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ORIGINAL ARTICLE

Deflection estimate of reinforced concrete beams by the lumped damage mechanics

Estimativa de flecha em vigas de concreto armado pela mecânica do dano concentrado

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Abstract: The evaluation of the deflection in beams is an indispensable step of the structural design. Currently, many standard codes adopt the Branson's model. However, the Branson's model underestimates the deflection of beams with a reinforcement rate of less than 1%. Therefore, this study proposes a new alternative to quantify the deflections in reinforced concrete beams, based on the Lumped Damage Mechanics (LDM). LDM is a nonlinear theory, which uses concepts from Fracture and Damage Mechanics combined with plastic hinges. The viability of the proposed model was verified through comparisons with results from experimental works developed by other authors and the application of the Branson's model. The obtained results showed that the proposed calculation model had a good approximation of the experimental data with satisfactory accuracy and equivalent values to the Branson's model in the investigated scenarios.

Keywords: reinforced concrete beams, deflection, lumped damage mechanics, plastic hinge, structural design.

Resumo: A avaliação de flechas em vigas é uma etapa indispensável no projeto estrutural. Atualmente, muitas normas adotam o modelo de Branson. Entretanto, o modelo de Branson subestima a flecha de vigas com taxa de armadura menor do que 1%. Desta forma, este estudo propõe uma alternativa para quantificar flechas em vigas de concreto armado com base na Mecânica do Dano Concentrado (MDC). A MDC é uma teoria não linear que utiliza conceitos das Mecânicas da Fratura e do Dano combinados com rótulas plásticas. A viabilidade do modelo proposto foi verificada por meio da comparação com resultados experimentais obtidos por outros autores bem como a aplicação do modelo de Branson. Os resultados obtidos mostram que o modelo proposto tem boa aproximação aos dados experimentais com acurácia satisfatória e valores equivalentes ao modelo de Branson nos cenários investigados.

Palavras-chave: vigas de concreto armado, flecha, mecânica do dano concentrado, rótula plástica, projeto estrutural.

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1 INTRODUCTION

Structural response in service is an important issue in civil engineering. Design codes around the world estimate the immediate deflection of reinforced concrete (RC) beams based on the equation proposed by Branson [1], [2] for estimating the equivalent inertia moments of cracked RC members (e.g., [3]–[6]). Such design codes present small variations for the formulation proposed by Branson [1], [2].

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Recently, other researchers have initiated improvements to Branson's model, especially for concrete reinforced with other materials, such as fibre-reinforced polymers [7]–[20]. Gribniak et al. [21] presented a statistical study of the immediate deflections of RC beams evaluated by different design codes [3], [22], [23] and smeared crack numerical analysis with several finite elements. Despite the accuracy of the finite element analysis (FEA) presented in [21], its application to design engineering practice is unfeasible.

Lumped damage mechanics (LDM) appears as an interesting alternative to FEA because of its use of few finite elements resulting from its combination of key concepts from classic fracture [24] and damage mechanics [25] with plastic hinges. For a review on LDM, see [26].

Hence, this study mainly aims to propose a simplified formulation for estimating the immediate deflections of RC beams based on the LDM framework. Different from the work in [1], [2], which is solely based on experimental observations, the model proposed in the current study is supported by the popular and widely accepted concepts of fracture and damage mechanics, such as effective stress, strain equivalence hypothesis and the Griffith energy criterion.

2 DEFLECTION OF REINFORCED CONCRETE BEAMS

To describe this nonlinear behaviour of RC beams, Branson [1] performed an experimental study on rectangular and 'T' beams applied with uniformly distributed short-term loads.

A formulation was subsequently proposed to calculate the immediate deflection based on an effective moment of inertia. This formula establishes a proportional relationship between the moment of inertia of the gross concrete section about the centroidal axis I_g; and the moment of inertia of the cracked section transformed into concrete I_{cr}. Based on a multiplier factor, the ratio between the first cracking moment M_{cr} and the maximum moment in the beam due to service loads at the deflection stage M_a is calculated. Branson's model [1] is expressed by Equation 1.

$$I_{eff} = (M_{cr}/M_a)^m I_g + [1 - (M_{cr}/M_a)^m] I_{cr} \tag{1}$$

Exponent m equal to 3 is adopted for the calculation of a reference section for the entire span. The calculation considers the sum of the effects of the loss of stiffness and the contribution of the concrete in the traction area between cracks, in the cracked region of the span, and also the region without visible cracks. For the calculation of an individual section, exponent m is assigned with a value of 4 [27].

According to Bischoff [15], Branson's Model [1] works well for RC beams with a reinforcement rate between 1% and 2%, which was the standard reinforcement rate in the past. However, the equation underestimates the deflection of RC beams with a reinforcement rate below 1%; corroborating the results obtained in this study.

3 LUMPED DAMAGE MECHANICS

Consider the beam element depicted in Figure 1, where L denotes the span. The deformed shape of such beam can be described by two relative rotations at edges i and j i.e. ϕ_i and ϕ_j, respectively (Figure 1a). These relative rotations, now called generalised deformations [28], are conjugated to two bending moments (m_i and m_j) named generalised stresses [28] (Figure 1a).

The transverse displacement along the beam element is represented by a cubic polynomial function w(x) (Figure 1b). Then, the boundary conditions are:

$$w(0) = w(L) = 0 \qquad -w_{,x}|_{x=0} = \phi_i \qquad -w_{,x}|_{x=L} = \phi_j \tag{2}$$

Therefore, the transverse displacement field is described as follows:

$$w(x) = (-x^3/L^2 + 2x^2/\mathrm{L} - x)\phi_i + (-x^3/L^2 + x^2/\mathrm{L})\phi_j \tag{3}$$

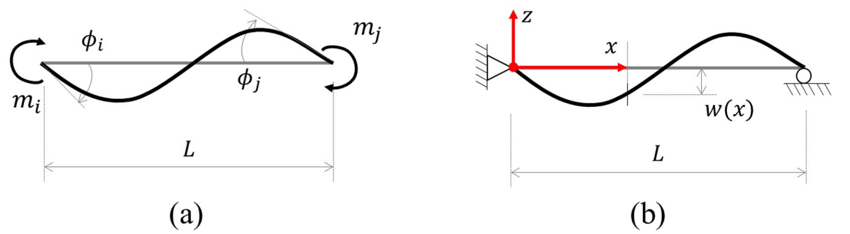

Figure 1. Deformed shape of a beam element: (a) generalised deformations and stresses; (b) transverse displacement field.

Now, considering that the generalised deformations are elastic, i.e. ϕ_i^e and ϕ_j^e, the bending moment distribution along the beam element can be written as:

$$M(x) = EI_g w_{,xx} = EI_g\left[(-6x/L^2 + 4/L)\phi_i^e + (-6x/L^2 + 2/L)\phi_j^e\right] \tag{4}$$

As the bending moments at the edges of the beam element are m_i and m_j (Figure 1a), then:

$$M(0) = \left(4EI_g/L\right)\phi_i^e + \left(2EI_g/L\right)\phi_j^e = m_i \tag{5}$$

$$M(L) = -\left(2EI_g/L\right)\phi_i^e - \left(4EI_g/L\right)\phi_j^e = -m_j \tag{6}$$

Equations 5, 6 can be rewritten in terms of generalised deformations, i.e.

$$\phi_i^e = \left(L/3EI_g\right)m_i - \left(L/6EI_g\right)m_j \tag{7}$$

$$\phi_j^e = -\left(L/6EI_g\right)m_i + \left(L/3EI_g\right)m_j \tag{8}$$

Equations 7, 8 can be expressed in matrix form, as:

$$\{\boldsymbol{\Phi}^e\} = [\mathbf{F_0}]\{\mathbf{M}\} \tag{9}$$

where $\{\boldsymbol{\Phi}^e\} = \{\phi_i^e\ \phi_j^e\}^T$ is the matrix of elastic generalised deformations, $\{\mathbf{M}\} = \{m_i\ m_j\}^T$ is the matrix of generalised stresses, $[\mathbf{F_0}]$ is the elastic flexibility matrix, described by:

$$[\mathbf{F_0}] = \begin{bmatrix} L/3EI_g & -L/6EI_g \\ -L/6EI_g & L/3EI_g \end{bmatrix} \tag{10}$$

and the superscript T means ‘transpose of’.

LDM states that a beam element is understood as a composition of an elastic beam with two inelastic hinges at its edges (Figure 2a). Therefore, such hinges are responsible for inelastic effects.

Under the deformation equivalence hypothesis [26], the matrix of generalised deformations $\{\boldsymbol{\Phi}\}$ can be expressed as a sum of three parts:

$$\{\boldsymbol{\Phi}\} = \{\boldsymbol{\Phi}^e\} + \{\boldsymbol{\Phi}^d\} + \{\boldsymbol{\Phi}^p\} \tag{11}$$

being $\{\mathbf{\Phi^e}\}$ the elastic part, $\{\mathbf{\Phi^p}\} = \{\phi_i^p\ \phi_j^p\}^T$ the plastic part, accounting for reinforcement yielding at the hinges (Figure 3b), and $\{\mathbf{\Phi^d}\}$ the damaged one, expressed by [29]:

$$\{\boldsymbol{\phi}^d\} = [\mathbf{C}(\mathbf{D})]\{\mathbf{M}\} = \begin{bmatrix} \frac{Ld_i}{3EI(1-d_i)} & 0 \\ 0 & \frac{Ld_j}{3EI(1-d_j)} \end{bmatrix}\{\mathbf{M}\} \tag{12}$$

where, $[\mathbf{C(D)}]$ is the matrix of additional flexibility due to damage variables at the hinges (d_i and d_j); it represents concrete cracking (Figure 2c).

Finally, the following expression is obtained by substituting Equations 9 and 12 in (11):

$$\{\boldsymbol{\phi} - \boldsymbol{\phi}^p\} = [\mathbf{F}(\mathbf{D})]\{\mathbf{M}\} \tag{13}$$

where $[\mathbf{F(D)}]$ is the flexibility matrix of a damaged beam element; it is described as:

$$[\mathbf{F}(\mathbf{D})] = [\mathbf{F_0}] + [\mathbf{C}(\mathbf{D})] = \begin{bmatrix} \frac{L}{3EI(1-d_i)} & -\frac{L}{6EI} \\ -\frac{L}{6EI} & \frac{L}{3EI(1-d_j)} \end{bmatrix} \tag{14}$$

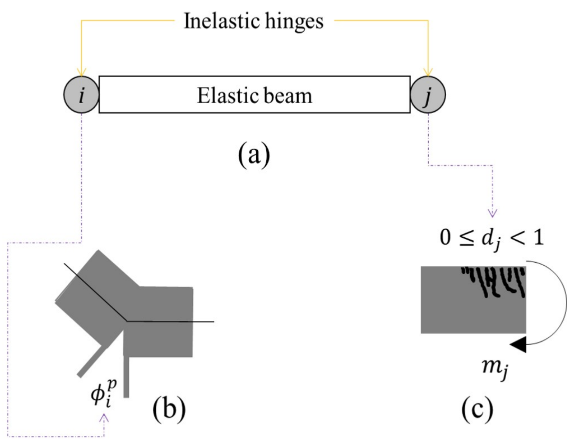

Figure 2. Lumped damage mechanics for RC beams: (a) elastic beam with inelastic hinges, (b) reinforcement yielding and (c) concrete cracking.

Note that both terms of the main diagonal of $[\mathbf{F(D)}]$ present the inertia moment penalised by a damage variable i.e. $I_g(1 - d_i)$ and $I_g(1 - d_j)$. Henceforth, this study focuses on only one of the hinges. The damage variable of such hinge is described herein without an index (d).

The concept of effective inertia moment (I_{eff}) is then introduced as a function of d [30]:

$$I_{eff} = I_g(1 - d) \tag{15}$$

Cippolina et al. [29] experimentally presented a simple way to quantify the damage variable. Such experiment consisted of a simply supported beam, such as that depicted in Figure 3a.

During the test, unloading-reloading cycles were performed to quantify the beam stiffness (Figure 3b). For the first unloading-reloading cycle, the applied load was lower than the threshold for concrete cracking as an alternative to measure elastic stiffness (S_0). Thereafter, the stiffness values for concrete cracking were obtained (Figure 3b), i.e. $S(d)$. The damage variable d for any cycle is then calculated [29]:

$$d = 1 - S(d)/S_0 \tag{16}$$

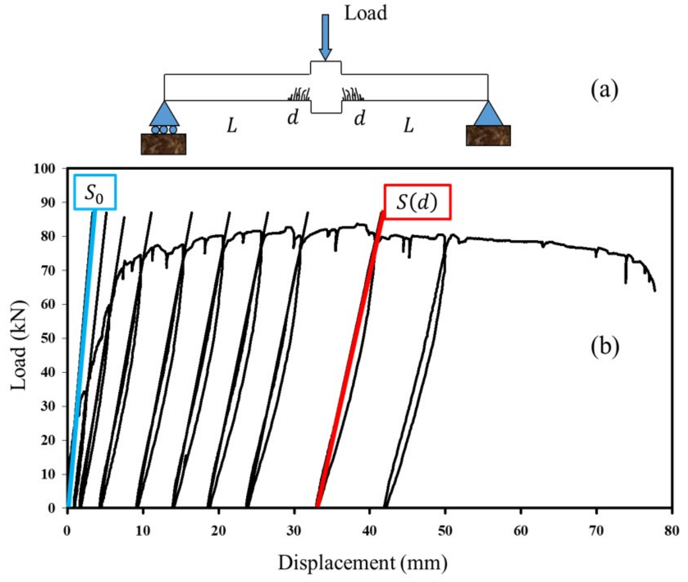

Figure 3. Graphical representation of damage measurement (adapted from [29]).

Experimental observations (e.g. [26], [29], [30]) show that the damage variable can be easily associated with the plastic bending moment (M_p) and ultimate bending moment (M_u), both of which are known quantities of classic RC theory.

Despite its accuracy at the load bearing condition of structural elements, the necessity of a lumped damage approach to analyse deflection in beams was observed by the application of the classic LDM for reinforced concrete structures [26] to a deflection test. In order to illustrate this issue, note that the classic LDM cracking evolution criterion is based on the generalised Griffith criterion, where the energy release rate (G) is equal to a crack resistance function (R), both defined as follows [26]:

$$G = R \;\Rightarrow\; m^2L/\left[6EI_g(1-d)^2\right] = R_0 + q\ln(1-d)/(1-d) \tag{17}$$

being R_0 the initial crack resistance and q a parameter associated to the longitudinal reinforcement.

Then, three conditions are known by the bending moment vs. damage obtained by Equation 17 i.e. by the classic LDM (see Figure 4):

$$m|_{d=0} = M_r \;\Rightarrow\; M_{cr}^2L/\left(6EI_g\right) = R_0 \text{ (18)}$$

$$m|_{d=d_u} = M_u \;\Rightarrow\; M_{cr}^2L/\left[6EI_g(1-d_u)^2\right] = M_{cr}^2L/\left(6EI_g\right) + q\ln(1-d_u)/(1-d_u) \tag{19}$$

$$\left.\frac{\partial m}{\partial d}\right|_{d=d_u} = 0 \;\Rightarrow\; (1-d_u)\,M_{cr}^2L/\left(3EI_g\right) + q[1+\ln(1-d_u)] = 0 \tag{20}$$

where d_u is the ultimate damage i.e. the damage value at the ultimate condition.

For an experimental analysis carried out by Álvares [31] (Figure 4), the conditions in (18-20) result in: $R_0 = 0.32$kNmm, $q = -106.37$kNmm and $d_u = 0.63$. The experimental damage, depicted in Figure 4, is obtained by the following relation:

$$d_{experiment} = 1 - w_{elastic}/w_{experiment} \tag{21}$$

being w the deflection.

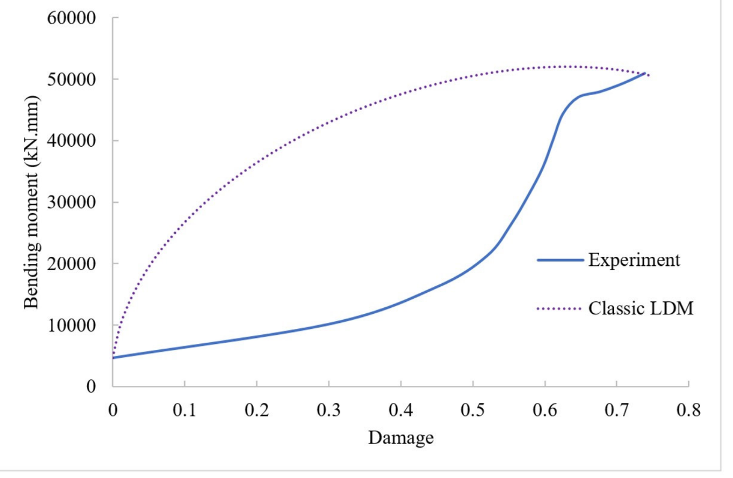

Figure 4. Experimental bending moment vs. damage results from Álvares [31] compared with the classic LDM [26].

4 PROPOSED MODEL

The proposed formulation for calculating deflection is derived by inserting an effective moment of inertia I_{eff}, calculated according to Equation 2.

$$I_{eff} = I_g(1 - d) \tag{22}$$

However, differently from classic LDM, the proposed approach must present a bending moment vs. damage relation closer to experimental analysis in order to evaluate deflected beams in service.

Therefore, an exponential expression for the acting moment (M_a) is proposed i.e.

$$M_a = M_{cr} + (M_u - M_{cr})\exp\left[-\left(I_g^2/I_{cr}^2\right)(1 - d/d_u)\right] \tag{23}$$

In classic LDM, d_u is numerically obtained by solving the system composed by Equations 19, 20. However, for practical applications, the following equation is a satisfactory approximation for d_u:

$$d_u = 0.5\left[d_p + 0.5\left(d_p + 1\right)\right] \tag{24}$$

where d_p is the plastic damage i.e. the damage when the reinforcement is about to yield.

Again, for practical applications, a reasonable approximation for d_p is:

$$d_p = 1 - I_{cr}/I_g \tag{25}$$

Therefore, by substituting Equations 24, 25 in Equation 23, d is the acting damage on the structural element is calculated according to Equation 26:

$$d = \left\{\ln[(M_a - M_{cr})/(M_u - M_{cr})] + I_g^2/I_{cr}^2\right\}\left(1 - 3I_{cr}/4I_g\right) I_{cr}^2/I_g^2 \tag{26}$$

where M_a is the acting moment, M_{cr} is the first cracking moment, M_u is the ultimate moment.

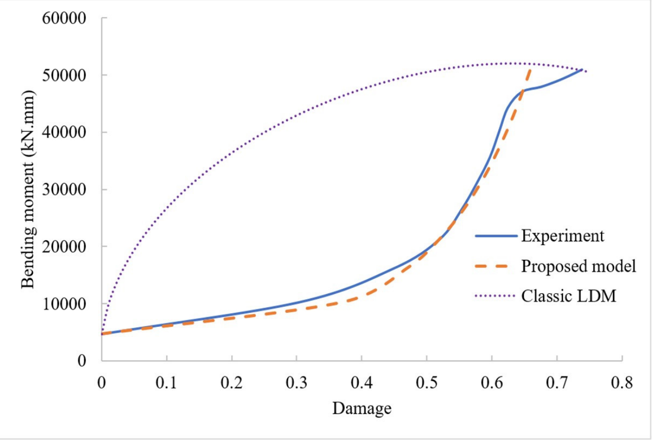

Figure 5. Experimental bending moment vs. damage results from Álvares [31] compared with the classic LDM [26] and the proposed model.

5 RESULTS

To analyse the proposed model, literature data including beams with different dimensions, strengths, elasticity modules and reinforcement rates were used. The selected works provided the results of the increase in displacements with the applied loads, in addition to the necessary information for the application in both Proposed and Branson's models.

As an illustration, the experimental results of Álvares [31] and Fernandes [32] are shown in Figures 4 and 5, respectively. The application of the Proposed Model to such experiments provided a satisfactory behaviour (Figures 4 and 5).

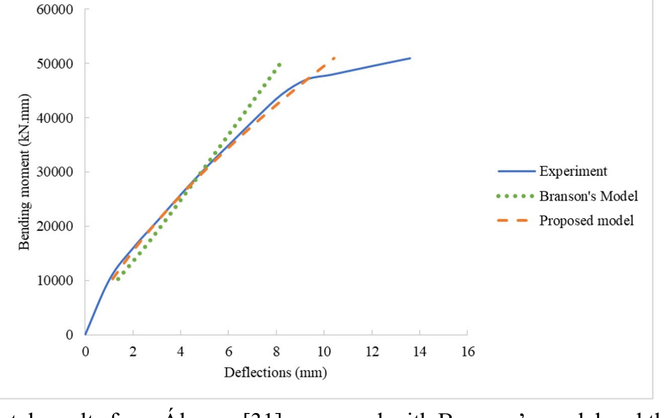

Figure 6. Experimental results from Álvares [31] compared with Branson's model and the proposed model.

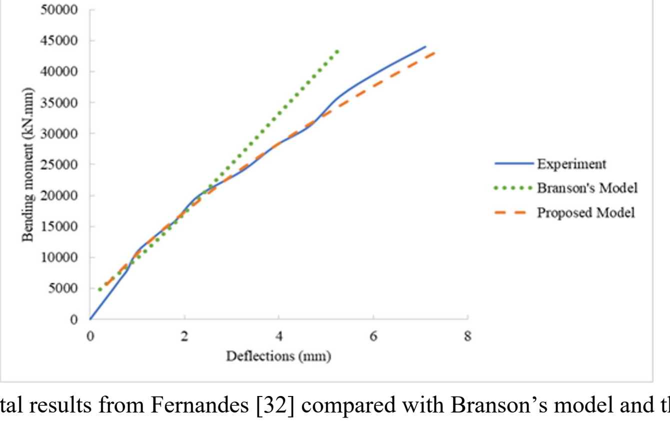

Figure 7. Experimental results from Fernandes [32] compared with Branson's model and the proposed model.

Subsequently, the Proposed Model was compared with 72 experiments by several authors [33]–[41] that were brought together in the work of Melo [42]. These works provided the necessary properties for the calculation of the deflection by the methods studied in this work, in addition to the values of force versus displacement for service situation. The properties of beams are depicted in Appendix A (Tables A1 to A6).

In order to analyse these results, it was necessary to group them according to the compressive strength (f_c) and the reinforcement rate. Figures 6 and 7 show the results of the deflections of reinforced concrete beams with f_c between 20 and 50 MPa. Figures 8, 9, 10, 11, 12 and 13, show the results of the deflections of the reinforced concrete beams with f_c between 50 and 90 MPa.

To statistically compare the Proposed Model results with the responses provided by the experiments considered and by Branson's Model, the most appropriate multiple comparison tests were used for each situation, according to the normality and homoscedasticity of the data considered. The normality test employed was the Shapiro-Wilk one, considering normality when the p-value was greater than the 5% significance level. In relation to the homoscedasticity test, both Bartlett and Levene were used. The Bartlett test was applied when the sample had a normal distribution; otherwise, Levene test was adopted. The studied samples met the conditions of homoscedasticity when the p-value was above the 5% significance level.

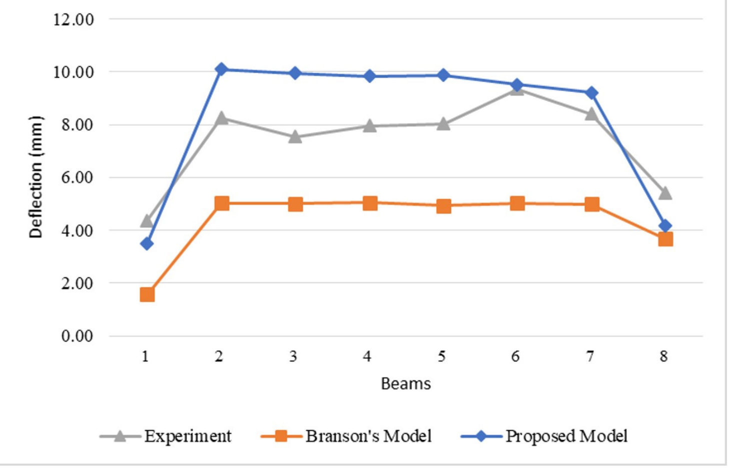

Figure 8. Beams with reinforcement rate from 0 to 1%.

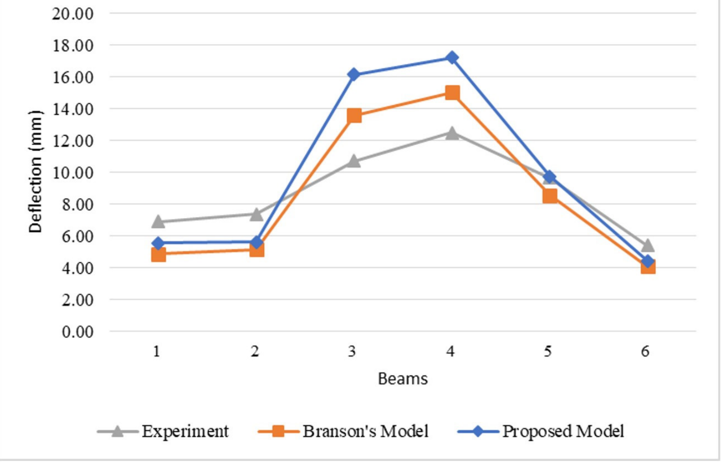

Figure 9. Beams with reinforcement rate from 1 to 2%.

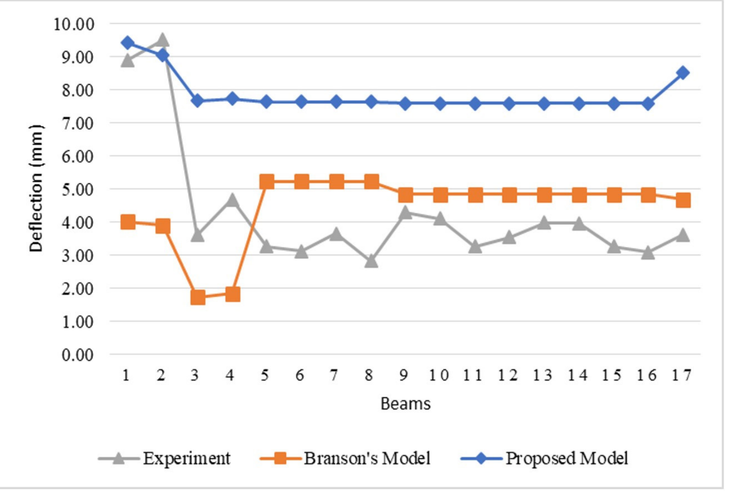

Figure 10. Beams with reinforcement rate from 0 to 1%.

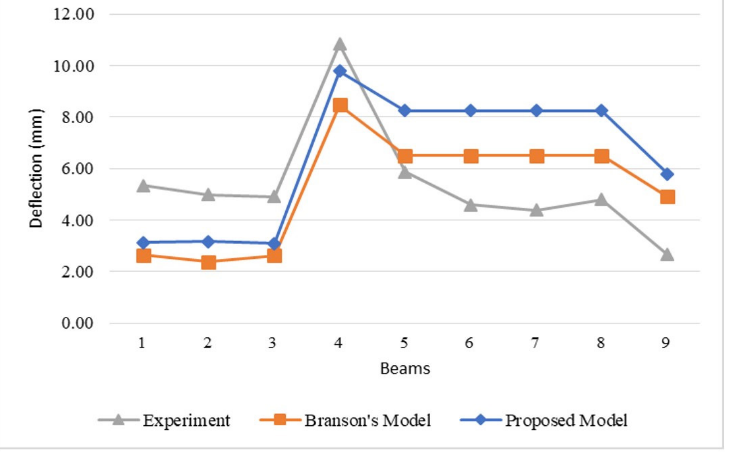

Figure 11. Beams with reinforcement rate from 1 to 2%.

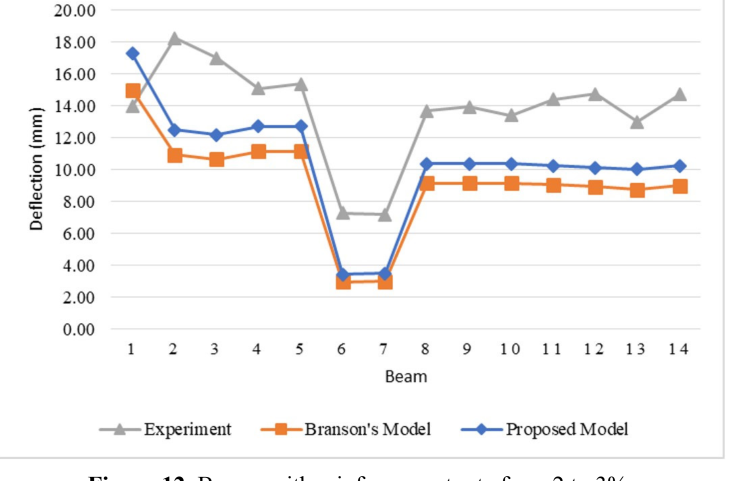

Figure 12. Beams with reinforcement rate from 2 to 3%.

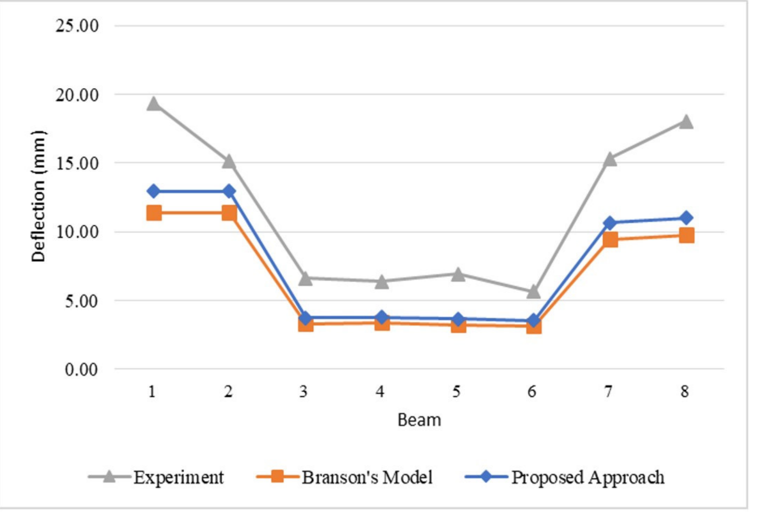

Figure 13. Beams with reinforcement rate greater than 3%.

Regarding the equivalence test between groups, ANOVA test was applied when the conditions of normality and homoscedasticity were satisfied, with the identification of difference between groups by the Fisher-Bonferroni test, considering that for p-value greater than 0.05 no difference was found. When the hypotheses related to the ANOVA test were not verified, the non-parametrical Kruskal-Wallis test was used, adopting a significance level of 5%.

To easier the analysis, the results were divided according to the concrete resistance f_c and the reinforcement rate, as specified in Tables 1 to 3.

Table 1. Beams analysis with f_c from 20 to 50 MPa with reinforcement rate from 0 to 1% and reinforcement rate from 1% to 2%.

f_c (MPa)	Reinforcement ratio (%)	Group	N° of beams by group	Normality (p-value)	Homoscedasticity (p-value)	Comparison of means/ medians	Difference	Difference (p-value)
20 – 50	0 – 1	Experiment	8	0.1267	0.5244	Experiment - Branson	No	0.0161
		Proposed		< 0.001		Branson - Proposed	Yes	
		Branson		< 0.001		Experiment - Proposed	No	
	1 – 2	Experiment	6	0.8571	0.2934	Experiment - Branson	No	0.8796
		Proposed		0.1235		Branson - Proposed	No	
		Branson		0.1631		Experiment - Proposed	No	

Table 2. Beams analysis with f_c from 50 to 90 MPa with reinforcement rate from 0 to 1% and reinforcement rate from 1% to 2%.

f_c (MPa)	Reinforcement ratio (%)	Group	N° of beams by group	Normality (p-value)	Homoscedasticity (p-value)	Comparison of means/ medians	Difference	Difference (p-value)
50 – 90	0 – 1	Experiment	17	< 0.001	0.2051	Experiment - Branson	No	< 0.001
		Proposed		< 0.001		Branson - Proposed	Yes	
		Branson		< 0.001		Experiment - Proposed	Yes	
	1 – 2	Experiment	21	0.0601	0.4582	Experiment - Branson	No	0.3215
		Proposed		0.0656		Branson - Proposed	No	
		Branson		0.0761		Experiment - Proposed	No	

Table 3. Beams analysis with f_c from 50 to 90 MPa with reinforcement rate from 2% to 3% and reinforcement rate greater than 3%.

f_c (MPa)	Reinforce-ment ratio (%)	Group	N° of beams by group	Normality (p-value)	Homoscedasticity (p-value)	Comparison of means/ medians	Difference	Difference (p-value)
50 – 90	2 – 3	Experiment	12	0.1067	0.9018	Experiment - Branson	Yes	< 0.001
		Proposed		0.0014		Branson - Proposed	No	
		Branson		0.0020		Experiment - Proposed	Yes	
	Greater than 3	Experiment	8	0.0499	0.0298	Experiment - Branson	No	0.1023
		Proposed		0.0104		Branson - Proposed	No	
		Branson		< 0,001		Experiment - Proposed	No	

According to the results obtained by the statistical analysis presented in Tables 1 to 3, it is possible to verify that:

- For f_c from 20 to 50 MPa and rates from the studied reinforcement, the statistical tests applied to the samples
- showed not only equivalent variance but also equal mean values between deflection obtained by either the experiments or the proposed model, derived from TDC. For reinforcement rates from 1% to 2%, it was also found that the normal distribution is associated with the results of the experimental deflections and the proposed formulation forecast.
- For f_c from 20 to 50 MPa and rates from the studied reinforcement, the statistical tests applied to the samples showed that the average deflection provided by the proposed model and Branson's model are equivalent, with adherence to the respective experimental results averages.
- For f_c from 50 to 90 MPa and reinforcement rates from 1% to 2% and greater than 3%, statistical tests applied to the samples showed not only equivalent variance values but also equal mean values between deflection obtained by either the experiments or the proposed model. At these reinforcement rates, it was also found that the average deflection values provided by both proposed and Branson models are equivalent, with adherence to the respective experimental results means.
- For the f_c in the range of 50 to 90 MPa, for reinforcement rates from 0 to 1%, the statistical tests applied to the samples showed the equality of the medians of the deflection values obtained by the experiments and the Branson's Model, both being different medians that obtained through the proposed formulation. However, it should be noted that, for these rates, the non-normality of the data for all groups was observed and the difference in the variance of the experimental results in relation to the variance of the values from the Branson's Model and the proposed formulation.

6 CONCLUSIONS

The main objective of this work was to propose an effective moment of inertia for the calculation of deflection in reinforced concrete beams, using the formulations of the Lumped Damage Mechanics as a basis, and to evaluate the proposed calculation model, comparing it with experimental results and with the Branson's Model.

The Proposed Model presented a satisfactory behaviour when the evolution of the deflection was verified and compared with the experimental works of Álvares [31] and Fernandes [32].

The statistical tests used showed that the reinforced concrete beams with the f_c in the range of 20 to 50 MPa, there is an equivalence of variance and equality of means between the Proposed Model and the experimental response, both in the reinforcement rates between 0 and 1% as well as the reinforcement rate of 1% to 2%. In this range of f_c, the Proposed Model also proved to be equivalent to the Branson's Model.

In the f_c range between 50 and 90 MPa and reinforcement rate from 0 to 1%, the statistical tests showed the equality of the medians between the deflection values of the experiment and the Branson's Model, but both are different from the median of the Proposed Model. In this group of beams the non-normality of the data and difference in variance between the experiment and the calculation methods studied were also verified.

Through the study, it was possible to verify that the application of the results of the Proposed Model provides a good approximation of the experimental response, equivalent to that provided by the Branson's Model in most of the investigated scenarios.

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APPENDIX A

Table A1. Beams data with f_c from 20MPa to 50MPa and reinforcement rate from 0 a 1%.

Authors	Beam	b (mm)	h (mm)	Service load (kN)	f_c (MPa)	ρ (%)	Experimental deflection (mm)	Proposed Model (mm)	Branson's Model (mm)
Sharifi [39]	SCCB1	200	300	13.27	31.60	0.51	4.38	3.52	1.59
Gribniak [37]	S-1	280	300	65.68	47.30	0.37	8.25	10.10	5.03
	S-1R	281	299	65.68	47.30	0.37	7.53	9.95	5.02
	S-2	280	300	67.20	48.70	0.37	7.96	9.83	5.05
	S-2R	282	300	66.46	48.20	0.37	8.02	9.86	4.93
	S-3	277	300	61.74	41.10	0.36	9.35	9.52	5.03
	S-3R	281	299	62.69	41.20	0.36	8.41	9.21	4.98
Silva [40]	V25A Sub	150	150	26.30	27.40	0.70	5.40	4.17	3.68

Table A2. Beams data with f_c from 20MPa to 50MPa and reinforcement rate from 1% to 2%.

Author	Beam	b (mm)	h (mm)	Service load (kN)	f_c (MPa)	ρ (%)	Experimental deflection (mm)	Proposed Model (mm)	Branson's Model (mm)
Sharifi [39]	SCCB2	200	300	33.19	32.84	1.05	6.90	5.54	4.86
	SCCB3	200	300	43.72	28.84	1.44	7.35	5.62	5.15
Ashour et al. [33]	B-N2	200	250	53.18	48.61	1.02	10.70	16.15	13.56
	B-N3	200	250	76.19	48.61	1.53	12.47	17.21	15.00
Rashid and Mansur [35]	A211	250	400	324.39	42.80	1.96	9.64	9.73	8.55
Silva [40]	V25A Super	150	150	37.00	35.77	1.40	5.40	4.41	4.08

Table A3. Beams data with f_c from 50MPa to 90MPa and reinforcement rate from 0 to 1%.

Authors	Beam	b (mm)	h (mm)	Service load (kN)	f_c (MPa)	ρ (%)	Experimental deflection (mm)	Proposed Model (mm)	Branson's Model (mm)
Gribniak [37]	S-4	277	300	62.21	54.20	0.36	8.90	9.41	4.01
	S-4R	283	301	63.62	54.20	0.36	9.50	9.04	3.89
Elrakib [38]	B503	250	400	61.50	52.00	0.30	3.59	7.66	1.72
	B753	250	400	77.51	73.00	0.38	4.67	7.73	1.83
Mousa [41]	A1	150	200	37.84	55.00	0.75	3.26	7.64	5.22
	A2	150	200	37.84	55.00	0.75	3.11	7.64	5.22
	A3	150	200	37.84	55.00	0.75	3.64	7.64	5.22
	A4	150	200	37.84	55.00	0.75	2.82	7.64	5.22
	B1	150	200	38.13	65.00	0.75	4.29	7.59	4.83
	B2	150	200	38.13	65.00	0.75	4.09	7.59	4.83
	B3	150	200	38.13	65.00	0.75	3.25	7.59	4.83
	B4	150	200	38.13	65.00	0.75	3.53	7.59	4.83
	B5	150	200	38.13	65.00	0.75	3.97	7.59	4.83
	B6	150	200	38.13	65.00	0.75	3.96	7.59	4.83
	B7	150	200	38.13	65.00	0.75	3.26	7.59	4.83
	B8	150	200	38.13	65.00	0.75	3.09	7.59	4.83
	B14	150	200	34.59	65.00	0.75	3.61	8.50	4.68

Table A4. Beams data with f_c from 50MPa to 90MPa and reinforcement rate from 1% to 2%.

Authors	Beam	b (mm)	h (mm)	Service load (kN)	f_c (MPa)	ρ (%)	Experimental deflection (mm)	Proposed Model (mm)	Branson's Model (mm)
Ashour et al. [33]	B-M2	200	250	55.00	78.50	1.02	10.73	15.42	11.89
	B-M3	200	250	80.29	78.50	1.53	13.14	16.46	13.92
Bernardo and Lopes [34]	B1	120	270	75.10	79.20	1.40	12.67	11.04	9.24
	B2	120	270	111.05	78.90	1.94	14.13	12.25	10.34
	B3	120	270	110.99	78.50	1.94	14.55	12.25	10.34
	C1	120	270	111.60	82.90	1.94	16.34	12.25	10.32
	C2	120	270	111.73	83.90	1.94	17.35	12.25	10.32
	D1	120	270	75.79	88.00	1.24	12.79	11.08	9.19
	A1	120	270	74.13	62.90	1.40	10.16	11.02	9.35
	A2	120	270	106.07	64.90	1.94	12.15	12.54	10.62
	A3	120	270	105.90	64.10	1.94	10.95	12.54	10.62
Maghsoudi and Bengar [36]	B2	200	300	187.73	70.50	1.05	5.75	3.15	2.29
	B3	200	300	277.85	70.80	1.70	5.33	3.13	2.63
	BC2	200	300	186.81	63.48	1.05	4.98	3.16	2.37
	BC3	200	300	275.61	63.21	1.70	4.90	3.09	2.61
Rashid and Mansur [35]	B211a	250	400	346.29	73.60	1.96	10.83	9.78	8.48
Mousa [41]	B9	150	200	64.73	65.00	1.34	5.86	8.24	6.50
	B10	150	200	64.73	65.00	1.34	4.59	8.24	6.50
	B11	150	200	64.73	65.00	1.34	4.37	8.24	6.50
	B12	150	200	64.73	65.00	1.34	4.80	8.24	6.50
Silva [40]	V50A Super	150	150	48.99	53.90	1.40	2.65	5.78	4.90

Table A5. Beams data with f_c from 50MPa to 90MPa and reinforcement rate from 2% to 3%.

Authors	Beam	b (mm)	h (mm)	Service load (kN)	f_c (MPa)	ρ (%)	Experimental deflection (mm)	Proposed Model (mm)	Branson's Model (mm)
Ashour et al. [33]	B-M4	200	250	104.09	78.50	2.04	13.94	17.28	14.98
Bernardo and Lopes [34]	C3	120	270	137.18	83.60	2.48	18.25	12.49	10.91
	C4	120	270	140.37	83.40	2.48	16.99	12.17	10.65
	A4	120	270	128.27	63.20	2.48	15.07	12.69	11.14
	A5	120	270	128.95	65.10	2.48	15.35	12.72	11.15
Maghsoudi and Bengar [36]	BC4	200	300	359.41	71.45	2.09	7.26	3.42	2.94
	B4	200	300	360.01	72.80	2.09	7.18	3.48	2.99
Rashid and Mansur [35]	B311	250	400	495.86	72.80	2.95	13.67	10.35	9.13
	B312	250	400	495.86	72.80	2.95	13.92	10.35	9.13
	B313	250	400	495.86	72.80	2.95	13.40	10.35	9.13
	B321	250	400	499.63	77.00	2.95	14.39	10.25	9.03
	B331	250	400	495.86	72.80	2.95	14.73	10.12	8.93
	C211	250	400	415.89	85.60	2.37	12.98	10.01	8.73
	C311	250	400	480.51	88.10	2.77	14.70	10.24	8.99

Table A6. Beams data with f_c from 50MPa to 90MPa and reinforcement rate greater than 3%.

Authors	Beam	b (mm)	h (mm)	Service load (kN)	f_c (MPa)	ρ (%)	Experimental deflection (mm)	Proposed Model (mm)	Branson's Model (mm)
Bernardo and Lopes [34]	D2	120	270	169.34	85.80	3.18	19.35	12.95	11.39
	D3	120	270	169.41	86.00	3.18	15.11	12.95	11.39
Maghsoudi and Bengar [36]	BC5	200	300	668.36	72.98	4.11	6.62	3.72	3.30
	B5	200	300	664.90	71.00	4.11	6.38	3.79	3.37
	BC6	200	300	669.11	73.42	4.11	6.90	3.65	3.24
	BC7	200	300	668.36	72.98	4.11	5.63	3.54	3.14
Rashid and Mansur [35]	C411	250	400	598.96	85.60	3.57	15.30	10.67	9.44
	C511	250	400	707.01	88.10	4.33	18.04	11.01	9.77