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MATHEMATICAL SCIENCES

Super-critical Hardy-Littlewood inequalities for multilinear forms

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Abstract: The multilinear Hardy-Littlewood inequalities provide estimates for the sum of the coefficients of multilinear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{R}$ (or $\mathbb{C}$) when $1/p_1 + \cdots + 1/p_m < 1$. In this paper we investigate the critical and super-critical cases; i.e., when $1/p_1 + \cdots + 1/p_m \geq 1$.

Key words: Multilinear forms, sequence spaces, inequalities, estimates.

INTRODUCTION

Littlewood's 4/3 theorem assures that for $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, we have

$$\left(\sum_{j_1,j_2=1}^{n} \left| A(e_{j_1}, e_{j_2}) \right|^{4/3} \right)^{3/4} \leq \sqrt{2}\, \|A\|$$

for all positive integers n and all bilinear forms $A : \ell_\infty^n \times \ell_\infty^n \to \mathbb{K}$, where as usual

$$\|A\| = \sup \{ |A(x,y)| : \|x\| \leq 1 \text{ and } \|y\| \leq 1 \}$$

and ℓ_p^n denotes $\mathbb{K}^n$ with the ℓ_p norm; the exponent 4/3 cannot be improved (i.e., cannot be replaced by a smaller one). Under an anisotropic viewpoint, the result can be generalized as follows (see Theorem 5.1 in Pellegrino et al. 2017): the inequality

$$\left(\sum_{j_1=1}^{n} \left(\sum_{j_2=1}^{n} \left| A(e_{j_1}, e_{j_2}) \right|^{a} \right)^{\frac{b}{a}} \right)^{\frac{1}{b}} \leq \sqrt{2}\, \|A\| \tag{1}$$

holds for all n whenever $a, b \in [1, \infty)$ satisfy

$$\frac{1}{a} + \frac{1}{b} \leq \frac{3}{2}.$$

Moreover, if $a, b \in [1, \infty)$ satisfy

$$\frac{1}{a} + \frac{1}{b} > \frac{3}{2},$$

then (1) is not possible, i.e., if

$$\left(\sum_{j_1=1}^{n} \left(\sum_{j_2=1}^{n} \left| A(e_{j_1}, e_{j_2}) \right|^{a} \right)^{\frac{b}{a}} \right)^{\frac{1}{b}} \leq C\, \|A\|,$$

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then the constant C must depend on n.

From now on, unless stated otherwise, the exponents involved in the inequalities are positive and can be even infinity (in this case the corresponding sum is replaced by the supremum). We also consider $1/\infty := 0$. The Hardy-Littlewood inequalities for bilinear forms were conceived in 1934 by Hardy and Littlewood (see Theorem 5 in Hardy & Littlewood 1934), as a natural generalization of Littlewood's 4/3 inequality. The results of the seminal paper of Hardy and Littlewood, in a modern and somewhat more general presentation, can be summarized by the following two theorems:

Theorem 1. *(see Osikiewicz & Tonge 2001 and Aron et al. 2017) Let $1 < q \leq 2 < p$, with $\frac{1}{p} + \frac{1}{q} < 1$. The following assertions are equivalent:*

(a) There is a constant $C \geq 1$ (not depending on n) such that

$$\left(\sum_{j_1=1}^{n} \left(\sum_{j_2=1}^{n} \left| A(e_{j_1}, e_{j_2}) \right|^{a} \right)^{\frac{b}{a}} \right)^{\frac{1}{b}} \leq C \left\| A \right\|$$

for all bilinear forms $A : \ell_p^n \times \ell_q^n \to \mathbb{K}$ and all positive integers n.

(b) The exponents a, b satisfy

$$(a, b) \in \left[\frac{q}{q-1}, \infty \right) \times \left[\frac{1}{1 - \left(\frac{1}{p} + \frac{1}{q} \right)}, \infty \right).$$

Moreover, the optimal constant C is 1.

Theorem 2. *(see Pellegrino et al. 2017) Let $p, q \in [2, \infty]$, with $\frac{1}{p} + \frac{1}{q} < 1$. The following assertions are equivalent:*

(a) There is a constant $C \geq 1$ (not depending on n) such that

$$\left(\sum_{j_1=1}^{n} \left(\sum_{j_2=1}^{n} \left| A(e_{j_1}, e_{j_2}) \right|^{a} \right)^{\frac{b}{a}} \right)^{\frac{1}{b}} \leq C \left\| A \right\|$$

for all bilinear forms $A : \ell_p^n \times \ell_q^n \to \mathbb{K}$ and all positive integers n.

(b) The exponents a, b satisfy

$$(a, b) \in \left[\frac{q}{q-1}, \infty \right) \times \left[\frac{1}{1 - \left(\frac{1}{p} + \frac{1}{q} \right)}, \infty \right)$$

and

$$\frac{1}{a} + \frac{1}{b} \leq \frac{3}{2} - \left(\frac{1}{p} + \frac{1}{q} \right). \tag{2}$$

Since (2) is trivially verified under the conditions of Theorem 1, we can unify the two theorems as follows:

Theorem 3. *Let $q \in (1, \infty]$ and $p \in [2, \infty]$, with $\frac{1}{p} + \frac{1}{q} < 1$. The following assertions are equivalent:*

(a) There is a constant $C \geq 1$ (not depending on n) such that

$$\left(\sum_{j_1=1}^{n}\left(\sum_{j_2=1}^{n}\left|A(e_{j_1},e_{j_2})\right|^{a}\right)^{\frac{b}{a}}\right)^{\frac{1}{b}} \leq C\,\|A\|$$

for all bilinear forms $A : \ell_p^n \times \ell_q^n \to \mathbb{K}$ and all positive integers n.

(b) The exponents a, b satisfy

$$(a,b) \in \left[\frac{q}{q-1},\infty\right) \times \left[\frac{1}{1-\left(\frac{1}{p}+\frac{1}{q}\right)},\infty\right)$$

and

$$\frac{1}{a}+\frac{1}{b} \leq \frac{3}{2}-\left(\frac{1}{p}+\frac{1}{q}\right).$$

In 1981, Praciano-Pereira (see Praciano-Pereira 1981) extended the Hardy-Littlewood inequalities to m-linear forms as follows: if $p_1, \ldots, p_m \in [1,\infty]$ and

$$\frac{1}{p_1}+\cdots+\frac{1}{p_m} \leq \frac{1}{2},$$

there exists a constant $C \geq 1$ (not depending on n) such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{n}\left|T(e_{j_1},\ldots,e_{j_m})\right|^{\frac{2m}{m+1-2\left(\frac{1}{p_1}+\cdots+\frac{1}{p_m}\right)}}\right)^{\frac{m+1-2\left(\frac{1}{p_1}+\cdots+\frac{1}{p_m}\right)}{2m}} \leq C\|T\|, \tag{3}$$

for all m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and for all positive integers n.

When

$$\frac{1}{2} \leq \frac{1}{p_1}+\cdots+\frac{1}{p_m} < 1,$$

Dimant and Sevilla-Peris (see Dimant & Sevilla-Peris 2016 and Cavalcante 2018) have proved that there exists a constant $C \geq 1$ (not depending on n) such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{n}\left|T(e_{j_1},\ldots,e_{j_m})\right|^{\frac{1}{1-\left(\frac{1}{p_1}+\cdots+\frac{1}{p_m}\right)}}\right)^{1-\left(\frac{1}{p_1}+\cdots+\frac{1}{p_m}\right)} \leq C\|T\|, \tag{4}$$

for all m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and for all positive integers n.

Both in (3) and (4) the exponents are sharp, i.e., they cannot be replaced by smaller exponents keeping the constant C not depending on n (this terminology will be used throughout the paper). However, there still remains the question: what about anisotropic versions of (3) and (4), i.e., variants with eventually different exponents associated to each index? Throughout this paper we shall address this question and related problems.

In Albuquerque et al. 2014, the anisotropic version of the result of Praciano-Pereira was finally settled (see also Santos & Velanga 2017 for a more complete version for the case $p_1, \ldots, p_m = \infty$):

Theorem 4. *(see Theorem 1.2 in Albuquerque et al. 2014 and Theorem 5.2 in Pellegrino et al. 2017) Let $p_1, ..., p_m \in [1, \infty]$ be such that*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} \leq \frac{1}{2}$$

and

$$q_1, ..., q_m \in \left[\frac{1}{1 - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_m} \right)}, 2 \right].$$

The following assertions are equivalent:

(a) There is a constant $C \geq 1$ (not depending on n) such that

$$\left(\sum_{j_1=1}^{n} \left(\cdots \left(\sum_{j_m=1}^{n} \left| A\left(e_{j_1}, \ldots, e_{j_m}\right) \right|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C \|A\|,$$

for all m-linear forms $A : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \longrightarrow \mathbb{K}$ and all positive integers n.

(b) The inequality

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} \leq \frac{m+1}{2} - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_m} \right)$$

is verified.

The anisotropic version of (4) is still not completely solved, but in Aron et al. 2017 the following partial answer (that also generalizes Theorem 1) was obtained:

Theorem 5. *(see Theorem 3.2 in Aron et al. 2017) Let $m \geq 2$ and $1 < p_m \leq 2 < p_1, ..., p_{m-1}$, with*

$$\frac{1}{p_1} + \cdots + \frac{1}{p_m} < 1.$$

The following assertions are equivalent:

(a) There is a constant $C \geq 1$ (not depending on n) such that

$$\left(\sum_{j_1=1}^{n} \left(\cdots \left(\sum_{j_m=1}^{n} \left| A\left(e_{j_1}, \ldots, e_{j_m}\right) \right|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C \|A\|,$$

for all m-linear forms $A : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \longrightarrow \mathbb{K}$ and all positive integers n.

(b) The exponents $q_1, ..., q_m$ satisfy

$$q_1 \geq \delta_m^{p_1, \ldots, p_m}, q_2 \geq \delta_{m-1}^{p_2, \ldots, p_m}, ..., q_{m-1} \geq \delta_2^{p_{m-1}, p_m}, q_m \geq \delta_1^{p_m},$$

with

$$\delta_{m-k+1}^{p_k, \ldots, p_m} := \frac{1}{1 - \left(\frac{1}{p_k} + \cdots + \frac{1}{p_m} \right)}.$$

The attentive reader may wonder why the case

$$\frac{1}{p_1}+\cdots+\frac{1}{p_m}\geq 1 \tag{5}$$

is not investigated in the previous results? The reason is simple, because in this case it is easy to prove that if there exists C (not depending on n) such that

$$\left(\sum_{j_1,\ldots,j_m=1}^{n}\left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{s}\right)^{\frac{1}{s}}\leq C\,\|T\|,$$

for all m-linear forms $T:\ell_{p_1}^{n}\times\cdots\times\ell_{p_m}^{n}\longrightarrow\mathbb{K}$ and all positive integers n, then $s=\infty$ (i.e., we are forced to deal with the sup norm, and the result becomes trivial). However, under the anisotropic viewpoint, as a matter of fact, there is no reason to avoid the case (5) and it constitutes a vast field yet to be explored. The first step in this direction is the following:

Theorem 6. *(see Theorem 1 in Paulino 2019) For all $m\geq 2$ we have*

$$\sup_{j_1}\left(\sum_{j_2=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}\left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_2}{q_3}}\right)^{\frac{1}{q_2}}\leq 2^{\frac{m-2}{2}}\,\|T\| \tag{6}$$

for all m-linear forms $T:\ell_m^n\times\cdots\times\ell_m^n\to\mathbb{K}$, and all positive integers n, with

$$q_k=\frac{2m(m-1)}{mk-2k+2}$$

for all $k=2,\ldots,m$. Moreover, $q_1=\infty$ and $q_2=m$ are sharp and, for $m>2$ the optimal exponents q_k satisfying (6) fulfill

$$q_k\geq\frac{m}{k-1},\ k=2,\ldots,m.$$

The case considered in Theorem 6 is called critical because it is a special case of (5), and from now on we shall call case (5) super-critical, which is the topic of the present paper. In the next sections we provide a partial solution to the super-critical case for 3-linear forms and we investigate what are the conditions needed to obtain m-linear Hardy-Littlewood inequalities in the super-critical case.

THE 3-LINEAR CASE

We begin this section by presenting two simple, albeit very useful, lemmas that will be used all along the paper.

Two multi-purpose lemmas

For $S=\{s_1,\ldots,s_k\}\subset\{1,\ldots,m\}$, we define

$$\hat{S}:=\{1,\ldots,m\}\backslash S$$

and by $\mathbf{i}_S$ we shall mean $(i_{s_1}, \dots, i_{s_k})$. If $S = \{s_1, \dots, s_k\}$ and $\mathbf{p} = (p_1, \dots, p_m) \in (0, \infty]^m$, we define

$$\left|\frac{1}{\mathbf{p}}\right|_S := \frac{1}{p_{s_1}} + \cdots + \frac{1}{p_{s_k}}.$$

The lemmas read as follows:

Lemma 7. *Let $k \in \{1, \dots, m\}$ and $\mathbf{p} = (p_1, \dots, p_m) \in [1, \infty]^m$. If there is a constant $C \geq 1$ (not depending on n) such that*

$$\left(\sum_{j_{s_1}=1}^{n}\left(\sum_{j_{s_2}=1}^{n}\cdots\left(\sum_{j_{s_m}=1}^{n}\left|T(e_{j_{s_1}}, \dots, e_{j_{s_m}})\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\,\|T\|$$

for all m-linear forms $T\colon \ell_{p_{s_1}}^n \times \cdots \times \ell_{p_{s_m}}^n \to \mathbb{K}$ and all positive integers n, then

$$\left(\sum_{j_{s_{k+1}}=1}^{n}\left(\sum_{j_{s_{k+2}}=1}^{n}\cdots\left(\sum_{j_{s_m}=1}^{n}\left|A(e_{j_{s_{k+1}}}, \dots, e_{j_{s_m}})\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_{k+1}}{q_{k+2}}}\right)^{\frac{1}{q_{k+1}}} \leq C\,\|A\|$$

for all $(m-k)$-linear forms $A\colon \ell_{p_{s_{k+1}}}^n \times \cdots \times \ell_{p_{s_m}}^n \to \mathbb{K}$ and all positive integers n.

Proof. To simplify the notation, we can suppose $(s_1, \dots, s_m) = (1, \dots, m)$.

Let suppose that there is a constant $C \geq 1$ such that

$$\left(\sum_{j_1=1}^{n}\left(\sum_{j_2=1}^{n}\cdots\left(\sum_{j_m=1}^{n}\left|T(e_{j_1}, \dots, e_{j_m})\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\,\|T\|$$

for all m-linear forms $T\colon \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$.

Given an $(m-k)$-linear form $S\colon \ell_{p_{k+1}}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$, we define the m-linear form $T\colon \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$, given by

$$T\left(x^{(1)}, x^{(2)}, \dots, x^{(m)}\right) = x_1^{(1)} \cdots x_1^{(k)} S\left(x^{(k+1)}, x^{(k+2)}, \dots, x^{(m)}\right).$$

It is obvious that $\|T\| = \|S\|$; then, by the above assumption there is a constant $C \geq 1$ such that

$$
\begin{aligned}
&\left(\sum_{j_{k+1}=1}^{n}\left(\sum_{j_{k+2}=1}^{n}\cdots\left(\sum_{j_m=1}^{n}\left|S(e_{j_{s_{k+1}}},\dots,e_{j_{s_m}})\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_{k+1}}{q_{k+2}}}\right)^{\frac{1}{q_{k+1}}}\\
&=\sup_{\mathbf{i}_{\{\widehat{k+1,\dots,m}\}}}\left(\sum_{j_{k+1}=1}^{n}\left(\sum_{j_{k+2}=1}^{n}\cdots\left(\sum_{j_m=1}^{n}\left|e_1^{(1)}\cdots e_1^{(k)}S(e_{j_{s_{k+1}}},\dots,e_{j_{s_m}})\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_{k+1}}{q_{k+2}}}\right)^{\frac{1}{q_{k+1}}}\\
&=\sup_{\mathbf{i}_{\{\widehat{k+1,\dots,m}\}}}\left(\sum_{j_{k+1}=1}^{n}\left(\sum_{j_{k+2}=1}^{n}\cdots\left(\sum_{j_m=1}^{n}\left|T(e_{j_1},\dots,e_{j_{s_m}})\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_{k+1}}{q_{k+2}}}\right)^{\frac{1}{q_{k+1}}}\\
&\leq\left(\sum_{j_1=1}^{n}\left(\sum_{j_2=1}^{n}\cdots\left(\sum_{j_m=1}^{n}\left|T(e_{j_1},\dots,e_{j_m})\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}\\
&\leq C\,\|T\|\\
&= C\,\|S\|\,.
\end{aligned}
$$

$\square$

Lemma 8. *Let $k \in \{1,\dots,m\}$ and $\mathbf{p} = (p_1,\dots,p_m) \in [1,\infty]^m$. Let $S = \{s_1,\dots,s_k\} \subset \{1,\dots,m\}$. If there is a constant $C \geq 1$ (not depending on n) such that*

$$\left(\sum_{j_{s_1}=1}^{n}\left(\sum_{j_{s_2}=1}^{n}\cdots\left(\sum_{j_{s_k}=1}^{n}\left|A(e_{j_{s_1}},\dots,e_{j_{s_k}})\right|^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\,\|A\|$$

for all k-linear forms $A\colon \ell_{p_{s_1}}^n \times \cdots \times \ell_{p_{s_k}}^n \to \mathbb{K}$ and all positive integers n, then

$$\sup_{\mathbf{i}_{\hat{S}}}\left(\sum_{j_{s_1}=1}^{n}\left(\sum_{j_{s_2}=1}^{n}\cdots\left(\sum_{j_{s_k}=1}^{n}\left|T(e_{j_1},\dots,e_{j_m})\right|^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\,\|T\|$$

for all m-linear forms $T\colon \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n. Moreover, if

$$\left|\frac{1}{\mathbf{p}}\right|_S < 1$$

and, for every $j \in \hat{S}$,

$$\left|\frac{1}{\mathbf{p}}\right|_{S\cup\{j\}} \geq 1,$$

the sup *cannot be improved (here and henceforth, this means that the sup cannot be replaced by any ℓ_p-sum).*

Proof. To simplify the notation, we can suppose $(s_1, ..., s_k) = (1, ..., k)$.

Let us fix the last $m-k$ variables and work with k-linear forms $A\colon \ell_{p_1}^n \times \cdots \times \ell_{p_k}^n \to \mathbb{K}$. Since

$$\left(\sum_{j_1=1}^{n}\left(\sum_{j_2=1}^{n}\cdots\left(\sum_{j_k=1}^{n}\left|A(e_{j_1},...,e_{j_k})\right|^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\,\|A\|$$

for all k-linear forms $A\colon \ell_{p_1}^n \times \cdots \times \ell_{p_k}^n \to \mathbb{K}$, we know that there is a constant $C \geq 1$, such that for any fixed vectors $e_{j_{k+1}}, ..., e_{j_m}$, we have

$$\begin{aligned}&\left(\sum_{j_{s_1}=1}^{n}\left(\sum_{j_{s_2}=1}^{n}\cdots\left(\sum_{j_{s_k}=1}^{n}\left|T(e_{j_1},...,e_{j_m})\right|^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}\\ &\leq C\left\|T\left(\cdot,\cdots,\cdot,e_{j_{k+1}},...,e_{j_m}\right)\right\|\end{aligned}$$

for all m-linear forms $T\colon \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$. Then, there is a constant $C \geq 1$, such that

$$\begin{aligned}&\sup_{\mathbf{i}_{\hat{S}}}\left(\sum_{j_{s_1}=1}^{n}\left(\sum_{j_{s_2}=1}^{n}\cdots\left(\sum_{j_{s_k}=1}^{n}\left|T(e_{j_1},...,e_{j_m})\right|^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}}\\ &\leq C\sup_{\mathbf{i}_{\hat{S}}}\left\|T\left(\cdot,\cdots,\cdot,e_{j_{k+1}},...,e_{j_m}\right)\right\|\\ &\leq C\,\|T\|\end{aligned}$$

for all m-linear forms $T\colon \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$.

Now let us show that the sup cannot be improved. In fact, in this case we have $m-k$ suprema, none of which can be improved. Otherwise there will exist $i \in \hat{S}$, $r \in (0,\infty)$ and $C \geq 1$ such that

$$\sup_{\mathbf{i}_{\widehat{S\cup\{i\}}}}\left(\sum_{j_i=1}^{n}\left(\sum_{j_{s_1}=1}^{n}\left(\sum_{j_{s_2}=1}^{n}\cdots\left(\sum_{j_{s_k}=1}^{n}\left|T(e_{j_1},...,e_{j_m})\right|^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{r}{q_1}}\right)^{\frac{1}{r}} \leq C\,\|T\|$$

for all m-linear forms $T\colon \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all n. Using the Lemma 7, this would imply the existence of a constant $C \geq 1$ such that

$$\left(\sum_{j_i=1}^{n}\left(\sum_{j_{s_1}=1}^{n}\left(\sum_{j_{s_2}=1}^{n}\cdots\left(\sum_{j_{s_k}=1}^{n}\left|A(e_{j_i},e_{j_{s_1}},...,e_{j_{s_k}})\right|^{q_k}\right)^{\frac{q_{k-1}}{q_k}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{r}{q_1}}\right)^{\frac{1}{r}} \leq C\,\|A\|$$

for all $(k+1)$-linear forms $A\colon \ell_{p_i}^n \times \ell_{p_{s_1}}^n \times \cdots \times \ell_{p_{s_k}}^n \to \mathbb{K}$. Considering $\rho = \max\{q_1, ..., q_k, r\}$, by the monotonicity of the ℓ_q norms we conclude that there is a constant $C \geq 1$ such that

$$\left(\sum_{j_i,j_{s_1},...,j_{s_k}=1}^{n}\left|A(e_{j_i},e_{j_{s_1}},...,e_{j_{s_k}})\right|^{\rho}\right)^{\frac{1}{\rho}} \leq C\,\|A\|$$

for all $(k+1)$-linear forms $A\colon \ell_{p_i}^n \times \ell_{p_{s_1}}^n \times \cdots \times \ell_{p_{s_k}}^n \to \mathbb{K}$. But this is impossible due to the hypothesis $\left|\frac{1}{\mathbf{p}}\right|_{S\cup\{i\}} \geq 1$. □

In the next sections, using Lemma 7 and Lemma 8, we obtain the super-critical versions of the Hardy-Littlewood inequalities presented in the introduction.

A first natural illustration of the usefulness of Lemma 7 and Lemma 8 leads us to an alternate proof of Proposition 6.3 in Pellegrino et al. 2017. In fact, if $q \in (1,\infty]$, it is well known that

$$\left(\sum_{j=1}^{n} \left|A\left(e_j\right)\right|^{a}\right)^{\frac{1}{a}} \leq \|A\|$$

for all bounded linear forms $A\colon \ell_q \to \mathbb{K}$, if, and only if, $a \geq \frac{q}{q-1}$. Thus, for $a,b \in (0,\infty]$, and $p,q \in (1,\infty]$ such that $\frac{1}{p}+\frac{1}{q} \geq 1$, we invoke Lemma 7 and Lemma 8 to obtain:

Proposition 9. *(see Proposition 6.3 in Pellegrino et al. 2017) Let $p,q \in (1,\infty]$ be such that $\frac{1}{p}+\frac{1}{q} \geq 1$. We have*

$$\left(\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\left|A(e_i,e_j)\right|^{a}\right)^{\frac{b}{a}}\right)^{\frac{1}{b}} \leq \|A\|$$

for all bilinear forms $A\colon \ell_p^n \times \ell_q^n \to \mathbb{K}$ and all n if, and only if, the exponents a,b satisfy

$$b=\infty \text{ and } a \geq \frac{q}{q-1}.$$

In this section we are mainly interested in the case of 3-linear forms.

By Theorem 6 used for 3-linear forms we have

$$\sup_{j_1}\left(\sum_{j_2=1}^{n}\left(\sum_{j_3=1}^{n}\left|T\left(e_{j_1},e_{j_2},e_{j_3}\right)\right|^{q_3}\right)^{\frac{q_2}{q_3}}\right)^{\frac{1}{q_2}} \leq \sqrt{2}\,\|T\| \tag{7}$$

for all 3-linear forms $T:\ell_3^n \times \ell_3^n \times \ell_3^n \to \mathbb{K}$, and all positive integers n, with $q_2=3$ and $q_3=12/5$. Moreover, the supremum cannot be replaced by an ℓ_p-sum and $q_2=3$ is sharp; besides, the optimal exponent q_3 satisfying (7) fulfills $q_3 \geq 3/2$.

As a consequence of Lemma 7 and Lemma 8, we complete the above result.

Proposition 10. *Let $p,r \in (1,\infty)$ and $q \in [2,\infty]$ be such that $\frac{1}{q}+\frac{1}{r}<1$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} \geq 1$. The following assertions are equivalent:*

(a) There is a constant $C \geq 1$ (not depending on n) such that

$$\left(\sum_{j_1=1}^{n}\left(\sum_{j_2=1}^{n}\left(\sum_{j_3=1}^{n}\left|T\left(e_{j_1},e_{j_2},e_{j_3}\right)\right|^{q_3}\right)^{\frac{q_2}{q_3}}\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\,\|T\|$$

for every 3-linear form $T:\ell_p^n \times \ell_q^n \times \ell_r^n \to \mathbb{K}$ and all n.

(b) The exponents q_1, q_2, q_3 satisfy

$$q_1 = \infty,\ q_2 \geq \frac{1}{1-\left(\frac{1}{r}+\frac{1}{q}\right)},\ q_3 \geq \frac{r}{r-1},$$

and

$$\frac{1}{q_2}+\frac{1}{q_3} \leq \frac{3}{2}-\left(\frac{1}{r}+\frac{1}{q}\right).$$

Proof. Since $\frac{1}{q}+\frac{1}{r}<1$, by Theorem 3 there is a constant $C \geq 1$ such that

$$\left(\sum_{j_2=1}^{n}\left(\sum_{j_3=1}^{n}\left|A\left(e_{j_2}, e_{j_3}\right)\right|^{q_3}\right)^{\frac{q_2}{q_3}}\right)^{\frac{1}{q_2}} \leq C\|A\|$$

for all bilinear forms $A\colon \ell_q^n \times \ell_r^n \to \mathbb{K}$ if, and only if,

$$q_3 \geq \frac{r}{r-1}, q_2 \geq \frac{1}{1-\left(\frac{1}{r}+\frac{1}{q}\right)}$$

and

$$\frac{1}{q_3}+\frac{1}{q_2} \leq \frac{3}{2}-\left(\frac{1}{r}+\frac{1}{q}\right).$$

We combine this equivalence with the fact $\frac{1}{q}+\frac{1}{r}<1$ and $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} \geq 1$, and then, we invoke Lemma 7 and Lemma 8 to conclude the proof. □

Corollary 11. *For all 3-linear forms $T: \ell_3^n \times \ell_3^n \times \ell_3^n \to \mathbb{K}$ and all n, we have*

$$\left(\sum_{j_1=1}^{n}\left(\sum_{j_2=1}^{n}\left(\sum_{j_3=1}^{n}\left|T\left(e_{j_1}, e_{j_2}, e_{j_3}\right)\right|^{q_3}\right)^{\frac{q_2}{q_3}}\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\|T\|$$

if, and only if, $q_1=\infty$, $q_2 \geq 3$, $q_3 \geq 3/2$, and $\frac{1}{q_2}+\frac{1}{q_3} \leq \frac{5}{6}$.

THE *m*-LINEAR CASE

Now we use Lemma 7 and Lemma 8 to obtain super-critical versions of Hardy-Littlewood inequalities for m-linear forms. Our main result is the following Theorem. Below, we use the notation $\lceil x \rceil$ to represent the smallest integer greater than to x, i.e., $\lceil x \rceil = \min\{n \in \mathbb{Z} \mid n > x\}$.

Theorem 12. *Let $m \geq 2$ be an integer, $p \in (1, 2m]$, $k := \max\{0, \lceil m-p \rceil\}$ and $A = \{i \in \{1, \ldots, m-1\} : i \leq k\}$. Then, there is a constant $C \geq 1$ (not depending on n) such that*

$$\sup_{j_i, i \in A}\left(\sum_{j_{k+1}, \ldots j_m=1}^{n}\left|T(e_{j_1}, \ldots, e_{j_m})\right|^{q}\right)^{\frac{1}{q}} \leq C\|T\|$$

for every m-linear form $T: \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ if, and only if,

$$q \geq \frac{p}{p-(m-k)}.$$

Moreover, the sup *cannot be improved.*

Proof. The case $k=0$ is precisely (4), so we shall assume $k \geq 1$. Since $p \in (m-k, m-k+1]$ we have

$$\frac{1}{m-k+1} \leq \frac{1}{p} < \frac{1}{m-k}$$

and thus

$$\frac{m-k}{m-k+1} \leq \frac{m-k}{p} < 1.$$

On the other hand we also have

$$1 \leq \frac{m-k+1}{p}.$$

By (4) there is a constant $C \geq 1$ such that

$$\left(\sum_{j_{k+1},\ldots j_m=1}^{n} \left| T(e_{j_{k+1}}, \ldots, e_{j_k}) \right|^q \right)^{\frac{1}{q}} \leq C \|T\|$$

for every $(m-k)$-linear form $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ if, and only if,

$$q \geq \frac{p}{p-(m-k)}.$$

By Lemma 8 with $S = \{k+1, k+2, \ldots, m\} \subset \{1, \ldots, m\}$, and Lemma 7 we conclude the proof. □

We finish this section with some super-critical results in the anisotropic setting, whose proofs we omit. We begin with a super-critical version of Theorem 5:

Theorem 13. *Let $m \geq 2$, $k \in \{1, \ldots, m-1\}$, $p_1, \ldots, p_k \in [1, \infty]$, $p_{k+1}, \ldots, p_{m-1} \in (2, \infty]$ and $p_m \in (1, 2]$, such that*

$$\frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m} < 1$$

and

$$\frac{1}{p_j} + \frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m} \geq 1$$

for all $j \in \{1, \ldots, k\}$. The following assertions are equivalent:

(a) *There is a constant $C \geq 1$ (not depending on n) such that*

$$\left(\sum_{j_1=1}^{n} \left(\sum_{j_2=1}^{n} \cdots \left(\sum_{j_m=1}^{n} \left| T(e_{j_1}, \ldots, e_{j_m}) \right|^{q_m} \right)^{\frac{q_{m-1}}{q_m}} \cdots \right)^{\frac{q_1}{q_2}} \right)^{\frac{1}{q_1}} \leq C \|T\|$$

for all m-linear forms $T : \ell_p^n \times \cdots \times \ell_p^n \to \mathbb{K}$ and all n.

(b) *The exponents satisfy*

$$q_1 = \cdots = q_k = \infty \text{ and } q_i \geq \frac{1}{1 - \left(\frac{1}{p_i} + \cdots + \frac{1}{p_m} \right)}, i = k+1, \ldots, m.$$

Analogously, using Lemma 7, Lemma 8 and Theorem 4 we have:

Theorem 14. *Let $p_1, \ldots, p_k \in [1,2]$ and $p_{k+1}, \ldots, p_m \in [2,\infty]$ be such that*

$$\frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m} \leq \frac{1}{2}$$

and

$$\frac{1}{p_j} + \frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m} \geq 1$$

for all $j \in \{1, \ldots, k\}$, and

$$q_{k+1}, \ldots, q_m \in \left[\frac{1}{1 - \left(\frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m}\right)}, 2\right].$$

The following assertions are equivalent:

(a) *There is a constant C (not depending on n) such that*

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}\left|T\left(e_{j_1}, \ldots, e_{j_m}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\|T\|,$$

for all m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \longrightarrow \mathbb{K}$ and all positive integers n.

(b) *$q_1 = \cdots = q_k = \infty$ and the inequality*

$$\frac{1}{q_{k+1}} + \cdots + \frac{1}{q_m} \leq \frac{(m-k)+1}{2} - \left(\frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m}\right)$$

is verified.

The next result shows that it is possible to avoid the condition $\frac{1}{p_j} + \frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m} \geq 1$, for all $j \in \{1, \ldots, k\}$:

Theorem 15. *Let $p_1, \ldots, p_k \in [1,2]$ and $p_{k+1}, \ldots, p_m \in [2,\infty]$ be such that*

$$\frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m} \leq \frac{1}{2}$$

and

$$q_{k+1}, \ldots, q_m \in \left[\frac{1}{1 - \left(\frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m}\right)}, 2\right]$$

with

$$\frac{1}{q_{k+1}} + \cdots + \frac{1}{q_m} = \frac{(m-k)+1}{2} - \left(\frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m}\right).$$

The following assertions are equivalent:

(a) *There is a constant C (not depending on n) such that*

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}\left|T\left(e_{j_1}, \ldots, e_{j_m}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\|T\|,$$

for all m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \longrightarrow \mathbb{K}$ and all positive integers n.

(b) $q_1 = \cdots = q_k = \infty$.

Proof. Suppose that (a) holds and $q_k < \infty$. In this case, Lemma 7 provides a constant C such that

$$\left(\sum_{j_k=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}\left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_k}{q_{k+1}}}\right)^{\frac{1}{q_k}} \leq C\|T\|$$

for all $(m-k+1)$-linear forms $T : \ell_{p_k}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n. For any $(m-k+1)$-linear form $T : \ell_{p_k}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$, we define an $(m-k+1)$-linear form S with the same rule of T, but different domain $\ell_2^n \times \ell_{p_{k+1}}^n \times \cdots \times \ell_{p_m}^n$. So, there is a constant C such that

$$\begin{aligned}
&\left(\sum_{j_k=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}\left|S\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_k}{q_{k+1}}}\right)^{\frac{1}{q_k}} \\
&= \left(\sum_{j_k=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}\left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_k}{q_{k+1}}}\right)^{\frac{1}{q_k}} \\
&\leq C\|T\| \\
&\leq C\|S\|.
\end{aligned} \tag{8}$$

for all $(m-k+1)$-linear forms $S : \ell_2^n \times \ell_{p_{k+1}}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$, and the exponents satisfy

$$\begin{aligned}
\frac{1}{q_k} + \frac{1}{q_{k+1}} + \cdots + \frac{1}{q_m} &= \frac{1}{q_k} + \frac{(m-k)+1}{2} - \left(\frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m}\right) \\
&> \frac{(m-k)+1}{2} - \left(\frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m}\right) \\
&= \frac{(m-k+1)+1}{2} - \left(\frac{1}{2} + \frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m}\right).
\end{aligned}$$

On the other hand, replacing the unimodular $(m-k+1)$-linear form of the Kahane-Salem-Zygmund inequality (see Lemma 6.1 in Albuquerque et al. 2014) in (8), we obtain

$$n^{\frac{1}{q_k}+\cdots+\frac{1}{q_m}} \leq C_m \cdot n^{\frac{(m-k+1)+1}{2}-\left(\frac{1}{2}+\frac{1}{p_{k+1}}+\cdots+\frac{1}{p_m}\right)}.$$

Since this is valid for all n, we conclude that

$$\frac{1}{q_k} + \cdots + \frac{1}{q_m} \leq \frac{(m-k+1)+1}{2} - \left(\frac{1}{2} + \frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m}\right),$$

and this is a contradiction. Hence $q_k = \infty$. Finally, the fact that $q_1 = \cdots = q_{k-1} = \infty$ is a consequence of Lemma 8, because

$$\frac{1}{p_j} + \frac{1}{p_k} + \frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m} \geq 1$$

for all $j \in \{1, \ldots, .k-1\}$ (recall that $p_1, \ldots, p_k \in [1, 2]$).

Finally, using Theorem 4 and Lemma 8 we prove that (b) implies (a). □

Remark 16. It is worth mentioning that the above theorems are independent. For instance, if $m = 4$, $k = 2$, $p_1 = p_2 = 2$ and $p_3 = p_4 = 8$, nothing can be inferred by Theorem 14. However, using Theorem 15, we conclude that if $q_3, q_4 \in [4/3, 2]$ and $\frac{1}{q_3} + \frac{1}{q_4} = \frac{5}{4}$ then there is a constant C (not depending on n) such that

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_4=1}^{n}\left|T\left(e_{j_1},\ldots,e_{j_4}\right)\right|^{q_4}\right)^{\frac{q_3}{q_4}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\,\|T\|,$$

for all 4-linear forms $T : \ell_2^n \times \ell_2^n \times \ell_8^n \times \ell_8^n \longrightarrow \mathbb{K}$ and all positive integers n if, and only if, $q_1 = q_2 = \infty$.

The following result was proved in Albuquerque & Rezende 2018 (in Corollary 2):

Theorem 17. *(see Corollary 2 in Albuquerque & Rezende 2018) Let m be a positive integer and $p_1, \ldots, p_m \in [1, 2m]$ and $\dfrac{1}{p_1} + \cdots + \dfrac{1}{p_m} < 1$. Then, there is a constant C (not depending on n) such that*

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}\left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq C\,\|T\|$$

for all m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n, with

$$\frac{1}{q_i} = \frac{1}{2} + \frac{(m-i+1)}{2m} - \left(\frac{1}{p_i} + \cdots + \frac{1}{p_m}\right),$$

for all $i = 1, \ldots, m$.

Again, Lemma 7 and Lemma 8 combined with the Kahane-Salem-Zygmund inequality (see Lemma 6.1 in Albuquerque et al. 2014) and Lemma 3.1 in Aron et al. 2017 give us the following super-critical version of the Theorem 17:

Theorem 18. *Let $m \geq 2$, $k \in \{1, \ldots, m-1\}$, $p_1, \ldots, p_k \in [1, \infty]$ and $p_{k+1}, \ldots, p_m \in [2, 2(m-k)]$, be such that*

$$\frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m} < 1$$

and

$$\frac{1}{p_j} + \frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m} \geq 1,$$

for all $j \in \{1, \ldots, k\}$. Then

$$\left(\sum_{j_1=1}^{n}\left(\cdots\left(\sum_{j_m=1}^{n}\left|T\left(e_{j_1},\ldots,e_{j_m}\right)\right|^{q_m}\right)^{\frac{q_{m-1}}{q_m}}\cdots\right)^{\frac{q_1}{q_2}}\right)^{\frac{1}{q_1}} \leq 2^{\frac{m-k-1}{2}}\,\|T\| \qquad (9)$$

for all m-linear forms $T : \ell_{p_1}^n \times \cdots \times \ell_{p_m}^n \to \mathbb{K}$ and all positive integers n, with $q_1 = \cdots = q_k = \infty$ and

$$\frac{1}{q_i} = \frac{1}{2} + \frac{(m-i+1)}{2(m-k)} - \left(\frac{1}{p_i} + \cdots + \frac{1}{p_m}\right),$$

for all $i = k+1, \ldots, m$. Moreover, $q_1 = \cdots = q_k = \infty$, and the optimal exponents q_i satisfying (9) are such that

$$q_i \geq \frac{1}{1 - \left(\frac{1}{p_i} + \cdots + \frac{1}{p_m}\right)}, i = k+1, \ldots, m,$$

and the inequality

$$\frac{1}{q_{k+1}} + \cdots + \frac{1}{q_m} \leq \frac{(m-k)+1}{2} - \left(\frac{1}{p_{k+1}} + \cdots + \frac{1}{p_m}\right)$$

is verified.

Remark 19. When $k = 1$ and $p_1 = \cdots = p_m = m$ we recover Theorem 6.

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