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Topological equivalence for multiple saddle connections

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ABSTRACT

We study the topological equivalence between two vector fields defined in the neighborhood of the skeleton of a normal crossings divisor in an ambient space of dimension three. We deal with singularities obtained from local ones by ambient blowing-ups: we impose thus the non-degeneracy condition that they are all hyperbolic without certain algebraic resonances in the set of eigenvalues. Once we cut-out the attractors, we get the result if the corresponding graph has no cycles. The case of cycles is of another nature, as the Dulac Problem in dimension three.

Key words: saddle connections, singular vector fields, topological equivalence, blowing ups.

1 INTRODUCTION

This note is motivated by the study of the topological equivalence between a real analytic three-dimensional germ of vector field and its principal part given by the Newton diagram. The work (Brunella and Miari 1990) solves the problem in dimension two. One of the main difficulties in dimension three is the presence of *rigid* saddle connections in the intersections of the irreducible components of the divisor after reduction of singularities. Here we deal with saddle connections along the skeleton of a divisor. The complete study of the topological equivalence for a vector field and its principal part will be given in a forthcoming paper.

Let M be a real analytic ambient space and $D \subset M$ be a normal crossings divisor on M. The *skeleton* $\mathcal{K}$ of D is the union of the intersections of pairs of irreducible components of D. The *corners* $\mathcal{C}$ of D are the points of triple intersections: we assume they are finitely many. Let us

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consider two vector fields ξ and ξ' defined on a neighborhood of the skeleton and assume that the following properties are fulfilled:

1. The irreducible components of D are invariant for ξ and ξ'.
2. The equilibrium points of ξ and ξ' are just the corners $\mathcal{C}$ of D and they are hyperbolic saddle singularities.
3. At each corner $C \in \mathcal{C}$ the eigenvalues of ξ and ξ' corresponding to each line in the skeleton are the same ones.

Also we will consider a generic hypothesis $\mathcal{H}$ on the whole set of the eigenvalues: this condition $\mathcal{H}$ is given by a finite number of algebraic relations, in particular it will be satisfied in the case that all the eigenvalues are algebraically independent.

The *skeleton* $\mathcal{K}$ supports a graph whose vertices are the corners $\mathcal{C}$. The main result in this paper is the following one:

THEOREM 1. *Assume $\mathcal{H}$ is satisfied and the skeleton $\mathcal{K}$ has no cycles. Then there are two neighborhoods V and V' of $\mathcal{K}$ and a topological equivalence $h : V \to V'$ between ξ and ξ' that respects the divisor and induces the identity over the skeleton.*

Let us made some comments on the hypothesis. After blowing-up, we generically get hyperbolic singularities, but not necessarily of saddle type nor just corners. Anyway, we can make a excision by cutting a small ball around attractor-type singularities to get only saddle type singularities. The topological equivalence extends in a standard way to the attractors. Also, we can do the same procedure with a non-corner saddle such that the invariant variety of dimension two is contained in the divisor. Moreover, if we have a saddle not in a corner, we can add the (transversal) invariant variety of dimension two as a new *small* component of the divisor to get in this way a corner.

The hypothesis that the eigenvalues for ξ and ξ' are the same ones is satisfied in the case of a vector field and its principal part. In fact, one can get more accurate properties defined in terms of the distribution of eigenvalues, see (Alonso 2002) that guarantee the existence of a topological equivalence and that give even a complete invariant system for the topological moduli.

The case of two opposite saddles is completely solved in (Alonso 2002) in a framework that is exactly the complement of (Bonatti and Dufraine 2001), where they need to consider complex eigenvalues.

2 DIRECT PROPAGATION OF THE EQUIVALENCE

Let us first give a description of the *equivalence neighborhoods* V and V'. We construct V in reference to ξ and V' will be constructed in the same way in reference to ξ'.

Around each corner $C \in \mathcal{C}$ we fix a coordinate system (x, y, z) such that the divisor is locally given by $xyz = 0$ and we have the *push properties*

$$d(x^2 + y^2) < 0, \text{ outside x=y=0; } dz^2 > 0, \text{ outside z=0.}$$

Working in a small box $\mathrm{B}_+(a, b) = \left\{0 \le z \le a; 0 \le x^2 + y^2 \le b^2\right\}$, we define the *chimney* $\mathrm{Ch}_+(a, b, \rho)$ to be the saturation by the flow of ξ of the *top* of the chimney $\Omega_+ = \left\{z = a; x^2 + y^2 \le \rho^2\right\}$. In a similar way we get the chimney $\mathrm{Ch}_-(a, b, \rho)$ in $z \le 0$. The region $\Sigma = \mathrm{Ch}_+(a, b, \rho) \cap \left\{x^2 + y^2 = b^2\right\}$ is called the *fence* and $\Delta = \mathrm{Ch}_+(a, b, \rho) \cap \{z = 0\}$ the *basis* of the chimney.

PROPOSITION 2 (M.I. Camacho 1985). *Any homeomorphism $\bar{h} : \Sigma \to \Sigma'$ between the fences of two chimneys associated respectively to ξ and ξ' extends to a topological equivalence between ξ and ξ' in the chimneys. Moreover, if $\bar{h}$ respects the divisor and induces the identity over the skeleton, then h can be taken also with this property.*

Now we put a small cylinder that connects two chimneys along each *edge* of the skeleton: the edges being the segments of the skeleton connecting two adjacent corners. In this way we get the equivalence neighborhood V.

Our results of topological equivalence will be first obtained in the *basic zones*, that are the closures of the connected components in the equivalence neighborhood of the complement of the divisor. Two basic zones intersect either in a point, or in an edge, or in a *wall*. A *wall* W is defined as follows: we consider an irreducible component D of the divisor and we take W to be one of the connected components in D of the complement of the other components of the divisor.

The propagation of the topological equivalence will be compatible with a global *push function* g defined in the equivalence neighborhood. The function g restricts to a wall, acting there as a push function for the restrictions of the vector fields. In this way, to know the equivalence in a wall it is enough to know it in the transversal sections between attractor type points (that is, the wall contains the corresponding invariant variety of dimension two) or/and the *ends* of the wall. Also, we always get the identity over the skeleton.

The basic result of direct propagation of the topological equivalence given in proposition 2 allows us to get a topological equivalence in the case we have not a connection of saddles by the invariant varieties of dimension one. To be precise, each edge Γ of the skeleton connecting two corners C_1 and C_2 is of one of the following types:

A. The invariant variety of dimension one in C_1 coincides with Γ and Γ is contained in the invariant variety of dimension two in C_2.

B. The invariant varieties of dimension two in C_1 and C_2 contain both Γ. (Here there are two possible positions: following the fact that the invariant varieties either coincide or cut transversely along Γ).

C. The edge Γ is transversal to the two invariant varieties of dimension two and hence it coincides with the corresponding invariant varieties of dimension one. This edge will be called an *essential saddle connection edge*.

PROPOSITION 3. *Let Z be a basic zone for ξ and Z' the corresponding basic zone for ξ' without essential saddle connection edges. Let us fix a topological equivalence h_W for each wall in the frontier of Z. Then there is a topological equivalence $h : Z \to Z'$ between ξ and ξ' that respects the given h_W.*

Let us give an idea of the proof of this result. We construct an oriented graph $\mathcal{A}(Z)$ as follows. Each edge Γ of the type A will be provided of an arrow going from C_1 to C_2, in the case that C_1 is the vertex (corner) such the invariant variety of dimension one coincides with Γ. In this way, a topological equivalence in a pair of chimneys at C_1 will propagate to a partial topological datum for a homeomorphism in the fences of a pair of chimneys at C_2: always respecting the frontier data. Several data like this can be integrated in a homeomorphism between the fences at C_2, that can be extended to the chimneys and so on. In the edges of the type B, we chose the arrow arbitrarily: this choice will determine our global procedure of propagation of the topological equivalence. Also a topological equivalence at the lower vertex in the edge will provide a partial topological datum for a homeomorphism in the fences of the higher vertex in the edge, that can eventually be integrated in a homeomorphism between the fences.

We know that $\mathcal{A}(Z)$ is an oriented tree, since the skeleton has no cycles. Hence we can propagate a topological equivalence from the minimal points to the whole zone Z.

COROLLARY 4. *If there are not essential saddle connection edges, we get a topological equivalence $h : V \to V'$ between ξ and ξ' that respects the divisor and induces the identity over the skeleton.*

In fact, the basic zones are like in the proposition. We can do the topological equivalence zone by zone, respecting the frontiers.

We say that a zone Z without edges of the type C is a *direct propagation zone* or a zone of type I.

3 ESSENTIAL SADDLE CONNECTIONS. FIRST ZONES

Let us deal with a skeleton having edges of the type C and assume in this section that there are no edges of the type B. Then, we have two types of basic zones: direct propagation zones as above and basic zones having at least one edge of the type C and the other ones of the type A, that we call *zones of type* II. In this section we will give an idea of the proof of the following result.

THEOREM 5. *Let Z be a basic zone of type II for ξ and Z' the corresponding basic zone for ξ'. Consider a topological equivalence h_W for each wall W of Z. Then there is a topological equivalence $h : Z \to Z'$ between ξ and ξ' that respects the given h_W.*

And it produces as above the corollary.

COROLLARY 6. *If there are no edges edges of type B, we get a topological equivalence $h : V \to V'$ between ξ and ξ' that respects the divisor and induces the identity over the skeleton.*

Since the skeleton has no cycles, a zone of type II has exactly one edge of type C. Moreover a wall W contains at most one corner corresponding to the basis of a chimney, that is, of attractor type. The walls without attractor type points are exactly those containing the edge of type C. In order to fix a topological equivalence in a wall W is enough to give the data at a small circle around the attractor type point or at a transversal section at the edge of type C.

The above theorem 5 has already been proven in (Alonso 2002) for the special case that the basic zone has just one edge of type C and two corners. There it was used a pair of *standard foliations* of the disc. For general case we will need to introduce the more complicated notions of *mills* of foliations and *seeds* of homeomorphisms.

Let us work on the first quadrant $\mathbb{D}_{++}$ of the disc and choose coordinates x, y. The *standard foliation of weight* $\alpha > 0$ is by definition the foliation $\mathcal{F}_\alpha$ defined by $xdy - \alpha ydx = 0$, whose leaves are $y = cx^\alpha$. Consider the weighted blowing-up

$$\pi_\alpha : [0, 1] \times [0, 1] \to \mathbb{D}_{++}$$

given by $(s, t) \mapsto (s \cos t\pi/2, s^\alpha \sin t\pi/2)$. Note that $\mathcal{F}_\alpha$ is the direct image by π_α of the horizontal foliation $ds = 0$. We say that a topological foliation $\mathcal{H}$ over $\mathbb{D}_{++}$ is α-*compatible* if it is the direct image by π_α of a foliation $\widetilde{\mathcal{H}}$, where $\widetilde{\mathcal{H}}$ is the direct image of $ds = 0$ by a homeomorphism of $[0, 1] \times [0, 1]$ respecting the borders.

Let us recall the basic fact on the extension of a homeomorphism at the top of a chimney to a topological equivalence in the chimney. Consider two first quadrant chimneys C_{++} and C'_{++} associated at a point with ξ and ξ' respectively. Let us denote by Ω_{++} and Ω'_{++} the corresponding tops of the chimneys, that we identify in coordinates compatible with the chimneys with the disc $\mathbb{D}_{++}$. Denote by α, α' the quotient of the eigenvalues in the basis of the chimney (invariant variety of dimension two) for ξ and ξ'.

Proposition 7 (Alonso 2002). *A homeomorphism $h : \Omega_{++} \to \Omega'_{++}$ extends to a topological equivalence in C_{++} and C'_{++} between ξ and ξ' if and only if $h_*\mathcal{F}_\alpha$ is an α'-compatible foliation. Equivalently, it sends any α-compatible foliation to an α'-compatible foliation.*

Another tool in our proof is the remark that the flow along the chimney C_{++} modifies the β-compatible foliations in a controlled way. Actually, assume that $\beta \neq \alpha$, then any β-compatible foliation $\mathcal{G}$ at the top of the chimney concentrates in a $\widetilde{\beta}$-compatible foliation in one of the two *windows* of the fence, depending on the fact that $\beta < \alpha$ or $\beta > \alpha$. Moreover $\widetilde{\beta} = (\beta - \alpha)/\alpha\mu$ where μ corresponds to the eigenvalue associated to the invariant variety of dimension one (in the case $\beta > \alpha$). The *windows* are the regions of the fence at the intersection points with the skeleton. Conversely, any γ-compatible foliation at the fence goes by the flow to a γ^*-compatible foliation at the top, where γ^* depends algebraically from γ, α and μ.

A *mill* $\mathcal{M}$ of length n consists in a list of weights $\mathcal{W} = \{\beta_i\}_{i=1}^n$ such that $0 < \beta_i < \beta_{i+1}$ and a list of points $\mathcal{P} = \{P_i = (x_i, y_i) \in \mathbb{S}_{++}\}_{i=1}^n$ such that $0 < x_i < x_{i+1} < 1$. Associated to each index i we consider the leaf $\mathcal{L}_i$ given by $y = c_i x^{\beta_i}$ and passing through the point P_i. The leaves

$\mathcal{L}_0$ and $\mathcal{L}_{n+1}$ are respectively the vertical and the horizontal semi-axes. The region $\mathcal{R}_i$ is the region of $\mathbb{D}_{++}$ contained between $\mathcal{L}_{i-1}$ and $\mathcal{L}_{i+1}$, for $i = 1, 2, \ldots, n$. As a corollary of proposition 7 we have that

Assume that $\alpha = \beta_i$, then any homeomorphism of the top of the chimney that respects the standard foliation $\mathcal{F}_\alpha$ on the region $\mathcal{R}_i$ extends to a homeomorphism of the chimney that gives a topological equivalence between ξ and ξ'.

Here we identify, without loss of genericity the chimneys corresponding to ξ and ξ'. Recall also that we have the same eigenvalues for both vector fields.

Define by $\mathcal{A}_i$, for $i = 0, 1, \ldots, n+1$ the arc joining P_i and P_{i+1} over the frontier of the disc. A *seed* $\mathcal{S}$ of homeomorphisms for the mill is a list of homeomorphisms:

$$\tau_i : \mathcal{L}_i \to \mathcal{L}_i; \;\; \rho_j : \mathcal{A}_j \to \mathcal{A}_j,$$

for $i = 0, 1, \ldots, n+1$ and $j = 0, 1, \ldots, n$.

Let us detail now one of the key remarks for our construction of the topological equivalence. For each index $i = 1, 2, \ldots, n$, the seed $\mathcal{S}$ provides a homeomorphism $\phi_i : [0, 1] \to [0, 1]$ as follows. First we do the β_i-weighted blowing-up π_{β_i}. We recover ϕ_i from τ_{i-1}, ρ_i, ρ_{i+1} and τ_i on the exceptional divisor $[0, 1]$, by knowing that $\mathcal{F}_{\beta_i}$ transforms into the horizontal foliation, the leaves $\mathcal{L}_{i-1}$ and $\mathcal{L}_{i+1}$ are transverse to it and they go one to the top and the other one to the bottom of the exceptional divisor. Assume that $\beta_i = \alpha$, if we have a homeomorphism $h : \mathbb{D}_{++} \to \mathbb{D}_{++}$ that respects the region $\mathcal{R}_i$ and the foliation $\mathcal{F}_\alpha$ on it, then h extends to a topological equivalence H between ξ and ξ' in the chimney, moreover

The restriction of H to the border of the basis of the chimney is conjugated to the morphism ϕ_i by the flow of the vector fields, after a weighted blowing-up with center the invariant variety of dimension one. In particular the morphism ϕ_i is determined by the restriction of H to that border and conversely.

The next lemma allows us to do this argument in a multi-valuate way.

LEMMA 8. *There is a homeomorphism $h : \mathbb{D}_{++} \to \mathbb{D}_{++}$ that respects the points P_i, the leaves $\mathcal{L}_i$, that extends the seed data and such that for each region $\mathcal{R}_i$ it respects the pair of foliations $\mathcal{F}_{\beta_i}$, $\mathcal{F}_{\beta_{i+1}}$ (when the index has a sense).*

Such a homeomorphism h is said to *respect* the mill $\mathcal{M}$ and the seed data $\mathcal{S}$.

Now let us give the construction of a mill $\mathcal{M}$ and a seed $\mathcal{S}$ associated to a basic zone Z of type II, jointly with a homeomorphism data in the walls. Let us put a transversal section Ω_{++} at the edge of type C, that we identify with the disc-quadrant $\mathbb{D}_{++}$. Given each corner in the zone, we have a weight corresponding to the quotient of the eigenvalues in the saddle. The corners ‘‘goes’’ naturally to the transversal section Ω_{++} following the edges of the type A (the one dimensional invariant varieties) and the weights vary as we have explained before. We get a list of weights β_i

at Ω_{++}. We enrich it by putting $\beta_i(-) < \beta_i < \beta_i(+)$ close enough to β_i, in such a way that the corresponding regions $\mathcal{R}_i(-)$, $\mathcal{R}_i$ and $\mathcal{R}_i(+)$ are such that $\mathcal{R}_i$ and $\mathcal{R}_j$ do not intersect for $i \neq j$. The data h_W of the walls determine one of the seed datum ϕ_i as we explained above. We complete with $\phi_i(-)$, $\phi_i(+)$ in order to get the complete seed data $\mathcal{S}$. Let us remark that our hypothesis on genericity of the eigenvalues allow us to suppose that all the weights β_i are *different*.

The next proposition extends the proposition 7.

Proposition 9. A homeomorphism $h : \Omega_{++} \to \Omega_{++}$ that respects the mill $\mathcal{M}$ and the seed data $\mathcal{S}$ above extends to a topological equivalence in the whole zone Z between ξ and ξ' that respects the fixed homeomorphisms in the walls.

Now the theorem 5 follows straightforward.

4 THE GENERAL SITUATION

Let us allow now edges of the type B in the skeleton. We consider a zone Z of type III: it contains at least one edge of type C and edges of the type A and B. We can do the same statements as above in order to end the proof of the main theorem, but this time the zone Z has a composed structure. Let us describe it in a non-detailed way.

First we have not necessarily just one edge of the type C. To each type C edge Γ we associated its *direct influence zone* $\mathrm{I}(\Gamma)$ defined by the edges and corners that connect to Γ by a sequence of edges of the type A. The *outer edges* of $\mathrm{I}(\Gamma)$ are necessarily of type B. We call a *cloud* $\mathcal{N}$ the maximal unions of direct influence zones connected just by one edge of type B. The basic zone Z is then a union of clouds by parts that behave like the direct propagation zones of type I.

The main difficulty in the construction of the topological equivalence is then inside each cloud $\mathcal{N}$, one cloud being quite independent from another one. We construct mills in a sequential way at each edge Γ of type C by considering the travelling of the weights in the saddles as explained before: one mill can receive weights from another direct influence zone. The first mill we construct interfers also in the next one, by creating a *connection sector* that accumulates the information coming from the previous mills. In this way a saddle in the cloud is controlled just by one mill and we assure the compatibility in the extension of a homeomorphism from the other ones. This allows us to get a topological equivalence at the cloud that respects data on the walls and on the B edges connecting with the other parts of the zone Z. We extend then to the whole zone.

RESUMO

Estudamos a equivalência topológica entre dois campos de vetores na vizinhança do esqueleto de um divisor com cruzamento normal, num ambiente de dimensão três. Consideramos singularidades obtidas por explosões a partir de uma singularidade local: isto justifica a condição de hiperbolicidade e não ressonância no conjunto dos autovalores. O resultado principal se obtém quando, depois de retirar os atratores, o grafo resultante não tem ciclos. O caso dos ciclos é de natureza semelhante ao problema de Dulac em dimensão três.

Palavras-chave: ligações de selas, campos de vetores singular, equivalência topológica, explosões.

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