



MATHEMATICAL SCIENCES

Holomorphic vector fields tangent to foliations in dimension three

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Abstract: We consider a singular holomorphic vector field in a neighborhood of $0 \in \mathbb{C}^3$ and suppose that there is a singular holomorphic foliation of codimension one (outside its singular set, given by a holomorphic decomposition of this neighborhood into complex surfaces, called leaves) to which it is tangent. This means that, when both objects are non-singular, the orbits of the vector field are contained in the leaves of the foliation. First we consider the desingularizations of both objects, trying to relate their final models. Then we analyse the situation where the vector field is tangent to three independent foliations.

Key words: Holomorphic foliations, holomorphic vector fields, pencil of foliations, invariant varieties.

INTRODUCTION

The main goal of this article is to study, at the origin of \mathbb{C}^3 , germs of holomorphic vector fields that are *tangent* to holomorphic foliations of codimension one. If X is a germ of holomorphic vector field at $(\mathbb{C}^3, 0)$, inducing a germ of singular holomorphic foliation of dimension one \mathcal{F} , and ω is a germ of holomorphic 1-form which satisfies the integrability condition $\omega \wedge d\omega = 0$, inducing a germ of singular holomorphic foliation of codimension one \mathcal{G} , we say that X (or \mathcal{F}) is *tangent* to ω (or to \mathcal{G}) if the one-dimensional leaves induced by X are entirely contained in the two-dimensional leaves of ω , wherever both objects are defined. We also say that ω (or \mathcal{G}) is *invariant* by X (or by \mathcal{F}). In algebraic terms, this is identified by the vanishing of the contraction of ω by X , that is, $i_X\omega = 0$.

One interesting point is that, due to the integrability condition of ω , not every germ of holomorphic vector field at $(\mathbb{C}^3, 0)$ is tangent to a holomorphic foliation. Examples of this situation are presented in two recent studies where the configuration of tangency between a vector field and a foliation is considered. The first one (Cano & Roche 2014), asserts that a germ of holomorphic vector field X at $(\mathbb{C}^3, 0)$ tangent to a foliation has a reduction of singularities. This means that, after a finite sequence of blow-ups with invariant centers (points or regular invariant curves in general position with the reduction divisor), the one-dimensional foliation induced by X is transformed into one for which all singularities are *elementary*, meaning that they are locally defined by vector fields with non-nilpotent linear part. Vector fields in a family proposed by F. Sanz and F. Sancho (Cano & Roche 2014) do not admit a reduction of singularities as above and, hence, are not tangent to any germ of holomorphic

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foliation. In the second one (Cerveau & Lins Neto 2019), it is proved that a germ of holomorphic vector field with isolated singularity at $0 \in \mathbb{C}^3$ that is tangent to a holomorphic foliation always admits a separatrix, that is, an invariant analytic curve. As a consequence, vector fields in the family of X. Gómex-Mont and I. Luengo (Gómez-Mont & Luengo 1992), which do not possess separatrices, are not tangent to holomorphic foliations.

The aforementioned studies suggest that consequences of geometric nature arise when there is tangency between a vector field and a foliation. This perception is the main motivation for this article and shall be developed in two different and independent approaches. First, we present the concept of strongly non-resonant germ of vector field, meaning that its linear part has eigenvalues that do not satisfy any non-trivial relation of linear dependency with integer coefficients. We characterize the 1-forms that are tangent to vector fields of this type (Prop. 4). Then we consider a germ of integrable holomorphic 1-form ω at $(\mathbb{C}^3, 0)$ that leaves invariant a germ of holomorphic vector field X , putting the following hypotheses on the reduction of singularities of X (which exists by Cano-Roche's result): it is composed only by punctual blow-ups (we say in this case X has an *absolutely isolated* singularity $0 \in \mathbb{C}^3$), the divisor associated with this sequence of blow-ups is invariant by the transformed foliation (that is, X is *non-dicritical*) and that all final models are strongly non-resonant singularities. Under these assumptions on X , we prove in Theorem 6 that ω defines a foliation of codimension one which has no saddle-nodes in its reduction of singularities — we say that ω belongs to the family of *CH-foliations*, which is an extension of the widely studied concept of *generalized curve* foliations in dimension two (see definitions in the next section).

Finally, in the last section of this article, we investigate the situation where a germ of holomorphic vector field at $(\mathbb{C}^3, 0)$ is tangent to three independent foliations, induced by germs of integrable holomorphic 1-forms ω_1, ω_2 and ω_3 . We prove that, in this case, up to multiplication by germs of functions in \mathcal{O}_3 , these 1-forms define a *pencil of integrable 1-forms*, that is, a two-dimensional linear space — which becomes one-dimensional when projectivized — in the space of integrable holomorphic 1-forms. As a consequence, X is tangent to the infinitely many integrable 1-forms in this pencil. In Theorem 11, we present a geometric characterization of pencil of integrable 1-forms, stated in the more general context of foliations at $(\mathbb{C}^n, 0)$, $n \geq 3$, which is a local version of the one given in (Cerveau 2002). It asserts that either there is a closed meromorphic 1-form ϑ such that $d\omega = \vartheta \wedge \omega$ for all 1-forms ω in the pencil or the *axis foliation* — that is, the unique foliation of codimension two that is tangent to all 1-forms in the pencil — has a meromorphic first integral. As a consequence, the axis foliation always admits an invariant hypersurface. This, translated into our original three-dimensional context, gives us that a germ of vector field at $(\mathbb{C}^3, 0)$ that is tangent to three independent germs of holomorphic foliations leaves invariant a germ of analytic surface.

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PRELIMINARIES

In local analytic coordinates (x_1, x_2, x_3) at $(\mathbb{C}^3, 0)$, we denote a germ of holomorphic vector field at $(\mathbb{C}^3, 0)$ by

$$X = A \frac{\partial}{\partial x_1} + B \frac{\partial}{\partial x_2} + C \frac{\partial}{\partial x_3},$$

where $A, B, C \in \mathcal{O}_3$. We also consider the germ of singular one-dimensional foliation \mathcal{F} whose leaves are the orbits of X . Thus, in order to avoid superfluous singularities, we suppose that A, B and C are without common factors. The singular set (of X or \mathcal{F}) is denoted by $\text{Sing}(X) = \text{Sing}(\mathcal{F}) = \{A = B = C = 0\}$, being an analytic set of dimension at most one. We denote a germ of holomorphic 1-form at $(\mathbb{C}^3, 0)$ by

$$\omega = adx_1 + bdx_2 + cdx_3,$$

where $a, b, c \in \mathcal{O}_3$, which are also supposed to be without common factors. If ω is integrable in the sense of Frobenius, that is, $\omega \wedge d\omega = 0$, it induces a germ of singular holomorphic foliation of codimension one, denoted by \mathcal{G} . We have $\text{Sing}(\omega) = \text{Sing}(\mathcal{G}) = \{a = b = c = 0\}$, also an analytic set of dimension at most one. A *separatrix* for a local holomorphic foliation is an irreducible germ of invariant analytic variety of the same dimension of the foliation. For germs of holomorphic vector fields and integrable 1-forms at $(\mathbb{C}^3, 0)$, separatrices are germs of invariant curves and surfaces, respectively.

We say that X (or \mathcal{F}) is tangent to ω (or to \mathcal{G}) if

$$i_X\omega = aA + bB + cC = 0,$$

which is equivalent to saying that, outside $\text{Sing}(X) \cup \text{Sing}(\omega)$, the one-dimensional leaves of \mathcal{F} are contained in the two-dimensional leaves of \mathcal{G} . If the germ of vector field X is tangent to the integrable 1-form ω , then $\text{Sing}(\omega)$ is invariant by X (see, for instance, (Mol 2011, Th. 1)). Thus, the one-dimensional components of $\text{Sing}(\omega)$ — if they exist — are separatrices of X .

One particular example of this configuration of tangency is provided by vector fields with first integrals. A non-constant germ of meromorphic — or holomorphic — function Φ at $(\mathbb{C}^3, 0)$ is a *first integral* for X if, in a small neighborhood of $0 \in \mathbb{C}^3$, Φ is constant along the orbits of X . This is equivalent to saying that X is tangent to the foliation defined by Φ , which is induced by the holomorphic 1-form obtained by cancelling the components of zeros and poles of the meromorphic 1-form $d\Phi$ (we also say that Φ is a first integral for this foliation). Note that in this case, any fiber of Φ accumulating to the origin is an X -invariant surface. We should mention that, in the article (Pinheiro & Reis 2014), the authors study germs of vector fields at $(\mathbb{C}^3, 0)$ that are *completely integrable* — that is, that have two independent holomorphic first integrals — and prove that this property is not a topological invariant.

In dimension two, a germ of holomorphic foliation is induced in local analytic coordinates (x_1, x_2) at $(\mathbb{C}^2, 0)$ by a germ of holomorphic vector field $X = A\partial/\partial x_1 + B\partial/\partial x_2$ with isolated singularity at the origin (or by the dual 1-form $\omega = Bdx_1 - Adx_2$). Recall that such a foliation has a reduction of singularities, that is, a finite sequence of blow-ups transforms it into one having a finite number of singularities which are *simple* or *reduced* (Seidenberg 1968). Such a singularity is locally induced by a vector field whose linear part is non-nilpotent, having two eigenvalues λ_1 and $-\lambda_2$ such that, if both non-zero, do not satisfy any non-trivial relation of the kind $m_1\lambda_1 + m_2\lambda_2 = 0$ with $m_1, m_2 \in \mathbb{Z}_{\geq 0}$. These simple singularities are called *non-degenerate*, whereas the ones having one zero eigenvalue are called *saddle-nodes*. A foliation at $(\mathbb{C}^2, 0)$ is said to be of *generalized curve* type (Camacho et al. 1984) if it has no saddle-nodes in some (and hence in any) reduction of singularities. Several geometric properties of a foliation of generalized curve type can be read in its separatrices. For example, a sequence of blow-ups that desingularizes its set of separatrices is also a reduction of singularities for the foliation itself (Camacho et al. 1984).

A foliation of codimension one at $(\mathbb{C}^3, 0)$ also admits a reduction of singularities (Cano & Cerveau 1992, Cano 2004). This means that after a finite sequence of blow-ups with invariant centers — points and regular curves satisfying a condition of general position with respect to the divisor — the foliation is transformed into one whose singularities are all *simple* or *reduced*, which, in analogy with the two-dimensional case, are essentially of two kinds: simple non-degenerate singularities and simple saddle-node singularities (see, for instance, the description in (Cuzzuol & Mol 2018)). Recall that the *dimensional type* of a codimension one foliation, $\tau \geq 1$, is the smallest number of variables needed to express its defining equation in some system of analytic coordinates. We say that a simple singularity is non-degenerate if there are analytic coordinates (x_1, x_2, x_3) at $0 \in \mathbb{C}^3$ in which the foliation is defined by a holomorphic 1-form ω whose terms of lowest order are:

- $x_1x_2 \left(\lambda_1 \frac{dx_1}{x_1} + \lambda_2 \frac{dx_2}{x_2} \right)$, if $\tau = 2$,
- $x_1x_2x_3 \left(\lambda_1 \frac{dx_1}{x_1} + \lambda_2 \frac{dx_2}{x_2} + \lambda_3 \frac{dx_3}{x_3} \right)$, if $\tau = 3$,

where the residues $\lambda_i \in \mathbb{C}^*$ are *non-resonant*, meaning that there are no non-trivial relations of the kind $m_1\lambda_1 + m_2\lambda_2 = 0$ (for $\tau = 2$) or $m_1\lambda_1 + m_2\lambda_2 + m_3\lambda_3 = 0$ (for $\tau = 3$), with $m_i \in \mathbb{Z}_{\geq 0}$. Evidently, the model for $\tau = 2$ corresponds to a simple non-degenerate singularity of a foliation in dimension two.

Now, a foliation \mathcal{G} at $(\mathbb{C}^3, 0)$ induced by an integrable holomorphic 1-form ω is of type *CH* if it satisfies one of the two equivalent properties:

- There exists a reduction of singularities of *CH*-type for \mathcal{G} , that is, one for which all final models are simple non-degenerate (in particular, a simple non-degenerate singularity is of type *CH*). In this case, every reduction of singularities of \mathcal{G} will be of *CH*-type.
- for every holomorphic map $\varphi : (\mathbb{C}^2, 0) \rightarrow (\mathbb{C}^3, 0)$ generically transversal to \mathcal{G} (that is, such that $\varphi^*\omega$ has an isolated singularity at $0 \in \mathbb{C}^2$) the foliation $\pi^*\mathcal{F}$ induced by $\varphi^*\omega$ is of generalized curve type.

The equivalence of these properties is proved in (Cano et al. 2015), where these foliations are called *complex hyperbolic*. Here we would rather use the initials *CH*, which makes a reference to this terminology, also carrying the idea of “generalized curve in higher dimension”. In the non-dicritical case (that is, when the reduction divisor is invariant by the transformed foliation), (Fernández-Sánchez & Mozo-Fernández 2009) call these foliations *generalized surfaces*. They also prove that a foliation of this kind becomes reduced once its set of separatrices is desingularized.

STRONGLY NON-RESONANT VECTOR FIELDS

In this section we work with vector fields whose linear parts have eigenvalues satisfying a stronger hypothesis of non-resonance. We call them *strongly non-resonant* vector fields. By assuming hypotheses on the reduction of singularities of a germ of vector field at $(\mathbb{C}^3, 0)$ that is tangent to a holomorphic foliation of codimension one — among them, that the final models belong to this family of strongly non-resonant vector fields — we can conclude that the foliation is of *CH*-type.

Recall that a vector $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$, $n \geq 2$, is *non-resonant* if there are no relations of the form $\alpha_j = \ell_1 \alpha_1 + \dots + \ell_n \alpha_n = 0$, with $\ell_1, \dots, \ell_n \in \mathbb{Z}_{\geq 0}$ satisfying $\sum_{j=1}^n \ell_j \geq 2$. A classical result asserts that a germ of complex analytic vector field X at $(\mathbb{C}^n, 0)$ whose associated eigenvalues (i.e. those of its linear part $DX(0)$) are non-resonant is linearizable in formal coordinates. We remark that these linearizing coordinates can be taken to be analytic if these eigenvalues belong to the Poincaré domain — i.e. the set of vectors $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ such that the origin $0 \in \mathbb{C}^n$ is not in the convex hull of $\{\alpha_1, \dots, \alpha_n\}$ (see (Ilyashenko & Yakovenko 2008), sections I.4 and I.5).

In this section, we work with the following notion: a vector $(\alpha_1, \dots, \alpha_n) \in \mathbb{C}^n$ is said to be *strongly non-resonant* if there are no non-trivial relations of the form $\ell_1 \alpha_1 + \dots + \ell_n \alpha_n = 0$, with $\ell_1, \dots, \ell_n \in \mathbb{Z}$. Such a non-trivial relation will be called *strong resonance*. A germ of holomorphic vector field at $(\mathbb{C}^n, 0)$, $n \geq 2$, is said to be *strongly non-resonant* if its associated eigenvalues are strongly non-resonant.

We denote the family of strongly non-resonant vector fields by \mathfrak{X}_{nr} . Clearly, a vector field in \mathfrak{X}_{nr} satisfies the usual condition of non-resonance, being linearizable in formal coordinates. Further, the associated eigenvalues are non-zero and pairwise distinct, implying that its linear part is diagonalizable. We also say that a germ of one-dimensional foliation \mathcal{F} at $(\mathbb{C}^n, 0)$ is strongly non-resonant if it is induced by a vector field in \mathfrak{X}_{nr} . Evidently, this definition does not depend on the choice of the vector field inducing \mathcal{F} .

In the sequel, we restrain ourselves to ambient dimension $n = 3$. Thus, if X is a germ of holomorphic vector field at $(\mathbb{C}^3, 0)$ in \mathfrak{X}_{nr} , we can take local formal coordinates (x_1, x_2, x_3) such that

$$X = \alpha_1 x_1 \frac{\partial}{\partial x_1} + \alpha_2 x_2 \frac{\partial}{\partial x_2} + \alpha_3 x_3 \frac{\partial}{\partial x_3}, \tag{1}$$

where $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}^*$ are pairwise distinct. Note that if we choose numbers $b_1, b_2, b_3 \in \mathbb{C}$, not all of them zero, satisfying $\alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 = 0$, then X is tangent to the formal meromorphic 1-form

$$\omega = x_1 x_2 x_3 \left(b_1 \frac{dx_1}{x_1} + b_2 \frac{dx_2}{x_2} + b_3 \frac{dx_3}{x_3} \right).$$

We start by proving a simple lemma:

Lemma 1. *A vector field in $X \in \mathfrak{X}_{nr}$ has exactly three formal smooth separatrices, which correspond to the coordinate axes in its diagonalized form.*

Proof. We take X in its diagonal form (1). Suppose, without loss of generality, that an X -invariant curve γ is parametrized as $\gamma(t) = (at + f(t), g(t), h(t))$ where $a \in \mathbb{C}^*$ and $f, g, h \in \hat{\mathcal{O}}_1$ are non-units, with f of order at least two. We have to prove that $g = h = 0$. Suppose, for instance, $g \neq 0$. The condition of invariance is expressed as

$$\Phi(t)(a + f'(t), g'(t), h'(t)) = (\alpha_1(at + f(t)), \alpha_2 g(t), \alpha_3 h(t)),$$

for some $\Phi \in \hat{\mathcal{O}}_1$ of the form $\Phi(t) = \alpha_1 t + \rho(t)$, where $\nu_0(\rho) \geq 2$. The above equation gives us

$$\frac{\alpha_2}{\Phi(t)} = \frac{g'(t)}{g(t)}.$$

Comparing residues in this formula, we find $\alpha_2/\alpha_1 = m$, where $m = \nu_0(g)$. This, however, gives a strong resonance for the vector $(\alpha_1, \alpha_2, \alpha_3)$, in contradiction with our hypothesis. Therefore $g = 0$ and, in a similar way, $h = 0$, giving that γ is contained in the x_1 -axis. □

Blow-ups preserve the family of strongly non-resonant vector fields. More precisely, we have the following:

Lemma 2. *Let $X \in \mathfrak{X}_{nr}$. Then the strict transform of the one-dimensional foliation induced by X by a blow-up with smooth invariant center is locally given by vector fields in \mathfrak{X}_{nr} .*

Proof. Take formal diagonalizing coordinates for X . For a punctual blow-up at $0 \in \mathbb{C}^3$, consider coordinates $x_1^* = x_1, x_2^* = x_2/x_1$ and $x_3^* = x_3/x_1$. In these coordinates, the strict transform of X is

$$\tilde{X} = \alpha_1 x_1^* \frac{\partial}{\partial x_1^*} + (\alpha_2 - \alpha_1) x_2^* \frac{\partial}{\partial x_2^*} + (\alpha_3 - \alpha_1) x_3^* \frac{\partial}{\partial x_3^*},$$

having an isolated singularity at $(x_1^*, x_2^*, x_3^*) = (0, 0, 0)$. We only have to check that the eigenvalues of \tilde{X} are strongly non-resonant. However, a relation of the kind

$$0 = c_1 \alpha_1 + c_2 (\alpha_2 - \alpha_1) + c_3 (\alpha_3 - \alpha_1) = (c_1 - c_2 - c_3) \alpha_1 + c_2 \alpha_2 + c_3 \alpha_3,$$

for $c_1, c_2, c_3 \in \mathbb{Z}$, is possible if and only if $c_1 = c_2 = c_3 = 0$, since the eigenvalues of X are strongly non-resonant.

In the case of a monoidal blow-up, its smooth invariant center must be one of the coordinate axes, by Lemma 1. For instance, fixing the x_3 -axis as the blow-up center and taking blow-up charts $x_1 = x_1^*, x_2 = x_2^*$ and $x_3^* = x_3/x_2$, the strict transform of X is

$$\tilde{X} = \alpha_1 x_1^* \frac{\partial}{\partial x_1^*} + \alpha_2 x_2^* \frac{\partial}{\partial x_2^*} + (\alpha_3 - \alpha_2) x_3^* \frac{\partial}{\partial x_3^*}.$$

Again, the absence of strong resonances for the eigenvalues of \tilde{X} follows from that of X . □

As a consequence, we have:

Corollary 3. *The only formal separatrices of a vector field in \mathfrak{X}_{nr} are those corresponding to the coordinate axes in its diagonal form.*

Proof. Let γ be a formal separatrix for $X \in \mathfrak{X}_{nr}$. If γ is smooth, this has been proved in Lemma 1. If γ is singular, we desingularize it through a sequence of punctual blow-ups (the existence of a desingularization can be seen in (Cano 2012)). By the previous lemma, the strict transform of the foliation induced by X has local models in \mathfrak{X}_{nr} . Its smooth invariant curves are either in the desingularization divisor or are contained in the strict transforms of the coordinate axis in diagonalizing coordinates for X . The transform of γ is evidently not in the desingularization divisor. This means that γ is smooth contained in one of the coordinate axes, which is not our case. □

Our objective now is to prove the following result:

Proposition 4. *Let ω be a germ of integrable holomorphic 1-form at $(\mathbb{C}^3, 0)$ with $\text{codim Sing}(\omega) \geq 2$. Suppose that there exists a vector field in $X \in \mathfrak{X}_{nr}$ that is tangent to the foliation defined by ω . Then, in formal diagonalizing coordinates for X and up to multiplication by a unit in $\hat{\mathcal{O}}_3$, we have either*

$$\omega = x_1 x_2 \left(b_1 \frac{dx_1}{x_1} + b_2 \frac{dx_2}{x_2} \right) \tag{I}$$

or

$$\omega = x_1x_2x_3 \left(b_1 \frac{dx_1}{x_1} + b_2 \frac{dx_2}{x_2} + b_3 \frac{dx_3}{x_3} \right), \tag{11}$$

where $b_1, b_2, b_3 \in \mathbb{C}^*$.

Proof. Fix (x_1, x_2, x_3) formal diagonalizing coordinates for X as in (1) and write

$$\omega = adx_1 + bdx_2 + cdx_3, \tag{2}$$

where $a, b, c \in \hat{\mathcal{O}}_3$ are without common factors. Since X is tangent to ω , the contraction of ω by X gives

$$0 = i_X\omega = \alpha_1x_1a + \alpha_2x_2b + \alpha_3x_3c. \tag{3}$$

The integrability condition in turn reads

$$0 = \omega \wedge d\omega = a(c_{x_2} - b_{x_3}) + b(-c_{x_1} + a_{x_3}) + c(b_{x_1} - a_{x_2}). \tag{4}$$

The differentiation of (3) with respect to each of the variables x_1, x_2 and x_3 produces the following set of equations:

$$\alpha_1a + \alpha_1x_1a_{x_1} + \alpha_2x_2b_{x_1} + \alpha_3x_3c_{x_1} = 0; \tag{5}$$

$$\alpha_1x_1a_{x_2} + \alpha_2b + \alpha_2x_2b_{x_2} + \alpha_3x_3c_{x_2} = 0; \tag{6}$$

$$\alpha_1x_1a_{x_3} + \alpha_2x_2b_{x_3} + \alpha_3x_3c_{x_3} + \alpha_3c = 0. \tag{7}$$

We have the following (this was shown to us by M. Fernández-Duque):

Assertion 1. *In the above conditions, X leaves invariant each ratio of coefficients of ω .*

Proof of the Assertion. In fact,

$$\begin{aligned} b^2X(a/b) &= bX(a) - aX(b) \\ &= b(\alpha_1x_1a_{x_1} + \alpha_2x_2a_{x_2} + \alpha_3x_3a_{x_3}) - a(\alpha_1x_1b_{x_1} + \alpha_2x_2b_{x_2} + \alpha_3x_3b_{x_3}) \\ &= b(-\alpha_1a - \alpha_2x_2b_{x_1} - \alpha_3x_3c_{x_1}) + b\alpha_2x_2a_{x_2} + b\alpha_3x_3a_{x_3} - a\alpha_1x_1b_{x_1} \\ &\quad - a\alpha_2x_2b_{x_2} - a\alpha_3x_3b_{x_3} \quad (\text{by (5)}) \\ &= b\alpha_3x_3(a_{x_3} - c_{x_1}) - ab\alpha_1 - b\alpha_2x_2b_{x_1} + b\alpha_2x_2a_{x_2} - a\alpha_1x_1b_{x_1} \\ &\quad - a\alpha_2x_2b_{x_2} - a\alpha_3x_3b_{x_3} \\ &= b\alpha_3x_3(a_{x_3} - c_{x_1}) + a\alpha_3x_3(c_{x_2} - b_{x_3}) + ab(\alpha_2 - \alpha_1) \\ &\quad + b\alpha_2x_2(a_{x_2} - b_{x_1}) + a\alpha_1x_1(a_{x_2} - b_{x_1}) \quad (\text{by (6)}) \\ &= \alpha_3x_3(c(a_{x_2} - b_{x_1})) + ab(\alpha_2 - \alpha_1) + (a_{x_2} - b_{x_1})(a\alpha_1x_1 + b\alpha_2x_2) \quad (\text{by (4)}) \\ &= ab(\alpha_2 - \alpha_1) + (a_{x_2} - b_{x_1})(\alpha_1ax_1 + \alpha_2bx_2 + \alpha_3cx_3) \\ &= ab(\alpha_2 - \alpha_1) \quad (\text{by (3)}). \end{aligned}$$

That is, $X(a/b) = (\alpha_2 - \alpha_1)a/b$. In a similar way, we find $X(a/c) = (\alpha_3 - \alpha_1)a/c$ and $X(b/c) = (\alpha_3 - \alpha_2)b/c$, proving the assertion. \square

We have just found that

$$X(a/b) = \mu_1a/b, \quad X(a/c) = \mu_2a/c \quad \text{and} \quad X(b/c) = \mu_3b/c,$$

where $\mu_1 = \alpha_2 - \alpha_1$, $\mu_2 = \alpha_3 - \alpha_1$ and $\mu_3 = \alpha_3 - \alpha_2$ are in \mathbb{C}^* . These equations can be rewritten as

$$bX(a) - aX(b) = \mu_1 ab, \quad cX(a) - aX(c) = \mu_2 ac \quad \text{and} \quad cX(b) - bX(c) = \mu_3 bc. \tag{8}$$

The first of these equations is equivalent to $bX(a) = a(X(b) + \mu_1 b)$, where we can see that the factors of a that do not divide b do divide $X(a)$. Similarly, from the second equation we have $cX(a) = a(X(c) + \mu_2 c)$, allowing us to conclude that the factors of a that do not divide c do divide $X(a)$. Since a , b and c do not have common factors we conclude that a divides $X(a)$. Analogously, b divides $X(b)$ and c divides $X(c)$. Therefore, we can find functions $R_1, R_2, R_3 \in \hat{\mathcal{O}}_3$ such that

$$X(a) = R_1 a, \quad X(b) = R_2 b \quad \text{and} \quad X(c) = R_3 c.$$

From equations (8) we have

$$R_1 - R_2 = \mu_1, \quad R_1 - R_3 = \mu_2 \quad \text{and} \quad R_2 - R_3 = \mu_3.$$

For $i = 1, 2, 3$, we write $R_i = (\lambda_i + f_i)$, where $\lambda_i \in \mathbb{C}$ and $f_i \in \hat{\mathcal{O}}_3$ is a non-unit. From the above equations, we have

$$\lambda_1 - \lambda_2 = \mu_1 = \alpha_2 - \alpha_1, \quad \lambda_1 - \lambda_3 = \mu_2 = \alpha_3 - \alpha_1, \quad \lambda_2 - \lambda_3 = \mu_3 = \alpha_3 - \alpha_2$$

and $f_1 = f_2 = f_3 = f$.

Suppose that $a \neq 0$ and denote by a_ν be its initial part, that is, the homogeneous part of order $\nu = \nu_0(a)$ of its Taylor series. Taking initial parts in both sides of $X(a) = (\lambda_1 + f)a$ and considering the fact that the derivation by X preserves the degree – actually, the multidegree – of each monomial, we have

$$X(a_\nu) = \lambda_1 a_\nu.$$

Further, if $x = b_1 x_1^i x_2^j x_3^k$ is a non-zero monomial in a_ν , where $b_1 \in \mathbb{C}^*$, then $X(x) = \lambda_1 x$, which gives $\lambda_1 = i\alpha_1 + j\alpha_2 + k\alpha_3 \neq 0$. Note that the fact that $(\alpha_1, \alpha_2, \alpha_3)$ is strongly non-resonant implies that $a_\nu = x$.

Assertion 2. a_ν divides a .

Proof of the Assertion. Write the power series $a = \sum_{\ell \geq \nu} a_\ell$, where a_ℓ assembles the homogeneous terms of degree ℓ . We will show by induction that $a_\nu = b_1 x_1^i x_2^j x_3^k$ divides each a_ℓ . There is nothing to prove for $\ell = \nu$. Let $m > \nu$ and suppose that a_ℓ is divisible by a_ν for all $\ell = \nu, \dots, m - 1$. Let ϱ be a monomial of a_m . We have $X(\varrho) = \lambda \varrho$ for some $\lambda \in \mathbb{C}$. Considering the calculation in the above paragraph, since λ_1 has already been determined by the multidegree of a_ν , we must have $\lambda \neq \lambda_1$. On the other hand, separating all monomials of the same multidegree of ϱ in the expression $X(a) = \lambda_1 a + fa$, we have

$$X(\varrho) = \lambda_1 \varrho + \tilde{\varrho}, \tag{9}$$

where $\tilde{\varrho}$ assembles all monomials coming from fa . Notice that $\tilde{\varrho}$ can be seen as a combination of monomials of a of order smaller than m having monomials of f as coefficients. Hence, by the induction hypothesis, $\tilde{\varrho}$ is divisible by a_ν . Rewriting (9) as $\lambda \varrho = \lambda_1 \varrho + \tilde{\varrho}$, we find $(\lambda - \lambda_1)\varrho = \tilde{\varrho}$, from where we deduce that ϱ is also divisible by a_ν . We then conclude that a_ν divides a_m , proving the general step of the induction. □

Suppose that the coefficients $a, b,$ and c of ω are non-zero. By Assertion 2, we can write

$$a = x_1^{i_1} x_2^{j_1} x_3^{k_1} (b_1 + g_1), \quad b = x_1^{i_2} x_2^{j_2} x_3^{k_2} (b_2 + g_2) \quad \text{and} \quad c = x_1^{i_3} x_2^{j_3} x_3^{k_3} (b_3 + g_3),$$

where $b_1, b_2, b_3 \in \mathbb{C}^*$ and $g_1, g_2, g_3 \in \hat{\mathcal{O}}_3$ are non-units. Since X is tangent to ω , we have

$$\alpha_1 b_1 x_1^{i_1+1} x_2^{j_1} x_3^{k_1} + \alpha_2 b_2 x_1^{i_2} x_2^{j_2+1} x_3^{k_2} + \alpha_3 b_3 x_1^{i_3} x_2^{j_3} x_3^{k_3+1} = 0,$$

which implies that

$$i_1 + 1 = i_2 = i_3, \quad j_1 = j_2 + 1 = j_3 \quad \text{and} \quad k_1 = k_2 = k_3 + 1.$$

Since a, b, c do not have common factors, we find directly that $i_1 = j_2 = k_3 = 0$, giving

$$i_2 = i_3 = 1, \quad j_1 = j_3 = 1 \quad \text{and} \quad k_1 = k_2 = 1.$$

Now we can write

$$\begin{aligned} \omega &= x_2 x_3 (b_1 + g_1) dx_1 + x_1 x_3 (b_2 + g_2) dx_2 + x_1 x_2 (b_3 + g_3) dx_3 \\ &= x_1 x_2 x_3 \left((b_1 + g_1) \frac{dx_1}{x_1} + (b_2 + g_2) \frac{dx_2}{x_2} + (b_3 + g_3) \frac{dx_3}{x_3} \right). \end{aligned} \tag{10}$$

Dividing equation (10) by $1 + g_1/b_1$, we can rewrite, by abuse of notation,

$$\omega = b_1 x_2 x_3 dx_1 + x_1 x_3 (b_2 + g_2) dx_2 + x_1 x_2 (b_3 + g_3) dx_3. \tag{11}$$

Let us apply the relation $X(b/a) = -\mu_1 b/a$ of (8) to this writing of ω . Write

$$\frac{b}{a} = \frac{x_1}{b_1 x_2} (b_2 + g_2) = \sum_{i, k \geq 0, j \geq -1} \alpha_{ijk} x_1^i x_2^j x_3^k$$

as a sum of meromorphic monomials. Since the derivation by X preserves monomials, we have that $\alpha_1 i + \alpha_2 j + \alpha_3 k = -\mu_1$ whenever $\alpha_{ijk} \neq 0$. Again, the fact that $(\alpha_1, \alpha_2, \alpha_3)$ is free from strong resonances implies that b/a is a monomial. Thus $g_2 = 0$ and

$$\frac{b}{a} = \frac{b_2 x_1}{b_1 x_2},$$

implying $b = b_2 x_1 x_3$. In an analogous way, we can also prove that $c = b_3 x_1 x_2$. This leads to the form (II) in the statement of the proposition. The case where one of the coefficients of ω is zero, for example, $c = 0$, is treated following the same steps above, giving form (I) in the statement. \square

Proposition 5. *Let ω be a germ of integrable holomorphic 1-form at $(\mathbb{C}^3, 0)$ invariant by a vector field in $X \in \mathfrak{X}_{nr}$. Then ω is of type CH.*

Proof. We apply Proposition 4. If the dimensional type is two, it is straightforward to see that ω is simple non-degenerate and, hence, of type CH. If the dimensional type is three, then, except for a possible resonance of its residues, ω has the form of a simple non-degenerate singularity. However, these resonances can be eliminated by punctual or monoidal blow-ups (Cano 2004, Fernández Duque 2015), obtaining simple non-degenerate singularities. That is, there exists a reduction of singularities of CH-type for ω , leading us to conclude that ω is of type CH. \square

Next result exemplifies how vector fields and codimension one foliations satisfying a relation of tangency can be geometrically entwined. Before stating it, we set a definition: a germ of holomorphic vector field — or its associated one-dimensional holomorphic foliation — at $(\mathbb{C}^3, 0)$ has an *absolutely isolated* singularity at $0 \in \mathbb{C}^3$ if it admits a reduction of singularities having only punctual blow-ups. We call the corresponding composition of blow-up maps an *absolutely isolated* reduction of singularities.

Theorem 6. *Let \mathcal{F} be a germ of one-dimensional holomorphic foliation at $(\mathbb{C}^3, 0)$ admitting a non-dicritical absolutely isolated reduction of singularities whose associated final models are all strongly non-resonant. If \mathcal{G} is a germ of foliation of codimension one invariant by \mathcal{F} , then \mathcal{G} is of type CH.*

Proof. The proof goes by induction on n , the minimal length of all absolutely isolated reductions of singularities for \mathcal{F} as in the theorem's assertion. If $n = 0$ the result follows from Proposition 5. Suppose then that $n > 0$ and that the result is true for one-dimensional foliations having non-dicritical absolutely isolated reductions of singularities with strongly non-resonant final models of length less than n . Denote by $\pi : (M, E) \rightarrow (\mathbb{C}^3, 0)$ the first punctual blow-up of the corresponding reduction of singularities of \mathcal{F} . If $\mathcal{G}_1 = \pi^*\mathcal{G}$ were non-singular over the divisor $E = \pi^{-1}(0) \simeq \mathbb{P}^2$, then $\text{Sing}(\mathcal{G})$ would be an isolated singularity at $0 \in \mathbb{C}^3$. As a consequence of Malgrange's Theorem (Malgrange 1976), in this case \mathcal{G} would have a holomorphic first integral, being of type CH. We can then suppose that $\text{Sing}(\mathcal{G}_1) \cap E \neq \emptyset$ and pick $p \in \text{Sing}(\mathcal{G}_1) \cap E$. If $p \in \text{Sing}(\mathcal{F}_1)$, where $\mathcal{F}_1 = \pi^*\mathcal{F}$, then, by the induction hypothesis, we must have that \mathcal{G}_1 is of CH-type at p . Suppose then that p is regular for \mathcal{F}_1 . In this case, since \mathcal{F}_1 is tangent to \mathcal{G}_1 , the foliation \mathcal{G}_1 has dimensional type two at p and the leaf of \mathcal{F}_1 at p is a curve contained in the one-dimensional analytic set $\text{Sing}(\mathcal{G}_1)$. Since E is invariant by \mathcal{F}_1 , the component of $\text{Sing}(\mathcal{G}_1)$ containing this leaf is contained in E and, hence, it is an algebraic curve in $E \simeq \mathbb{P}^2$, that we denote by γ . Now, the sum of Camacho-Sad indices of $\mathcal{F}_1|_E$ along γ is the self-intersection number $\gamma \cdot \gamma > 0$ (Camacho & Sad 1982, Suwa 1995). This assures the existence of a singularity $q \in \text{Sing}(\mathcal{F}_1|_E)$, which is obviously a singularity of \mathcal{F}_1 . By the induction hypothesis, q is of CH-type for \mathcal{G}_1 . In view of this, the transversal model of \mathcal{G}_1 along (the generic point of) γ is of CH-type, leading to the conclusion that \mathcal{G}_1 is of type CH at p . We have found that each singularity of \mathcal{G}_1 over E is of type CH, admitting a CH-reduction of singularities. This means that \mathcal{G} itself has a CH-reduction of singularities, being a foliation of type CH. \square

INTEGRABLE PENCILS OF 1-FORMS

The goal of this section is to characterize the situation in which a germ of holomorphic vector field at $(\mathbb{C}^3, 0)$ is tangent to three independent holomorphic foliations. We show that, when this happens, the vector field is tangent to infinitely many foliations and it leaves invariant a germ of analytic surface. To this end, we work with the notion of pencil of integrable 1-forms or pencil of foliations. We will formulate our results in the broader context of holomorphic foliations of codimension one at $(\mathbb{C}^n, 0)$, $n \geq 3$, that leave invariant foliations of codimension two.

We start with a definition. Let ω_1 and ω_2 be independent germs of holomorphic 1-forms at $(\mathbb{C}^n, 0)$, that is, such that $\omega_1 \wedge \omega_2 \neq 0$. The *pencil of 1-forms* with *generators* ω_1 and ω_2 is the linear subspace $\mathcal{P} = \mathcal{P}(\omega_1, \omega_2)$ of the complex vector space of germs of holomorphic 1-forms at $(\mathbb{C}^n, 0)$ formed by all

1-forms $\omega_{(a,b)} = a\omega_1 + b\omega_2$, where $a, b \in \mathbb{C}$. We have the following lemma, which is a local version of (Mol 2011, Lem. 2):

Lemma 7. *If the independent germs of holomorphic 1-forms ω_1 and ω_2 do not have common components of codimension one in their singular sets, then the 1-forms $a\omega_1 + b\omega_2 \in \mathcal{P} = \mathcal{P}(\omega_1, \omega_2)$ have singular sets of codimension at least two, except possibly for a finite number of values $(a : b) \in \mathbb{P}^1$.*

Proof. Suppose, by contradiction, that the result is false. Then, for infinitely many values of $t \in \mathbb{C}$, the 1-form $\omega_1 + t\omega_2 \in \mathcal{P}$ has some component of codimension one in its singular set, defined by and irreducible $g_t \in \mathcal{O}_n$. Let us consider these values of t . Writing $\omega_1 = \sum_{i=1}^n A_i dx_i$ and $\omega_2 = \sum_{i=1}^n B_i dx_i$, where $A_i, B_i \in \mathcal{O}_n$, we have that, for each pair i, j , with $1 \leq i < j \leq n$, both $A_i + tB_i = 0$ and $A_j + tB_j = 0$ are zero over $g_t = 0$, implying that $A_i B_j - A_j B_i = 0$ over this same set. However, the fact that $\text{Sing}(\omega_1)$ and $\text{Sing}(\omega_2)$ do not have a common component of codimension one implies that independent functions g_t are associated with different values of t . Hence $A_i B_j - A_j B_i \equiv 0$ and, consequently, $A_i/B_i \equiv A_j/B_j$ for each pair i, j . By setting $\Phi = A_i/B_i$ – which is independent of the chosen i – we have a germ of meromorphic function at $(\mathbb{C}^n, 0)$ such that $\omega_1 = \Phi\omega_2$. This contradicts the fact that ω_1 and ω_2 are independent 1-forms. \square

Consider a pair of germs of holomorphic 1-forms ω_1 and ω_2 as in the lemma. We say that $\mathcal{P} = \mathcal{P}(\omega_1, \omega_2)$ is a *pencil of integrable 1-forms* or an *integrable pencil* if all its elements are integrable 1-forms, that is, $\omega \wedge d\omega = 0$ for all $\omega \in \mathcal{P}$. This is equivalent to the following fact, which will be referred to as *pencil condition*:

$$\omega_1 \wedge d\omega_2 + \omega_2 \wedge d\omega_1 = 0. \tag{12}$$

Observe that, after possibly cancelling codimension one components in the singular set, we associate with each $\omega_{(a,b)} \in \mathcal{P}$ a germ of singular holomorphic foliation \mathcal{F}_t , where $t = (a : b) \in \mathbb{P}^1$. For this reason, we also treat this object as *pencil of holomorphic foliations*. The 2-form $\omega_1 \wedge \omega_2$ is also integrable, defining, after cancelling singular components of codimension one, a singular holomorphic foliation of codimension two which is tangent to all foliations (associated with 1-forms) in $\mathcal{P} = \mathcal{P}(\omega_1, \omega_2)$. This codimension two foliation is called *axis of \mathcal{P}* .

Example 8. (Logarithmic 1-forms) Take independent irreducible germs of functions $f_1, \dots, f_k \in \mathcal{O}_n$, where $k \geq 2$. Consider also $(\lambda_1, \dots, \lambda_k)$ and (μ_1, \dots, μ_k) two \mathbb{C} -linearly independent k -uples of numbers in \mathbb{C}^* . Then, the holomorphic 1-forms

$$\omega_1 = f_1 \cdots f_k \left(\lambda_1 \frac{df_1}{f_1} + \cdots + \lambda_k \frac{df_k}{f_k} \right) \quad \text{and} \quad \omega_2 = f_1 \cdots f_k \left(\mu_1 \frac{df_1}{f_1} + \cdots + \mu_k \frac{df_k}{f_k} \right)$$

are generators of a pencil of integrable 1-forms. The axis foliation is defined by the 2-form

$$\frac{1}{f_1 \cdots f_k} \omega_1 \wedge \omega_2 = \sum_{1 \leq i < j \leq k} (\lambda_i \mu_j - \lambda_j \mu_i) h_{ij} df_i \wedge df_j,$$

where $h_{ij} = f_1 \cdots \widehat{f_i} \cdots \widehat{f_j} \cdots f_k$ is the product of all germs f_ℓ with the exception of f_i and f_j . Remark that the germs of analytic hypersurfaces $\{f_i = 0\}$ are invariant by all foliations in the integrable pencil, as well as by the axis foliation.

Recall that the integrability of a holomorphic 1-form ω at $(\mathbb{C}^n, 0)$ is equivalent to the following fact: there exists a meromorphic 1-form ϑ such that $d\omega = \vartheta \wedge \omega$. The sufficiency of this condition is clear. To prove its necessity, it is enough to take a meromorphic vector field Y such that $i_Y\omega = 1$, contract by Y both sides of the integrability condition $\omega \wedge d\omega = 0$ and take $\vartheta = -i_Y d\omega$. Now, if $\mathcal{P} = \mathcal{P}(\omega_1, \omega_2)$ is an integrable pencil, we get meromorphic 1-forms ϑ_1 and ϑ_2 satisfying $d\omega_1 = \vartheta_1 \wedge \omega_1$ and $d\omega_2 = \vartheta_2 \wedge \omega_2$. This, inserted in the pencil condition (12), becomes $(\vartheta_1 - \vartheta_2) \wedge \omega_1 \wedge \omega_2 = 0$. Then, we find germs of meromorphic functions g_1, g_2 at $(\mathbb{C}^n, 0)$ such that $\vartheta_1 - \vartheta_2 = g_1\omega_1 - g_2\omega_2$ (see Proposition 9 below). If we define $\vartheta = \vartheta_1 - g_1\omega_1 = \vartheta_2 - g_2\omega_2$, it is easy to see that

$$d\omega = \vartheta \wedge \omega \quad \forall \omega \in \mathcal{P}. \tag{13}$$

The meromorphic 1-form ϑ is uniquely defined by equation (13). Its exterior derivative $d\vartheta$ is called *pencil curvature*, denoted by $k(\mathcal{P})$.

Before proceeding, we present the following result:

Proposition 9. *Let ω_1, ω_2 and ω_3 be independent germs of holomorphic 1-forms at $(\mathbb{C}^n, 0)$, $n \geq 3$. Suppose that there exists a non-zero holomorphic 2-form η , locally decomposable outside its singular set, that is tangent to each of these three 1-forms, i.e., $\eta \wedge \omega_i = 0$ for $i = 1, 2, 3$. Then there are germs of meromorphic functions λ_1 and λ_2 at $(\mathbb{C}^n, 0)$ such that $\omega_3 = \lambda_1\omega_1 + \lambda_2\omega_2$.*

Proof. Denote by $T_{ij} = \text{Tang}(\omega_i, \omega_j) = \{\omega_i \wedge \omega_j = 0\}$ the set of tangencies between ω_i and ω_j . Note that T_{ij} contains $\text{Sing}(\omega_i) \cup \text{Sing}(\omega_j)$. Consider the analytic set $S = \text{Sing}(\eta) \cup T_{12} \cup T_{13} \cup T_{23}$. In a small neighborhood of $0 \in \mathbb{C}^n$, for each $p \notin S$, $\omega_1(p), \omega_2(p)$ and $\omega_3(p)$ define hyperplanes which are pairwise transversal and contain the subspace of codimension two defined by $\eta(p)$ (η is locally decomposable). Then, by elementary linear algebra, for each p outside S , we can write $\omega_3 = \lambda_1\omega_1 + \lambda_2\omega_2$, for some uniquely defined $\lambda_1, \lambda_2 \in \mathbb{C}$. We thus have functions λ_1 and λ_2 defined outside S . Wedging the above expression by ω_2 , we find $\omega_3 \wedge \omega_2 = \lambda_1\omega_1 \wedge \omega_2$. Hence, λ_1 can also be obtained as a quotient between a coefficient of $\omega_3 \wedge \omega_2$ and the corresponding coefficient of $\omega_1 \wedge \omega_2$. This shows that it has a meromorphic extension to a neighborhood of $0 \in \mathbb{C}^n$, still denoted by λ_1 . The same reasoning applies to λ_2 . By analytic continuation, the relation $\omega_3 = \lambda_1\omega_1 + \lambda_2\omega_2$ holds in a neighborhood of $0 \in \mathbb{C}^n$, proving the proposition. \square

Next, in the framework of the previous result, we add integrability as an ingredient. We obtain that if a distribution of codimension two is tangent to three independent foliations, then it is tangent to infinitely many foliations that are in a pencil. More precisely, we have:

Proposition 10. *Let ω_1, ω_2 and ω_3 be independent germs of integrable 1-forms at $(\mathbb{C}^n, 0)$, $n \geq 3$, with singular sets of codimension at least two. Suppose that there exists a non-zero holomorphic 2-form η , locally decomposable outside its singular set, that is tangent to each ω_i , for $i = 1, 2, 3$. Then ω_1, ω_2 and ω_3 define foliations that are in a pencil. Furthermore, η is integrable, defining the axis foliation of this pencil.*

Proof. We start by applying Proposition 9, finding that $\omega_3 = \lambda_1\omega_1 + \lambda_2\omega_2$, where λ_1 and λ_2 are germs of meromorphic functions in $(\mathbb{C}^n, 0)$. Write $\lambda_i = \psi_i/\varphi_i$, $i = 1, 2$, with $\psi_i, \varphi_i \in \mathcal{O}_n$ without common factors. Let $\varphi = \text{lcm}(\varphi_1, \varphi_2)$, where lcm denotes the least common multiple. We then have

$$\varphi\omega_3 = \varphi\lambda_1\omega_1 + \varphi\lambda_2\omega_2.$$

Writing

$$\eta_1 = \varphi\lambda_1\omega_1 = \frac{\varphi}{\varphi_1}\psi_1\omega_1, \quad \eta_2 = \varphi\lambda_2\omega_2 = \frac{\varphi}{\varphi_2}\psi_2\omega_2 \quad \text{and} \quad \eta_3 = \varphi\omega_3,$$

we have three integrable 1-forms, defining the same foliations as ω_1 , ω_2 and ω_3 , satisfying

$$\eta_3 = \eta_1 + \eta_2, \tag{14}$$

so that the pencil condition (12) holds for η_1 and η_2 . Thus, $\mathcal{P}(\eta_1, \eta_2)$ will be an integrable pencil of 1-forms if its generic element has a singular set of codimension at least two. This will follow directly from Lemma 7 if we prove that $\text{Sing}(\eta_1)$ and $\text{Sing}(\eta_2)$ do not have common components of codimension one. Indeed, looking at (14), the possible common irreducible components of codimension one of $\text{Sing}(\eta_1)$ and $\text{Sing}(\eta_2)$ are also components of $\text{Sing}(\eta_3)$. Fix an equation for such a component. It must be a factor of φ , since $\text{codim Sing}(\omega_3) \geq 2$. By definition of least common multiple, it cannot be a factor of both φ/φ_1 and φ/φ_2 . Suppose, for instance, that it is not a factor of φ/φ_1 . Then it is evidently a factor of φ_1 and, since it defines a component of zeroes of η_1 , it must be also a factor of ψ_1 . This gives a contradiction, since ψ_1 and φ_1 were chosen without common factors. Finally, the distributions of codimension two subspaces induced by η and by the integrable 2-form $\eta_1 \wedge \eta_2$ coincide outside $\text{Sing}(\eta) \cup \text{Tang}(\eta_1, \eta_2)$, giving the last part of the proposition’s statement. \square

In the sequel we present a characterization of pencils of integrable 1-forms at $(\mathbb{C}^n, 0)$. It is a local version of a result by D. Cerveau on pencils of foliations in \mathbb{P}^3 (Cerveau 2002). The proof given here essentially follows the same arguments, adapting them to the local setting.

Theorem 11. *Let \mathcal{P} be a pencil of integrable 1-forms at $(\mathbb{C}^n, 0)$, $n \geq 3$. Then, at least one of the following conditions is satisfied:*

- (a) *There exists a closed meromorphic 1-form ϑ such that $d\omega = \vartheta \wedge \omega$ for every 1-form $\omega \in \mathcal{P}$. When ϑ is holomorphic, all foliations in \mathcal{P} admit holomorphic first integrals.*
- (b) *The axis foliation of \mathcal{P} is tangent to the levels of a non-constant meromorphic function.*

In particular, there exists a germ of hypersurface at $(\mathbb{C}^n, 0)$ that is tangent to the axis foliation of \mathcal{P} .

Proof. The two cases in the assertion correspond to the pencil curvature $k(\mathcal{P})$ being zero or non-zero.

Case 1: $k(\mathcal{P}) = 0$, that is, the 1-form ϑ in (13) is closed. In the purely meromorphic case, we can write, from (Cerveau & Mattei 1982),

$$\vartheta = \sum_{i=1}^k \lambda_i \frac{df_i}{f_i} + d \left(\frac{h}{f_1^{n_1} \cdots f_k^{n_k}} \right),$$

where $f_i \in \mathcal{O}_n$ are irreducible equations of the components of the polar set of ϑ , $h \in \mathcal{O}_n$, $\lambda_i \in \mathbb{C}$, $n_i \in \mathbb{N}$ for $i = 1, \dots, k$, with $\lambda_i = 0$ only if $n_i > 0$. Condition (13) then says that each hypersurface $f_i = 0$ is invariant by every $\omega \in \mathcal{P}$ and, hence, also by the axis of \mathcal{P} .

Suppose, on the other hand, that ϑ holomorphic. Since $d\vartheta = 0$, there exists $h \in \mathcal{O}_n$ such that $\vartheta = dh$ and hence $d\omega = dh \wedge \omega$ for every $\omega \in \mathcal{P}$. Setting $h' = \exp(h) \in \mathcal{O}_n^*$, we have $dh'/h' = dh$ and thus $d\omega = (dh'/h') \wedge \omega$. Then $d(\omega/h') = 0$, implying that there exists $f \in \mathcal{O}_n$ such that $\omega/h' = df$. Hence, each foliation in \mathcal{P} has a holomorphic first integral, and this implies that the axis of \mathcal{P} is

completely integrable, that is, it has two independent holomorphic first integrals. In particular, the axis foliation leaves invariant germs of analytic hypersurfaces.

Case 2: $k(\mathcal{P}) \neq 0$. Let ω_1 and ω_2 be generators of the pencil. We have the following:

Assertion. *There exists a germ of meromorphic function α at $(\mathbb{C}^n, 0)$ such that*

$$d\vartheta = \alpha\omega_1 \wedge \omega_2. \tag{15}$$

Proof of the Assertion. First note that, taking differentials in both sides of $d\omega_j = \vartheta \wedge \omega_j$, we have $d\vartheta \wedge \omega_i = 0$ for $i = 1, 2$. Now, let q be a point near $0 \in \mathbb{C}^n$ such that $(\omega_1 \wedge \omega_2)(q) \neq 0$. This means that ω_1 and ω_2 are non-singular and linearly independent at q , so that we can find analytic coordinates (x_1, x_2, \dots, x_n) at q such that $\omega_1 = A_1 dx_1$ and $\omega_2 = A_2 dx_2$, where A_1, A_2 are invertible germs of holomorphic functions at q . Write $d\vartheta = B_{12} dx_1 \wedge dx_2 + \sum_{(i,j) \neq (1,2)} B_{ij} dx_i \wedge dx_j$, where each B_{ij} is a germ of the meromorphic function at q . Since $d\vartheta \wedge \omega_1 = 0$ and $d\vartheta \wedge \omega_2 = 0$, we must have $B_{ij} = 0$ whenever $(i, j) \neq (1, 2)$. Then, at q ,

$$d\vartheta = B_{12} dx_1 \wedge dx_2 = \frac{B_{12}}{A_1 A_2} (A_1 dx_1) \wedge (A_2 dx_2) = \alpha\omega_1 \wedge \omega_2,$$

where α is a germ of the meromorphic function at q . In this way, we produce a meromorphic function α , defined outside the set of tangencies $\text{Tang}(\omega_1, \omega_2)$, which, by comparing coefficients of $d\vartheta$ and $\omega_1 \wedge \omega_2$ as we did in the proof of Proposition 9, can be extended to a meromorphic function defined in a neighborhood of $0 \in \mathbb{C}^n$. This proves the assertion. \square

Now, we split our analysis in two subcases:

Subcase 2.1: α is constant. We claim that the axis foliation of \mathcal{P} is tangent to the levels of a meromorphic (possibly holomorphic) function. Indeed, $\alpha \neq 0$, since $k(\mathcal{P}) \neq 0$, so that, from the exterior derivative of (15), we get that $d\omega_1 \wedge \omega_2 - \omega_1 \wedge d\omega_2 = 0$. This, together with the pencil condition (12), gives

$$\omega_2 \wedge d\omega_1 = \omega_1 \wedge d\omega_2 = 0.$$

Hence, using (13), we find that $\vartheta \wedge \omega_1 \wedge \omega_2 = 0$. Applying Proposition 9 to the 1-forms ω_1, ω_2 and $\omega_3 = \vartheta$ and to the 2-form $\eta = \omega_1 \wedge \omega_2$, we find germs of meromorphic functions μ_1 and μ_2 at $(\mathbb{C}^n, 0)$ satisfying

$$\vartheta = \mu_1 \omega_1 + \mu_2 \omega_2.$$

Inserting this in (13), we get

$$d\omega_1 = -\mu_2 \omega_1 \wedge \omega_2 \quad \text{and} \quad d\omega_2 = \mu_1 \omega_1 \wedge \omega_2.$$

If $\mu_1 = 0$ then $d\omega_2 = 0$, and hence $\omega_2 = d(g)$ for some $g \in \mathcal{O}_n$, which turns out to be a holomorphic first integral for the axis of \mathcal{P} . On the other hand, when $\mu_1 \neq 0$, the above equations give $d\omega_1 = -(\mu_2/\mu_1)d\omega_2$, which, by differentiation, yields

$$d(\mu_2/\mu_1) \wedge \omega_1 \wedge \omega_2 = 0.$$

When μ_2/μ_1 is non-constant, we have at once that μ_2/μ_1 is a meromorphic first integral for the axis of \mathcal{P} . When $\mu_2/\mu_1 = c$ for some $c \in \mathbb{C}$, we have $d\omega_1 = -cd\omega_2$. That is to say, $\omega_1 + c\omega_2$ is a 1-form

in \mathcal{P} which is closed, and hence exact, yielding once again a holomorphic first integral for the axis foliation.

Subcase 2.2: α is non-constant. Taking the exterior derivative of (15) and using (13), we obtain

$$\left(\frac{d\alpha}{2\alpha} + \vartheta\right) \wedge \omega_1 \wedge \omega_2 = 0. \tag{16}$$

Now, applying Proposition 9 to $\omega_1, \omega_2, \omega_3 = d\alpha/2\alpha + \vartheta$ and $\eta = \omega_1 \wedge \omega_2$, we can find k_1 and k_2 , germs of meromorphic functions at $(\mathbb{C}^n, 0)$, such that

$$\frac{d\alpha}{2\alpha} + \vartheta = k_1\omega_1 + k_2\omega_2. \tag{17}$$

Observe that, since $k(\mathcal{P}) \neq 0$, we have that either k_1 or k_2 is non-zero. Taking the exterior derivative and applying (13), we obtain

$$d\vartheta = (k_1\vartheta + dk_1) \wedge \omega_1 + (k_2\vartheta + dk_2) \wedge \omega_2.$$

This, wedged by ω_1 and ω_2 , gives, respectively,

$$\left(\vartheta + \frac{dk_2}{k_2}\right) \wedge \omega_1 \wedge \omega_2 = 0 \quad \text{and} \quad \left(\vartheta + \frac{dk_1}{k_1}\right) \wedge \omega_1 \wedge \omega_2 = 0. \tag{18}$$

Subtracting (16), we obtain, respectively,

$$\left(-\frac{1}{2} \frac{d\alpha}{\alpha} + \frac{dk_2}{k_2}\right) \wedge \omega_1 \wedge \omega_2 = 0 \quad \text{and} \quad \left(-\frac{1}{2} \frac{d\alpha}{\alpha} + \frac{dk_1}{k_1}\right) \wedge \omega_1 \wedge \omega_2 = 0. \tag{19}$$

This allows us to conclude that the meromorphic functions k_1^2/α and k_2^2/α are constant on the leaves of the axis foliation of \mathcal{P} . If one of these two functions is non-constant, we have a meromorphic first integral for the axis foliation and the proof of the theorem is accomplished. We claim that this actually happens. Indeed, suppose, by contradiction, that both k_1^2/α and k_2^2/α are constant. This would imply that k_1/k_2 (if $k_2 \neq 0$) is also constant. Writing $k_1/k_2 = c_1 \in \mathbb{C}$, we get, from (17),

$$\frac{d\alpha}{2\alpha} + \vartheta = k_2(c_1\omega_1 + \omega_2) \tag{20}$$

and, since k_2^2/α is constant, we find that

$$\vartheta = -\frac{dk_2}{k_2} + k_2(c_1\omega_1 + \omega_2). \tag{21}$$

Then, applying (13) to $c_1\omega_1 + \omega_2 \in \mathcal{P}$, we find

$$d(c_1\omega_1 + \omega_2) = -\frac{dk_2}{k_2} \wedge (c_1\omega_1 + \omega_2). \tag{22}$$

This gives that $k_2(c_1\omega_1 + \omega_2)$ is closed. By (21), we would have $k(\mathcal{P}) = d\vartheta = 0$, reaching a contradiction. □

Let us put the previous discussion in our initial three-dimensional context:

Theorem 12. *Let X be a germ of holomorphic vector field at $(\mathbb{C}^3, 0)$ tangent to three independent foliations of codimension one. Then there exists an integrable pencil \mathcal{P} such that X is tangent to all foliations in \mathcal{P} . Furthermore, at least one of the following two conditions holds:*

- i) there exists a closed meromorphic 1-form ϑ such that $d\omega = \vartheta \wedge \omega$ for every $\omega \in \mathcal{P}$;*
- ii) X has a non-constant meromorphic (possibly holomorphic) first integral.*

In particular, there exists a germ of analytic surface at $0 \in \mathbb{C}^3$ which is invariant by X .

We apply the theorem in the following:

Example 13. (Jouanolou's Example) Consider, for $m \geq 2$, the vector field $X = x_3^m \partial / \partial x_1 + x_1^m \partial / \partial x_2 + x_2^m \partial / \partial x_3$. Then $\omega = i_{Rj} \Omega$, where $\Omega = dx_1 \wedge dx_2 \wedge dx_3$ is the volume form and $R = x_1 \partial / \partial x_1 + x_2 \partial / \partial x_2 + x_3 \partial / \partial x_3$ is the radial vector field, is an integrable 1-form, with homogeneous coefficients, that is invariant by X . Since $i_R \omega = 0$, ω defines a foliation on the complex projective plane $\mathbb{P}_{\mathbb{C}}^2$ which leaves invariant no algebraic curve (Jouanolou 1979). This is equivalent to saying that, at $0 \in \mathbb{C}^3$, the vector field X leaves invariant no homogeneous surface — that is, a surface defined by the vanishing of a homogeneous polynomial in the coordinates (x_1, x_2, x_3) . Since X itself is a homogeneous vector field, this implies that X is not tangent to any germ of analytic surface at $0 \in \mathbb{C}^3$. By Theorem 12, X cannot be tangent to three independent foliations.

This example illustrates how Theorem 12 can be used to identify vector fields that are not tangent to three independent foliations. As remarked in the Introduction, the existence of a separatrix and of a desingularization are necessary conditions for a vector field to be tangent to (at least) one foliation. It would be interesting to specify conditions that characterize the tangency to (exactly) one or two foliations.

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