Positive Operator Valued Measures and the Quantum Monty Hall problem

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ABSTRACT

A quantum version of the Monty Hall problem, based upon the Positive Operator Valued Measures (POVM) formalism, is proposed. It is shown that basic normalization and symmetry arguments lead univocally to the associated POVM elements, and that the classical probabilities associated with the Monty Hall scenario are recovered for a natural choice of the measurement operators.

Key words: Quantum Measurements, Monty Hall Problem, Quantum Information Theory.

INTRODUCTION

The physics of information has been the focus of intensive research in recent years (Frieden 2004, Leff and Rex 2003, Nielsen and Chuang 2000, Plastino and Daffertshofer 2004, von Baeyer 2003). The discovery of novel and counter-intuitive ways of processing and transmitting (quantum) information (Nielsen and Chuang 2000) has greatly increased the interest in this emerging field. Probabilistic concepts play, of course, a prominent role in this multidisciplinary area of science. The quantum mechanical formalism itself can be regarded as a generalization of classical probability theory. From this point of view, classical probabilistic settings can be "quantized" (Meyer 1999). The extension of classical probability problems to quantum mechanical scenarios (and

the study of classical analogues of quantum mechanical information processes) illuminates fundamental issues concerning the relation between classical and quantum probabilities (Meyer 1999, Plastino and Daffertshofer 2004). Classical probability puzzles have also been used in order to clarify the application to statistical physics (and other fields) of subtle probabilistic ideas related to the concept of information (Tsallis et al. 1996).

The aim of the present contribution is to revisit the celebrated Monty Hall Problem (MHP) (Gillman 1992, Glazer and Work 2001, Hoffman 1999, von Baeyer 2003) and propose a new quantization scheme for it, based on positive operator valued measures (POVM) (Nielsen and Chuang 2000). The MHP (for an early and specially clear discussion see (Gillman 1992)) has aroused considerable interest among mathematicians, philosophers and psychologists because it touches on fundamental issues that revolve around the concept of probability and, further, on *how* people assess probabilities (von Baeyer 2003). The MHP is even becoming a staple illustration in the teaching of probability theory (see for instance (Glazer and Work 2001)).

More explicitly, in the present work we show how the POVM formalism leads to a rather "natural" formulation of a "minimalist" quantum version of the MHP.

POVM MEASUREMENTS

The von Neumann projective measurement discussed in standard textbooks of quantum mechanics does not constitute the most general kind of measurements allowed by the laws of Quantum Mechanics. General quantum measurements are described by a collection $\{M_m\}$ of measurement operators, the index m referring to the possible measurement outcomes of the concomitant experiment (Nielsen and Chuang 2000).

If the state of the quantum system is described, immediately before the measurement, by the density matrix ρ, then the probability that the result m occurs is given by

$$p(m) = tr(M_m^\dagger M_m \rho). \tag{1}$$

The density operator after obtaining the result m is

$$\rho_m = \frac{M_m \rho M_m^\dagger}{tr(M_m^\dagger M_m \rho)}. \tag{2}$$

The measurement operators satisfy the completeness equation, $\sum_m M_m^\dagger M_m = I$. The measurement operators are not necessary orthogonal and the number of available outcomes may differ from the dimension N of the Hilbert space. Let us now consider the hermitian, positive operators $E_m \equiv M_m^\dagger M_m$. It is plain that $\sum_m E_m = I$ and that $p(m) = tr(E_m \rho)$. Thus the set of operators E_m are sufficient to determine the probabilities $p(m)$ of the different measurement outcomes. The complete set $\{E_m\}$ is known as a POVM (Nielsen and Chuang 2000). Given an arbitrary set of hermitian positive operators $\{E_m\}$ such that $\sum_m E_m = I$, one can show that there exists at least one set of measurement operators M_m defining a measurement described by the POVM $\{E_m\}$. For instance, we have $M_m = \sqrt{E_m} \equiv \sum_{j=1}^N \lambda_j^{1/2} |j\rangle\langle j|$, where the $|j\rangle$'s and λ_j's, $j = 1, 2, \ldots N$,

are the eigenvectors and eigenvalues of E_m, respectively. Another possible set of measurement operators M_m is given by $M_m = U_m \sqrt{E_m}$, where the U_m are arbitrary unitary operators.

A QUANTIZATION SCHEME FOR THE MONTY HALL PROBLEM

The well-known classical Monty Hall problem was originally set in the context of a game show (von Baeyer 2003, D'Ariano et al. 2002).

1. Before the show starts a prize is hidden behind one of three closed boxes. The show master knows where the prize is but, of course, the candidate does not.

2. The candidate is asked to choose one of the three boxes, which is, however, not opened at this stage.

3. The show master opens another box, and shows that it is empty. (He can do this, because he knows where the prize is.)

4. The candidate can now open one of the remaining boxes to either collect her prize or lose.

The question is now, *whether the candidate should stick to her original choice or change to the other remaining box*. Classically, the optimum strategy is to alter her choice of box, thereby doubling her chance of winning the prize. Even if there is a general consensus that this conclusion is correct, the probabilistic reasoning leading to it is rather subtle, and has been the focus of heated controversy (von Baeyer 2003, Hoffman 1999). Like other probabilistic puzzles of this sort it can be "quantized", i.e., its key elements can be translated into a quantum mechanical language. Usually there are different ways of performing these quantizations, depending on what features of the problem are seen as essential, and also on how are the relevant information gathering processes formalized. The MHP is no exception and quantization schemes have already been proposed [see (Li et al. 2001, D'Ariano et al. 2002, Flitney and Abbott 2002)].

Our present purpose is to advance a *different* quantum version of the MHP based upon POVM. The main quantum variable will be the location of the prize which is a quantum mechanical particle lying in a three-dimensional Hilbert space. Thus, the particle's Hilbert space is spanned by the orthonormal basis $|0\rangle$, $|1\rangle$ and $|2\rangle$, the state $|i\rangle$ corresponding to the particle in the i-box. Person A then selects a superposition of these states as the initial state of the particle and person B selects a particular box, say $|0\rangle$. This initial choice ($|0\rangle$) stays fixed throughout the problem. Since now either box $|1\rangle$ or $|2\rangle$ can be opened (opening a box corresponds to a POVM measurement), there will be two possible measurement outcomes: employment of the POVM formalism entails that there will be two operators E_1 and E_2 that are, precisely, the POVM elements associated with measurement outcomes 1 and 2. In the present quantum mechanical scenario, person A performs the POVM measurement (E_1, E_2) in order to decide which box to open. If she gets the outcome E_1, she opens box $|2\rangle$ and if she gets outcome E_2, she opens box $|1\rangle$. From the properties of the POVM elements and from the fact that E_1 and E_2 form a complete set, it follows that (a) $E_1 + E_2 = I$, (b)

E_1 and E_2 are positive, (c) E_1 and E_2 are Hermitian. Also, since the problem is symmetric with respect to 1 and 2, it means that (d) E_1 and E_2 are symmetric in 1 and 2. The problem requires that the probability that box $|1\rangle$ is opened is zero when the particle is in state $|1\rangle$ and similarly for box $|2\rangle$. From that condition it follows that when the particle is in box $|1\rangle$, box $|2\rangle$ must be opened, that is, the probability of opening $|2\rangle$ must be one. Another condition is that, when the particle is in box $|0\rangle$, the probability of either box $|1\rangle$ or $|2\rangle$ being opened is the same, as neither box is preferred. Therefore, from the relation $E_m \equiv M_m^\dagger M_m$ and equation (1), one obtains $\langle 1| E_2 |1\rangle = \langle 2| E_1 |2\rangle = 0$, $\langle 1| E_1 |1\rangle = \langle 2| E_2 |2\rangle = 1$, and $\langle 0| E_1 |0\rangle = \langle 0| E_2 |0\rangle = \frac{1}{2}$. These constraints for E_1 and E_2 imply, after some algebra, that their associated matrices are

$$E_1 = \begin{pmatrix} \frac{1}{2} & c & -c \\ c^* & 1 & ib \\ -c^* & -ib & 0 \end{pmatrix}; \qquad E_2 = \begin{pmatrix} \frac{1}{2} & -c & c \\ -c^* & 0 & -ib \\ c^* & ib & 1 \end{pmatrix} \tag{3}$$

where c and b are complex numbers. The eigenvalues of $E_{1,2}$ are given by the third order equation

$$-\lambda^3 + \frac{3}{2}\lambda^2 + \lambda\left(-\frac{1}{2} + b^2 + 2|c|^2\right) - \frac{b^2}{2} - |c|^2 = 0.$$

The positiveness of $E_{1,2}$ then requires that $b^2 + 2|c|^2 \leq 0$, implying that $c = b = 0$. Therefore, the POVM operators $E_i = \frac{1}{2}|0\rangle\langle 0| + |i\rangle\langle i|$, $(i = 1, 2)$ are univocally determined by the basic normalization and symmetry features of the MHP. On the contrary, the measurement operators $M_{1,2}$ are not univocally determined. The general solutions for the equations $E_{1,2} = M_{1,2}^\dagger M_{1,2}$ determining the measurement operators $M_{1,2}$ compatible with the POVM operators $E_{1,2}$ are of the form $M_1 = \frac{1}{\sqrt{2}}|\phi_1\rangle\langle 0| + |\psi_1\rangle\langle 1|$ and $M_2 = \frac{1}{\sqrt{2}}|\phi_2\rangle\langle 0| + |\psi_2\rangle\langle 1|$, where $|\phi_1\rangle$, $|\psi_1\rangle$, $|\phi_2\rangle$, and $|\psi_2\rangle$ are arbitrary (normalized) states verifying $\langle \phi_1 | \psi_1 \rangle = 0$ and $\langle \phi_2 | \psi_2 \rangle = 0$.

On the basis of the above results we see that various quantum versions of the MHP are possible (depending on the choice of the initial state ρ_{in} and the measurement operators $M_{1,2}$) leading to different post-measurement outcomes, with different associated probabilities. Notice that the classical version of the MHP also admits several variants, corresponding to different initial probability distributions for the three boxes, the usual case being that of equiprobability. An initial equiprobable state in the quantum case would be described by a density matrix of the form

$$\rho_e = \lambda |\Psi\rangle\langle\Psi| + \frac{1}{3}(1 - \lambda)\mathbf{I}, \quad \text{where} \quad |\Psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle), \, 0 \leq \lambda \leq 1,$$

and $\mathbf{I}$ stands for the three-dimensional identity matrix. The quantum MHP proceeds as follows. Person A chooses an initial state ρ_{in} and then performs the quantum measurement $\{M_1, M_2\}$ (assuming that the box selected by B was 0). The state ρ_{aft} after the measurement is (depending on the measurement outcome) given by equation (2). The probabilities of finding the particle in one of the states $\{|0\rangle, |1\rangle, |2\rangle\}$ are $\langle k | \rho_{\mathrm{aft}} | k \rangle$, $k = 0, 1, 2$. Adopting the initial "equiprobable" state ρ_e and $M_{1,2} = \sqrt{E_{1,2}}$ it can be shown, after some algebra, that these final probabilities lead to

the same conclusion (i.e., person B must change her initial choice in order to double her chances of success) *as on the classical way if reasoning*. It is interesting that this result does not depend on the value of the parameter λ.

It is worth to stress the main features in which our *present* quantum version of the Monty Hall problem differs from the *other quantum versions* already discussed in the literature. First of all, ours is a direct quantization of the MHP without recourse to ancillary particles (as opposed, for instance, to the quantization advanced in (Li et al. 2001)). In this respect, the present approach has some similarities with the quantization scheme proposed in (Flitney and Abbott 2002).

However, our quantum version of the Monty Hall game also exhibits important differences from the one discussed in (Flitney and Abbott 2002). In this last case, the quantum game is modelled on the basis of a composite quantum system consisting of three qutrits: one associated with the location of the prize, a second one corresponding the candidate's choice and a third one associated with the master's choice regarding which box is to be open (Flitney and Abbott 2002). Our approach can be regarded as more economical. *The state of the system is here described by just one qutrit* representing the location of the prize, and the combined choices of the candidate and the master determine the structure of the generalized quantum measurement M_1, M_2 (and the associated POVM E_1, E_2).

CONCLUSIONS

A "minimalist" quantum version of the MHP, based upon the POVM formalism, has been advanced. The corresponding POVM elements $E_{1,2}$ were seen to be completely determined by the basic normalization and symmetry properties of the MHP scenario.

Starting from the an initial "equiprobable" state (with the natural choice $M_{1,2} = \sqrt{E_{1,2}}$ for the measurement operators) leads to identical conclusions, concerning the MHP puzzle, as in the classical probabilistic reasoning. It is remarkable that the argument leading to these conclusions seems to be more straightforward in the quantum than in the classical setting. It is perhaps not too farfetched to assert that the correct answer to the MHP puzzle is "embodied" in the very formalism of quantum mechanic's measurement theory.

RESUMO

Uma visão quântica do problema Monty Hall é proposta baseada no formalismo das Medidas Avaliadas do Operador Positivo (POVM). Demonstra-se que os argumentos de normalização básica e simetria levam de maneira inequívoca para elementos associados a POVM e que as probabilidades clássicas associadas ao cenário Monty Hall são recuperadas para uma escolha natural de medidas operadoras.

Palavras-chave: Medidas quânticas, Problema Monty Hall, Teoria quântica da informação.

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