j=0}^{n} (\tilde{F}_j + \tilde{Le}_j) = \tilde{F}_{n+2} + \tilde{Le}_{n+2} - [(n+1) + 2(\Theta + \Theta i + \Theta j + \Theta k)],$$

being $\Theta = 1 + i + \varepsilon + h$. ***Proof.*** *Using the Identity 3.1, we have:*

$$\begin{aligned}
\sum_{j=0}^{n} (\tilde{F}_j + \tilde{Le}_j) &= \sum_{j=0}^{n} \tilde{F}_j + \sum_{j=0}^{n} \tilde{Le}_j \\
&= \tilde{F}_{n+1} + \tilde{F}_n - (\Theta + \Theta i + \Theta j + \Theta k) + \tilde{Le}_{n+2} - [n + 1 + (\Theta + \Theta i + \Theta j + \Theta k)] \\
&= \tilde{F}_{n+2} + \tilde{Le}_{n+2} - [(n+1) + 2(\Theta + \Theta i + \Theta j + \Theta k)]
\end{aligned}$$

□

4 CONCLUSION

This work presents hybrid quaternions of Leonardo, based on the work of Dagdeviren and Kürüz (2020). Thus, by presenting the recurrence of hybrid quaternions of Leonardo, it was possible to explore his characteristic equation which has three real roots, two of which are equal to the Fibonacci hybrid quaternion equation. In addition to the roots of the characteristic equation, the hybrid Leonardo quaternions present a relationship with the hybrid Fibonacci quaternions and we conclude that this relationship is important for demonstrating the identities presented in the work. Furthermore, in this work, its generating function, recurrence for non-positive integer indices and identities around these numbers was shown.

For future work, one can extend the definitions, recurrence properties and related identities of hybrid quaternions to other possible integer sequences. Finally, this article makes it possible to contribute in the mathematical field and provide a study with knowledge about the set of hybrid numbers, sequence of Leonardo and its evolutionary process.

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