Surfaces in E^3 Invariant under a One Parameter Group of Isometries of E^3

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ABSTRACT

We develop a convenient surface theory in $\boldsymbol{E^3}$ in order to apply it to the class of the surfaces invariant under a one-parameter group of isometries of $\boldsymbol{E^3}$. In this way we derive intrinsic characterizations along with several results of subclasses of this class of surfaces that satisfy certain preassigned properties. In the process all results are also effortlessly derived. Among these subclasses are those with surfaces; of constant mean curvature, of constant Gaussian curvature, isothermic, with constant difference or ratio of the principal curvatures.

Key Words: Invariant, parameter, group, isometry, mean, Gaussian, principal, curvature.

INTRODUCTION

In this part we develop some general surface theory in E^3 to be applied to the surfaces invariant under a one parameter group of isometries of E^3, that is, generalized cylinders, surfaces of revolution and helicoidal surfaces. This theory develops some very useful results of surface theory in an easy and straightforward manner, some of which can apply to any surface and not necessarily to surfaces invariant under a one parameter group of isometries of E^3. Furthermore, we reach results concerning the class of surfaces with constant mean curvature.

When this theory is applied to the class of surfaces invariant under a one parameter group of isometries of E^3 we derive very easily some older known and many new results. Among the new results we distinguish: (1) Intrinsic characterization of those surfaces in this class with the difference of the principal curvatures constant. (2) Intrinsic characterization of the isothermic helicoidal surfaces. (3) New intrinsic characterization of the surfaces in this class with constant mean curvature. (4) Various interesting ordinary differential equations study worthy in their own sake.

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The study and the solutions of the ordinary differential equations involved can change the intrinsic results into explicit ones. These differential equations are very hard to crack down. They are of second order and highly non-linear. Even though they can easily be reduced to first order differential equations the integrals of these first order differential equations are far from being elementary. In the case of minimal surfaces the Gauss equation can be easily integrated and yields all known results in a very nice and straight way. Also, when the mean curvature is a non-zero constant we can integrate the same equation by making use of elliptic integrals and thus obtain an intrinsic characterization and some old and new results.

Finally, we examine the flat helicoidal surfaces with non-constant mean curvature (the ones with constant mean curvature being the right circular cylinders). These are exactly the tangential developable surfaces of circular helices. Apart from their “usual” parameters we find the principal and the natural parameters. Next, we examine the helicoidal surfaces with non-zero constant Gaussian curvature and the surfaces of revolution with constant Gaussian curvature. Last of all, we examine the helicoidal surfaces and the surfaces of revolution with ratio of principal curvatures constant. In all of the above cases we try to find all fundamental quantities of the surfaces as explicitly as the equations allow in a certain coordinate system that we call natural coordinates.

1. SOME GENERAL SURFACE THEORY IN E^3

A) We consider a surface M^2 in E^3 connected, oriented and of sufficient smoothness. We assume that over M^2 there is a well-defined field of orthonormal frames x, j_1, j_2, j_3 such that $x \in M^2$, $\{j_1, j_2\}$ is an orthonormal basis of the tangent plane of M^2 at x and j_3 is the unit normal vector to M^2 at x. On M^2 we consider the well known forms $\eta_i = dx \cdot j_i$, $i = 1, 2, 3$ which are the dual forms to the vectors j_1, j_2, j_3 and for $k, l = 1, 2, 3$ we define $\eta_{kl} = dj_k \cdot j_l$ which are the connection one-forms on M^2. All these forms satisfy the following well-known equations on M^2:

$$\begin{aligned}
&\eta_3 = 0 \quad \text{(since } j_3 \text{ is the unit normal vector to } M^2\text{)}\\
&\eta_{kl} = -\eta_{lk} \quad (\text{so, } \eta_{kk} = 0), \quad k, l = 1, 2, 3\\
&dx = \eta_1 j_1 + \eta_2 j_2\\
&dj_1 = \eta_{12} j_2 + \eta_{13} j_3\\
&dj_2 = \eta_{21} j_1 + \eta_{23} j_3\\
&dj_3 = \eta_{31} j_1 + \eta_{32} j_2
\end{aligned} \tag{1}$$

$$d\eta_k = \sum_{l=1}^{3} \eta_{kl} \wedge \eta_l, \quad k = 1, 2, 3 \quad \text{(first structural equation)} \tag{2}$$

$$d\eta_{kl} = \sum_{n=1}^{3} \eta_{kn} \wedge \eta_{nl}, \quad k, l = 1, 2, 3 \quad \text{(second structural equation)} \tag{3}$$

($\wedge$ is the symbol of the wedge product of forms and d is the exterior derivative of forms.)

Since $\{\eta_1, \eta_2\}$ is a basis of forms on M^2, for the connection form η_{12} we can write

$$\eta_{12} = p\eta_1 + q\eta_2 \tag{4}$$

for some smooth functions p, q. Then by (2) we get that

$$\begin{aligned} d\eta_1 &= p\eta_1 \wedge \eta_2 \\ d\eta_2 &= q\eta_1 \wedge \eta_2 \end{aligned} \tag{5}$$

Since $\eta_3 = 0$ on M^2 then $d\eta_3 = 0$ on M^2 and by (2) we have that $\eta_{13} \wedge \eta_1 + \eta_{23} \wedge \eta_2 = 0$. So, by Cartan's Lemma we can write that

$$\begin{aligned} \eta_{13} &= \alpha\eta_1 + \beta\eta_2 \\ \eta_{23} &= \beta\eta_1 + \gamma\eta_2 \end{aligned} \tag{6}$$

for some smooth functions α, β, γ.

We also have the following equations:

$$d\eta_{12} = -K\eta_1 \wedge \eta_2 \quad \text{(Gauss Equation)} \tag{GE}$$

$$\left.\begin{aligned} d\eta_{13} &= \eta_{12} \wedge \eta_{23} = -\beta d\eta_2 + \gamma d\eta_1 \\ d\eta_{23} &= \eta_{21} \wedge \eta_{13} \quad = \alpha d\eta_2 - \beta d\eta_1 \end{aligned}\right\} \quad \text{(Codazzi-Mainardi Equations)} \tag{CME}$$

The mean and Gaussian curvatures of M^2 are respectively

$$\begin{aligned} H &= \frac{1}{2}(\alpha + \gamma) \\ K &= \alpha\gamma - \beta^2 \end{aligned}$$

We will use the Hodge operator $*$ which rotates the frames of the tangent and cotangent space of M^2 by $\dfrac{\pi}{2}$. So, acting on the one forms we have that

$$*\eta_1 = \eta_2, \quad *\eta_2 = -\eta_1, \quad *^2 = -1$$

B) We will need the following result. Suppose that M^2 has orthogonal parameters (s, t) such that the first fundamental form is given by

$$I = \eta_1^2 + \eta_2^2 = Eds^2 + Gdt^2 \quad (E > 0,\ G > 0)$$

These parameters become isothermal, that is, we can rewrite I as

$$I = \lambda(x, y)(dx^2 + dy^2), \quad (\lambda > 0)$$

if and only if

$$\frac{\partial^2}{\partial s \partial t} \ln(\frac{E}{G}) = 0$$

(see Eisenhart 1909, Stephanidis 1987).

With $\eta_1 = \sqrt{E}ds$, $\eta_2 = \sqrt{G}dt$ and η_{12} the corresponding connection form, then this condition in terms of forms is easily proven to be equivalent to

$$d * \eta_{12} = 0 \tag{7}$$

(see Stephanidis 1987).

C) Now, we consider another typical field of frames on M^2, $x, e_1, e_2, e_3 = j_3$ with ω_1, ω_2 the coframe corresponding to e_1, e_2 and ω_{kl}, $k, l = 1, 2, 3$ the corresponding connection forms. We take $\{e_1, e_2\}$ to have the same orientation with $\{j_1, j_2\}$. Then we can find a branch of the angle ψ from e_1 to j_1, so that we can write

$$\begin{aligned} j_1 &= \cos\psi e_1 + \sin\psi e_2 \\ j_2 &= -\sin\psi e_1 + \cos\psi e_2 \end{aligned} \tag{8}$$

$$\begin{aligned} \eta_1 &= \cos\psi \omega_1 + \sin\psi \omega_2 \\ \eta_2 &= -\sin\psi \omega_1 + \cos\psi \omega_2 \end{aligned} \tag{9}$$

$$\begin{aligned} \eta_{13} &= \alpha\eta_1 + \beta\eta_2 = \cos\psi \omega_{13} + \sin\psi \omega_{23} \\ \eta_{23} &= \beta\eta_1 + \gamma\eta_2 = -\sin\psi \omega_{13} + \cos\psi \omega_{23} \end{aligned} \tag{10}$$

The forms η_{12} and ω_{12} are related by

$$\eta_{12} = d\psi + \omega_{12} \tag{11}$$

(The relations (10) and (11) follow from the previous equations by straightforward calculation.) Therefore, $d * \eta_{12} = d * d\psi + d * \omega_{12}$. We know that $d * d\psi = \Delta_2\psi \cdot dA$ with Δ_2 the Laplace-Beltrami operator and dA the area element of M^2 $\eta_1 \wedge \eta_2 = \omega_1 \wedge \omega_2$. Hence by (7) we obtain the following useful result:

If $\{\eta_1, \eta_2\}$ is derived from isothermal coordinates, then $\{\omega_1, \omega_2\}$ is also derived from isothermal coordinates if and only if $\Delta_2\psi = 0$. Therefore, the angle ψ between any two isothermal systems is harmonic.

D) Next, we are assuming that M^2 has no umbilic points ($\Longleftrightarrow H^2 > K$) and e_1, e_2 are the principal unit vectors corresponding to principal curvatures a, c. Since M^2 is connected we may assume

$a > c$. We have

$$\omega_{13} = a\omega_1, \quad \omega_{23} = c\omega_2 \tag{12}$$

$$H = \frac{a+c}{2} \quad (= \frac{\alpha+\gamma}{2}) \quad \text{(mean curvature)} \tag{13}$$

$$K = a \cdot c \quad (= \alpha\gamma - \beta^2) \quad \text{(Gaussian curvature)} \tag{14}$$

We set

$$J = \frac{a-c}{2} = \sqrt{H^2 - K} > 0 \Longleftrightarrow K = H^2 - J^2 \tag{15}$$

From , (9), (10), (12), (13) and (15) we can compute that

$$\left.\begin{aligned} \alpha &= J\cos 2\psi + H \\ \beta &= -J\sin 2\psi \\ \gamma &= -J\cos 2\psi + H \end{aligned}\right\} \tag{16}$$

We write the differentials

$$\begin{aligned} d\psi &= \psi_1\eta_1 + \psi_2\eta_2 \\ dH &= H_1\eta_1 + H_2\eta_2 \\ dJ &= J_1\eta_1 + J_2\eta_2 \end{aligned}$$

(thus defining ψ_1, ψ_2, H_1, H_2, J_1, J_2).

From (3), (4), (5), (6) and (16) we can solve for ψ_1, ψ_2 and find

$$\begin{aligned} \psi_1 &= \frac{1}{2}\left(-\frac{J_2}{J} - \frac{H_2}{J}\cos 2\psi - \frac{H_1}{J}\sin 2\psi + 2p\right) \\ \psi_2 &= \frac{1}{2}\left(\frac{J_1}{J} + \frac{H_2}{J}\sin 2\psi - \frac{H_1}{J}\cos 2\psi + 2q\right) \end{aligned}$$

So, the differential of ψ is given by

$$d\psi = \frac{1}{2}\left(\frac{J_1}{J}\eta_2 - \frac{J_2}{J}\eta_1\right) - \frac{1}{2}\sin 2\psi\left(\frac{H_1}{J}\eta_1 - \frac{H_2}{J}\eta_2\right) - \frac{1}{2}\cos 2\psi\left(\frac{H_2}{J}\eta_1 + \frac{H_1}{J}\eta_2\right) + (p\eta_1 + q\eta_2)$$

and hence

$$d(2\psi) = -\sin 2\psi\left(\frac{H_1}{J}\eta_1 - \frac{H_2}{J}\eta_2\right) - \cos 2\psi\left(\frac{H_2}{J}\eta_1 + \frac{H_1}{J}\eta_2\right) + *d\ln J + 2\eta_{12} \tag{17}$$

This equation (17) along with $dH = H_1\eta_1 + H_2\eta_2$ consist of an equivalent form of the Codazzi-Mainardi Equations when $J \neq 0$.

E) From (17) we can easily conclude two known facts about surfaces with constant mean curvature. From (17) we get that H is constant if and only if $d(2\psi) = *dlnJ + 2\eta_{12}$.

Taking its exterior derivative and using the Gauss Equation we get

$$0 = d * d \ln J + d2\eta_{12} = (\Delta_2 \ln J - 2K)\eta_1 \wedge \eta_2$$

So, *all surfaces with constant mean curvature and without umbilic points satisfy the following partial differential equation:*

$$\boldsymbol{\Delta_2 \ln J = \Delta_2 \ln \sqrt{H^2 - K} = 2K}, \quad \text{or} \quad \boldsymbol{\Delta_2 \ln(H^2 - K) = 4K}$$

(This equation was first observed by G. Ricci. Also see Tribuzy 1980). In addition, if the surface is minimal without umbilic points then $\Delta_2 \ln(-K) = 4K$.)

Furthermore, if H is constant and j_1, j_2 are the principal vectors e_1, e_2, then $\psi = 0 \pmod{\pi}$ and (11) with (17) give

$$\omega_{12} = \eta_{12} = \frac{-1}{2} * d \ln J \Longleftrightarrow *\omega_{12} = *\eta_{12} = \frac{1}{2} d \ln J$$

(This equation is of course valid for any ψ constant when H is constant.) Taking its exterior derivative we find $d * \omega_{12} = d * \eta_{12} = 0$.

Using (7) we conclude that

All surfaces with constant mean curvature and without umbilic points have principal coordinates that can become isothermal. (By definition, the surfaces for which their principal coordinates can become isothermal are called to be *isothermic surfaces*. (See, e.g., Eisenhart 1909).

Moreover, by putting $\psi = 0$ in (17) we straightforwardly obtain the following general characterization of isothermic surfaces in E^3: *A surface in E^3 without umbilic points is isothermic if and only if*

$$d\left(\frac{H_1}{J}\omega_1 - \frac{H_2}{J}\omega_2\right) = 0$$

where here H_1 and H_2 are determined by the relation $dH = H_1\omega_1 + H_2\omega_2$. With (u, v) principal coordinates, this relation is equivalent to the following second order hyperbolic homogeneous partial differential equation for H

$$H_{uv} - (\ln \sqrt{J})_v H_u - (\ln \sqrt{J})_u H_v = 0$$

Also, if $\omega_{12} = \rho\omega_1 + \sigma\omega_2$ we obtain that the geodesic curvatures of the principal curves when $J \neq 0$ are

$$\rho = \frac{1}{2}\frac{a_2}{J} = \frac{a_2}{a - c} \quad \text{and} \quad \sigma = \frac{1}{2}\frac{c_1}{J} = \frac{c_1}{a - c}$$

Therefore, if one of the principal curvatures is constant then the corresponding principal curve is a geodesic. If both principal curvatures are constants then $\rho = \sigma = 0$ and so $\omega_{12} = 0$. Therefore, $K = a \cdot c = 0$ and hence $a = 0$ or $c = 0$. (If $a = c = 0$ then $J = 0$ and the surface is a piece of the plane, if $a \neq 0$ constant and $c = 0$ then $J = a \neq 0$ and the surface is a piece of a right circular cylinder of radius $1/|a|$ and if $a = c \neq 0$ constant then $J = 0$ and the surface is a pience of a sphere of radius $1/|a|$.) Similar manipulations of the previous formulae yield various well-known results in surface theory (e.g., Liouville's formula, $p = \psi_1 + \cos\psi\rho + \sin\psi\sigma$, see Stephanidis 1987, applications to Bonnet surfaces etc.).

Remark and Note. *Except for the planes and the spheres all surfaces with constant mean curvature have isolated umbilic points.* So, the above two results hold true over an open dense subset of the surface whose complement consists of isolated umbilic points only. The isolatedness of the umbilic points when H is constant, is a well-known fact that follows from (the analyticity of such a surface and) the holomorphicity of the Hopf quadratic differential. (See Spivak 1979, Vol. V, Ch. 10 and Hopf 1956, pp. 136-139.)

2. APPLICATION TO SURFACES INVARIANT UNDER A ONE-PARAMETER GROUP OF ISOMETRIES

A) A surface M^2 invariant under a one-parameter group of isometries of E^3, is either a generalized cylinder or a surface of revolution or a helicoidal surface. The first fundamental form for these surfaces can be written as

$$I = E(s)(ds^2 + dt^2), \quad E(s) > 0$$

where t is parameter along the orbits of the group of the isometries (straight lines, circles, helices respectively) and s is parameter along the curves perpendicular to the orbits which are geodesics (see Baikoussis & Koufogiorgos 1997, 1998, Do Carmo & Dajczer 1982, Hitt & Roussos 1991, Eisenhart 1909, Soyuçok 1995). So, (s, t) are isothermal geodesic coordinates and we call them *natural coordinates*. We now let

$$e(s) = \sqrt{E(s)}$$

$$j_1 = \frac{\frac{\partial}{\partial s}}{e(s)}, \quad j_2 = \frac{\frac{\partial}{\partial t}}{e(s)}$$

$$\eta_1 = e(s)ds, \quad \eta_2 = e(s)dt$$

Hence,

$$\eta_{12} = \frac{e'(s)}{e(s)}dt = [\ln e(s)]'dt = \frac{e'(s)}{E(s)}\eta_2 \tag{18}$$

Also, we know that $H = H(s)$, $J = J(s)$, $\psi = \psi(s)$. We then write

$$H' = \frac{dH}{ds}, \quad J' = \frac{dJ}{ds}, \quad \psi' = \frac{d\psi}{ds}$$

Whereas the generalized cylinders and the surfaces of revolution may contain umbilic points the helicoidal surfaces contain no umbilic points (see Baikoussis & Koufogiorgos 1997 1998, Hitt &

Roussos 1991, Roussos 1988ab). But, where $J > 0$ from (17) and (18) we have that

$$d(2\psi) = -\sin 2\psi \frac{H'}{J} ds - \cos 2\psi \frac{H'}{J} dt + *(\ln J)' ds + 2(\ln e(s))' dt$$

So,

$$2\psi' ds = -\sin 2\psi \frac{H'}{J} ds + [-\cos 2\psi \frac{H'}{J} + (\ln J)' + (\ln E)'] dt$$

Therefore we have

$$\begin{aligned} 2\psi' &= -\sin 2\psi \frac{H'}{J} \\ [\ln(JE)]' &= \cos 2\psi \frac{H'}{J} \end{aligned} \tag{19}$$

B) For generalized cylinders and surfaces of revolution $\psi = 0, \mod \frac{\pi}{2}$. The first equation of (19) is then satisfied identically. The second equation of (19) gives

$$[\ln(JE)]' = \pm \frac{H'}{J} \tag{20}$$

This equation is satisfied by all generalized cylinders and surfaces of revolution without umbilic points and it implies that E is a global function. Moreover, if in this situation H is constant—Delaunay Surfaces—we observe that

$$JE = B, \quad B > 0 \quad \text{constant} \tag{21}$$

The Delaunay surfaces contain no umbilic points, for otherwise the umbilic points could not be isolated. Then $J > 0$ non-constant and the Gauss Equation with (15) give

$$\frac{1}{-2E}[(\ln E)_{ss} + (\ln E)_{tt}] = K = H^2 - J^2 \tag{22}$$

Here $(\ln E)_{tt} = 0$ and by (21) and (22) we get

$$J(\ln J)'' = 2B(H^2 - J^2) \tag{23}$$

C) By means of differential equations we can intrinsically determine the generalized cylinders and the surfaces of revolution for which $J = A > 0$ constant. By (20) we have

$$(\ln E)' = \frac{\pm H'}{A}$$

So,

$$E = Be^{\frac{\pm H}{A}}, \quad B > 0 \quad \text{constant}$$

The Gauss equation (22) becomes

$$H'' = \pm 2ABe^{\mp\frac{H}{A}}(H^2 - A^2) \tag{24}$$

From this differential equation we find H and then we find E. It is of course trivially satisfied for $H = \pm A$ $(= \pm J)$ constant, in which case we have the right circular cylinders. The second fundamental forms of these surfaces are given by

$$\begin{aligned} II &= Lds^2 + 2Mdsdt + Ndt^2\ , \quad \text{where} \\ L &= E(H \pm J) \\ M &= 0 \\ N &= E(H \mp J) \end{aligned}$$

REMARK. If $J \equiv 0$ for a surface then the surface is totally umbilical (the principal curvatures are equal to one another). The totally umbilical surfaces are known to be pieces of planes and/or spheres. Also, if J is a negative constant then a change of the orientation of a connected surface changes J to be a positive constant.

D) Now assume $\psi \neq 0$, mod $\dfrac{\pi}{2}$, that is, the surface is a helicoidal one. Equations (19) become

$$\left.\begin{aligned} \frac{H'}{J} = \frac{-2\psi'}{\sin 2\psi} &= (\ln|\cot\psi|)' \\ [\ln(EJ|\sin 2\psi|)]' &= 0 \end{aligned}\right\} \tag{25}$$

Let us assume without loss of generality that $0 < \psi < \dfrac{\pi}{2}$. Then

$$\left.\begin{aligned} \frac{H'}{J} &= [\ln(\cot\psi)]' = \frac{-2\psi'}{\sin 2\psi}, \quad 0 < \psi < \frac{\pi}{2} \\ E &= \frac{C}{J\sin 2\psi}, \quad C > 0 \quad \text{constant} \end{aligned}\right\} \tag{26}$$

(In a bit more general setting we may simply assume $JC > 0$.) We observe the following geometric characterization: *"A helicoidal surface has constant mean curvature if and only if the angle between the principal curves and the helices is constant."* (See Roussos 1988 ab)

The helicoidal surfaces with constant mean curvature have been completely determined in explicit form in Do Carmo & Dajczer 1982. Some additional properties of them are in Hitt & Roussos 1991. Here, by means of (26) we give an alternative intrinsic characterization, as follows:

When H and ψ are constant with $0 < \psi < \dfrac{\pi}{2}$ the Gauss equation given by (22) transforms into an equation in J as

$$J(\ln J)'' = \frac{2C}{\sin 2\psi}(H^2 - J^2) \tag{27}$$

The solution of this equation determines J and then E by (26). For the second fundamental form we have

$$\begin{aligned} II &= Lds^2 + 2Mdsdt + Ndt^2, \quad \text{where} \\ L &= EH + EJ\cos 2\psi = EH + C\cot 2\psi \\ M &= -EJ\sin 2\psi = -C \\ N &= EH - EJ\cos 2\psi = EH - C\cot 2\psi \end{aligned}$$

We observe that the constant C is the pitch of the helicoidal motion, h in Baikoussis & Koufogiorgos 1997, Do Carmo & Dajczer 1982, Soyuçok 1995, etc.

Changing ψ by a constant, for two constant values ψ_1 and ψ_2 of ψ we consider C_1 and C_2 constants satisfying

$$\frac{C_1}{\sin 2\psi_1} = \frac{C_2}{\sin 2\psi_2} \tag{28}$$

We then obtain two helicoidal surfaces with the same E, J and H (= constant). Therefore, they are non-trivially isometric (since $\psi_1 \neq \psi_2$) with the same constant mean curvature. For $\psi_2 = 0$ or $\frac{\pi}{2}$ relation (28) forces $C_2 = 0$ and then the corresponding surface is a surface of revolution with the same constant mean curvature H, i.e., it is a Delaunay surface. This describes the periodic deformation of a helicoidal surface with constant mean curvature through helicoidal surfaces of the same constant mean curvature. Moreover, all possible helicoidal surfaces with given H constant are obtained in this way via this periodic deformation of the Delaunay surfaces. See Do Carmo & Dajczer 1982, Hitt & Roussos 1991, and for their limit surfaces with respect to the parameters involved see Sasai 1996.

E) We observe that all cylinders and all surfaces of revolution are isothermic. We notice that, by (18) $d * \eta_{12} = 0$ and η_{12} corresponds to the principal coframe. Most helicoidal surfaces are not isothermic. So, here we are going to give an intrinsic characterization of the isothermic helicoidal surfaces.

For the isothermic helicoidal surfaces we have that both the (s, t) and principal coordinate systems are isothermal. Therefore, the angle ψ between them, as explained in section 1(C) is harmonic. Since $\psi = \psi(s)$, (s, t) is isothermal system of coordinates and $\Delta_2\psi = 0$ we get that

$$\psi'' = 0 \Longleftrightarrow \psi(s) = as + b, \quad \text{a, b constants}$$

If $a = 0$ then $\psi = b$ constant. This case was developed in the previous section (D) and we have that H is constant if and only if ψ is constant. Notice of course that all surfaces of constant mean curvature and without umbilic points are isothermic, as we have proved in section 1 (E). Therefore, in this section we will assume that $a \neq 0$. Then the constant b can be geometrically eliminated by replacing s by $s - \frac{b}{a}$. Hence, without loss of generality we shall assume that

$$\psi(s) = as, \quad a \neq 0 \text{ constant} \quad (\text{and } H = H(s) \text{ non-constant})$$

Now, (26) becomes

$$\left.\begin{aligned} \frac{H'}{J} &= \frac{-2a}{\sin 2as}, \quad 0 < s < \frac{\pi}{2a} \text{ if } a > 0, \frac{\pi}{2a} < s < 0 \text{ if } a < 0 \\ E &= \frac{C}{J\sin 2as} = \frac{-2aC}{H'\sin^2 2as} = -\gamma\frac{4a^2}{H'\sin^2 2as}, \text{ with } \gamma = \frac{C}{2a} \end{aligned}\right\} \tag{29}$$

Then the Gauss equation (22) becomes

$$\left(\frac{H''}{H'}\right)' - 2\gamma H' = \frac{4a^2}{\sin^2 2as}\left(2 - 2\gamma\frac{H^2}{H'}\right) \tag{30}$$

Notice that by the first equation of (29), in this setting, $H' < 0$ and thus H is strictly decreasing.

Equation (30) is of type Painlevé VI. Its integration is very involved and has been carried out in Bobenko & Eitner 1998, by means of elliptic integrals and certain hypergeometric transcendents. We refer the interested reader to this reference for the integration of this equation.

We note that in this situation J cannot be a constant. For otherwise, equations (29) and (30) are not compatible. By (29) we find that it would be

$$H = J\ln(\cot as) + A$$

with J, a, A constants. Then we plug this H into (30) and we see that it does not satisfy the equation. In such a situation J can be constant, only when H is constant. Then the helicoidal surface is a right circular cylinder considered as helicoidal surface.

Similarly, none of the following functions can be constant

$$\begin{aligned} K &= a\cdot c = H^2 - J^2 \\ a &= H + J \\ c &= H - J \\ \frac{a}{c} &= \frac{H+J}{H-J} \end{aligned}$$

unless again H is a constant and the surface is a right circular cylinder.

By (30) we find $H = H(s)$ and then from (29) we find $E = E(s)$. For the second fundamental form we have

$$\begin{aligned} II &= Lds^2 + 2Mdsdt + Ndt^2, \qquad \text{where} \\ L &= E(H + J\cos 2\psi) = EH + C\cot(2as) \\ M &= -EJ\sin 2\psi = -C \\ N &= E(H - J\cos 2\psi) = EH - C\cot(2as) \end{aligned}$$

Again $C = h$ is the pitch of the helicoidal motion that generates the helicoidal surface.

For the two parameter $(m, h = -C)$ family of isometric helicoidal surfaces (without preservation of the mean curvature, in general), as done in Baikoussis & Kouforgiorgos 1997 1998, Bour 1862, Do Carmo & Dajczer 1982, Hitt & Roussos 1991, we have that

$$H = \frac{\sqrt{m^2 E - m^4 (e')^2 - h^2}}{E^2} = \frac{\sqrt{4m^2 E^2 - m^4 (E')^2 - 4C^2 E}}{2\sqrt{E} E^2}, \quad (e = \sqrt{E})$$

Plugging this into (30) we find a differential equation for E of order four, much more complicated than (30) itself. Therefore, it is much better to deal with (30), as done in Bobenko & Eitner 1998, to find H and then find E, by (29). This is an effective way of determining explicitly the isothermic helicoidal surfaces.

F) We can also characterize the helicoidal surfaces with J constant ($\neq 0$) and H non-constant. As we have explained before in (E) these surfaces cannot be isothermic. From (25) we find that for $J > 0$ constant and $\psi \neq 0 \mod \dfrac{\pi}{2}$.

$$\left.\begin{aligned} H &= J \ln|\cot\psi| + A \ , \quad A \text{ constant} \\ E &= \frac{C}{J|\sin 2\psi|} \ , \quad C > 0 \text{ constant} \end{aligned}\right\} \tag{31}$$

Then the Gauss equation (22) gives

$$|\sin 2\psi|(\ln|\sin 2\psi|)'' = 2CJ[(\ln|\cot\psi| + a)^2 - 1] \tag{32}$$

$$\text{where } a = \frac{A}{J} \text{ constant.}$$

Solving this equation we find ψ and then E and H from (31).

The second fundamental form is computed as before, by

$$\begin{aligned} II &= L ds^2 + 2M ds dt + N dt^2, \quad \text{where} \\ L &= EH + EJ\cos 2\psi = EH \pm C\cot 2\psi \\ M &= -EJ\sin 2\psi = \mp C \\ N &= EH - EJ\cos 2\psi = EH \mp C\cot 2\psi \end{aligned}$$

REMARK. If both J and H are constant for a surface then both principal curvatures are constant. In such a case the surface is a piece of a plane, or a right circular cylinder, or a sphere.

G) *Further Known Facts.*

G1) All helicoidal surfaces with constant mean curvature can be isometrically and periodically deformed under preservation of the constant mean curvature to a surface of revolution of the same constant mean curvature (Delaunay surface). This deformation is differentiable and the intermediate surfaces are helicoidal of the same constant mean curvature. All helicoidal surfaces of a given constant mean curvature are obtained by this deformation. (See Do Carmo & Dajczer 1982, Hitt & Roussos 1991).

G2) The isothermic helicoidal surfaces with non-constant mean curvature accept a differentiable one-parameter family of non-trivial and geometrically distinct isometries that preserve the mean curvature from the surface to itself. Also, there are three other isometric associate surfaces to a given isothermic helicoidal surface with the same mean curvature at the corresponding points of the isometries.(See Bobenko & Eitner 1998, Cartan 1942, Roussos 1988b 1999a, Soyuçok 1995).

G3) All helicoidal surfaces admit the isometry $(s, t) \longrightarrow (s, -t)$ from the surface to itself. It is non-trivial when the orientation of the image surface of this isometry is kept to be the same with the original orientation of the given helicoidal surface. This isometry obviously preserves $H = H(s)$ at the corresponding points. (See Roussos 1988b 1999b, Soyuçok 1995).

From these three facts we conclude that all helicoidal surfaces are Bonnet surfaces (see Roussos 1988b). By definition, a surface M^2 in E^3 is called to be a Bonnet surface if it admits at least one non-trivial isometry from the surface to another surface or to itself that preserves the mean curvature (equivalent to, preseves each principal curvature). (See Bobenko & Eitner 1998, Cartan 1942, Roussos 1988b, 1999a, Soyuçok 1995, Voss 1993). It is a fact that if a surface admits two non-trivial and geometrically distinct isometries (that is, one is not the composition of the other followed by an isometry of the whole E^3) that preserve the mean curvature then it admits a whole one-parameter and differentiable family of such isometries and the surface is isothermic. (See the above references.)

G4) The helicoidal surfaces found in Baikoussis & Koufogiorgos 1997 satisfy that the ratio of their principal curvatures is constant $\neq 0, \pm 1$. Therefore, as we have seen in (E), they cannot be isothermic. So, they admit only one non-trivial isometry preserving the mean curvature, the one of (G3).

3. THE GAUSS EQUATIONS OF THE PREVIOUS SECTION

In this section we will discuss the Gauss equations found in the previous section. These are given by the second order ordinary differential equations:

$$(23) \quad J(\ln J)'' = 2B(H^2 - J^2) \quad H,\ B > 0 \text{ constants and } \quad J = J(s) > 0$$

$$(24) \quad H'' = \pm 2ABe^{\mp \frac{H}{A}}(H^2 - A^2), \quad A > 0, \quad B > 0 \text{ constants}$$

$$(27) \quad J(\ln J)'' = \frac{2C}{\sin 2\psi}(H^2 - J^2),\ H,\ C > 0,\ 0 < \psi < \frac{\pi}{2} \text{ constants}$$

$$(32) \quad |\sin 2\psi|(\ln|\sin 2\psi|)'' = 2CJ[(\ln|\cot\psi| + \alpha)^2 - 1],$$
$$C > 0,\ J > 0,\ \alpha \text{ constants}$$

(The Gauss equation (30) has been completely examined in Bobenko & Eitner 1998.)

A) We observe that (23) and (27) are the same differential equations. We can put the constants B and $\dfrac{C}{\sin 2\psi}$ under the same name A, so that both are written as

$$J(\ln J)'' = 2A(H^2 - J^2) \quad H,\ A > 0 \text{ constants} \tag{33}$$

This differential equation was derived for the surfaces in our considerations with H constant and under the assumption $J > 0$. By (21) and (26) we had that $EJ = A$, so that $A > 0$. We could make the whole consideration a bit more general by allowing $J \neq 0$ and $EJ = A$ constant such that $AJ > 0$. Then the Gauss equation becomes

$$J(\ln |J|)'' = 2A(H^2 - J^2), \quad H,\ A \neq 0 \text{ constants} \tag{34}$$

This differential equation can be expanded as

$$JJ'' - (J')^2 = 2A(H^2 - J^2)J \quad H,\ A \neq 0 \text{ constants and } \ AJ > 0 \tag{35}$$

We observe that in this new form $J = 0$ is allowed and we trivially obtain three constant solutions, namely

$$J = 0, \quad J = \pm H$$

From these three trivial solutions we can distinguish the three following cases:

$J = 0$, $H \neq 0$ constant: The surface is (a piece of) a sphere

$J = H = 0$: The surface is (a piece of) a plane

$J = \pm H \neq 0$ constant: The surface is (a piece of) a right circular cylinder

These three cases were known *a-priori* to be surfaces invariant under a one-parameter group of isometries of E^3 and with constant mean curvature H. Even though they are derived from (35) they cannot be derived by (33) and/or (34). So, equation (35) describes the surfaces in E^3 with constant mean curvature and invariant under a one-parameter group of isometries of E^3 in all generality. Apart from these three trivial cases we are going to examine the following harder ones: $J \neq 0$, $H = 0$ (minimal surfaces) and $J \neq 0$ $H \neq 0$ constant. In both cases, since H is constant and the surface is invariant under a one-parameter group of isometries of E^3 the isolatedness of the umbilic points implies that there cannot exist any umbilic points. So, on a connected surface here, either $J > 0$ or $J < 0$, i.e., J cannot change sign and cross zero at a point. Now we take up each of the above cases.

Case $J \neq 0$, $H = 0$ (minimal surfaces).
We write (33) as

$$JJ'' - (J')^2 = -2AJ^3, \text{ A constant such that } AJ > 0$$

Then

$$(\ln |J|)'' = -2AJ \tag{36}$$

The transformation $J = \dfrac{1}{AF^2}$, $F \neq 0$ function, turns this equation into

$$FF'' = (F')^2 + 1 > 0$$

so that neither F nor F'' can be zero at any point. Then F' cannot be zero in any open interval and so the last equation is equivalent to

$$\left[\ln\left[(F')^2 + 1\right]\right]' = (\ln F^2)'$$

which can be written as

$$\frac{kdF}{\sqrt{(kF)^2 - 1}} = \pm kds \quad \text{with } k \neq 0 \text{ constant}$$

The solutions of this equation are

$$F(s) = \frac{1}{k}\cosh(\pm ks + d) \quad , \quad k \neq 0, \quad d \text{ constants}$$

Since cosh is an even function, without loss of generality we have

$$F(s) = \frac{1}{k}\cosh(ks + c) \quad , \quad k \neq 0, \quad c \text{ constants}$$

Therefore

$$J(s) = \frac{1}{AF^2} = \frac{k^2}{A\cosh^2(ks + c)}$$

and

$$E(s) = \frac{A}{J(s)} = \frac{A^2\cosh^2(ks + c)}{k^2}$$

with $A \neq 0$, $k \neq 0$, c constants. The constant c may be geometrically eliminated by a translation of s, i.e., replace s by $s - \dfrac{c}{k}$.

The second fundamental forms of these surfaces in the (s, t) coordinates is given by

$$L = \pm EJ = \pm A \quad , \quad M = 0 \quad , \quad N = \mp EJ = \mp A$$

for the minimal surfaces of revolution—catenoids—and

$$\left.\begin{aligned} L = C\cot 2\psi &= A\cos 2\psi \\ M = -C &= -A\sin 2\psi \\ N = -C\cot 2\psi &= -A\cos 2\psi \end{aligned}\right\} 0 < \psi < \frac{\pi}{2}, \quad \text{mod } \pi, \text{ constant}$$

for the minimal helicoids.

We observe that by changing the constant ψ we get the periodic deformation of the helicoids into catenoids and vice versa, under preservation of the mean curvature $H = 0$. Because we were able to easily integrate the above Gauss equation we have completely described what happens in this case of minimal surfaces. (For another exposition of these surfaces, see Wunderlich 1952.)

We observe that $|J(s)|$ is bounded above by $\frac{k^2}{|A|} > 0$ and $J(s) \longrightarrow 0$ as $s \longrightarrow \pm\infty$. Also, $E(s)$ is bounded below by $\frac{A^2}{k^2}$ and $E(s) \longrightarrow +\infty$ as $s \longrightarrow \pm\infty$. Therefore, all these surfaces are complete and $J(s) \neq 0$ for all s which agrees with the fact that they contain no umbilic points. Moreover, $E(s)$ is a global function.

Case $J \neq 0$, $H \neq 0$ *constant.*
In this situation equation (35) is a bit difficult to integrate. We can reduce its order by making the transformation

$$y = (\frac{J'}{J})^2 \geq 0$$

Then

$$\frac{dy}{ds} = 2\frac{J'}{J} \cdot \frac{JJ'' - (J')^2}{J^2} = 2\frac{J'}{J} \cdot \frac{2A(H^2 - J^2)J}{J^2}$$

So

$$\frac{dy}{dJ} = 4A\frac{H^2 - J^2}{J^2} \text{ and thus } y = 4A(-\frac{H^2}{J} - J) + 4B \geq 0,$$

where, for convenience in what follows, we have put $4B$ as the constant of integration. Going back to J we find

$$\frac{dJ}{ds} = \pm\sqrt{J(-4AH^2 + 4BJ - 4AJ^2)} = \pm 2\sqrt{J(-AH^2 + BJ - AJ^2)}$$

A , B constants and $AJ > 0$.

As we see, the integral of the last equation is not elementary. In general, it can be computed in terms of an elliptic integral of the first kind whose lower limit of integration is zero and its upper limit varies in the interval $[0, \frac{\pi}{2}]$.

To solve the differential equation (35) we must compute the integral of

$$\frac{dJ}{2\sqrt{J(-AJ^2 + BJ - AH^2)}} \ , \quad A \neq 0 \ , H \neq 0 \text{ constants}$$

Since J cannot be zero at any point and the surface is connected we first assume that

$$J = J(s) > 0 \text{ and therefore } A > 0 \text{ constant.}$$

Then we need $-AJ^2 + BJ - AH^2 > 0$. Let

$$r_1 = \frac{B - \sqrt{B^2 - 4A^2H^2}}{2A} \ , \ r_2 = \frac{B + \sqrt{B^2 - 4A^2H^2}}{2A}$$

We must have J such that

$$J > 0 \text{ and } r_1 \leq J \leq r_2$$

For this, it must be $B > 0$ and $B > 2A|H|$. Under these conditions we have

$$0 < r_1 < r_2$$

and J is in the interval $[r_1, r_2]$. So we must compute

$$\int_{r_1}^{J} \frac{dx}{2\sqrt{x(-Ax^2 + Bx - AH^2)}} \ , \quad 0 < r_1 \leq J, x \leq r_2$$

Let $x = u^2$. Then $u = \sqrt{x} > 0$ and $\sqrt{r_1} \leq u \leq \sqrt{r_2}$. Then the integral transforms into

$$\int_{\sqrt{r_1}}^{u} \frac{2u\ du}{2\sqrt{u^2(-Au^4 + Bu^2 - AH^2)}} = \frac{1}{\sqrt{A}} \int_{\sqrt{r_1}}^{u} \frac{du}{\sqrt{(u^2 - r_1)(-u^2 + r_2)}}$$

Now, we let $u = \sqrt{r_1} \sec v$. Then

$$\cos v = \frac{\sqrt{r_1}}{u} \ , \quad 0 \leq v \leq \arccos\sqrt{\frac{r_1}{r_2}}$$

and the integral becomes

$$\frac{1}{\sqrt{A}} \int_0^v \frac{\sqrt{r_1} \sec v \tan v\ dv}{\sqrt{r_1 \tan^2 v(-r_1 \sec^2 v + r_2)}} = \frac{1}{\sqrt{A}} \int_0^v \frac{dv}{\sqrt{r_2 - r_1 - r_2 \sin^2 v}}$$

Finally, we let $\sqrt{r_2} \sin v = \sqrt{r_2 - r_1} \sin\phi$. Then

$$\sin\phi = \sqrt{\frac{r_2}{r_2 - r_1}} \sin v, \quad 0 \leq \phi \leq \frac{\pi}{2}$$

and the integral changes to

$$\frac{1}{\sqrt{A}} \int_0^{\phi} \frac{\sqrt{\frac{r_2 - r_1}{r_2}}\ \frac{\cos\phi}{\cos v} d\phi}{\sqrt{(r_2 - r_1) - (r_2 - r_1)\sin^2\phi}} = \frac{1}{\sqrt{Ar_2}} \int_0^{\phi} \frac{d\phi}{\sqrt{1 - \frac{r_2 - r_1}{r_2} \sin^2\phi}}$$

The constant

$$k^2 = \frac{r_2 - r_1}{r_2} = \frac{2\sqrt{B^2 - 4A^2H^2}}{B + \sqrt{B^2 - 4A^2H^2}} \ \text{ is in } (0, 1),$$

so that,

$$\int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}} := F(\phi, k), \quad 0 < k < 1 \text{ constant}$$

is an elliptic integral of the first kind in its Legendre's form. Also the constant Ar_2 is given as

$$Ar_2 = \frac{B + \sqrt{B^2 - 4A^2H^2}}{2}$$

and the integration of the differential equation leads to

$$\sqrt{\frac{2}{B + \sqrt{B^2 - 4A^2H^2}}} \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \pm s + c$$

c is a constant that can be geometrically eliminated, as a translation of the parameter s. Therefore, we are going to omit it. Hence we get

$$\phi = F^{-1}(\pm\sqrt{\frac{B + \sqrt{B^2 - 4A^2H^2}}{2}} s)$$

where, with k fixed, F^{-1} is the inverse function of $F(\phi, k)$ considered as function of ϕ. We also have

$$\phi = Arc \sin(\frac{1}{k}\sqrt{\frac{J - r_1}{J}})$$

Thus, using the definitions of r_1 and k we find that

$$J = \frac{r_1}{1 - k^2 \sin^2 \phi} = \frac{2AH^2}{B + \sqrt{B^2 - 4A^2H^2} \cos\left(2F^{-1}(\pm\sqrt{\frac{B + \sqrt{B^2 - 4A^2H^2}}{2}} s)\right)}$$

Then, we find the first fundamental form $I = E(s)(ds^2 + dt^2)$ by

$$E = \frac{A}{J} = \frac{B + \sqrt{B^2 - 4A^2H^2} \cos\left(2F^{-1}(\pm\sqrt{\frac{B + \sqrt{B^2 - 4A^2H^2}}{2}} s)\right)}{2H^2}$$

(Remind: $A > 0$, $B > 2A|H|$, $H \neq 0$ constants and F^{-1} is the inverse function of $F(\phi, k)$ considered as a function of ϕ. The constant k is kept fixed at a time.)

If we consider the case $J < 0$, then $A < 0$ constant and for the constant B we must have $B > 2|AH|$, $H \neq 0$ constant. The results again are exactly the same.

For the second fundamental form we do the same computations as we have already done in various places earlier (see parts of section 2).

As $F(\phi, k)$, $0 < k < 1$ constant, is never singular we may allow ϕ vary from $-\infty$ to $+\infty$. Therefore

$$\pm s = \sqrt{\frac{2}{B + \sqrt{B^2 - 4A^2H^2}}} \cdot F(\phi, k)$$

varies from $-\infty$ to $+\infty$, also. We notice that $E(s) = \dfrac{A}{J(s)}$ is a global function bounded below by the positive constant

$$\frac{B - \sqrt{B^2 - 4A^2H^2}}{2H^2} > 0$$

and bounded above by the positive constant

$$\frac{B + \sqrt{B^2 - 4A^2H^2}}{2H^2}$$

These constants are sharp, since they are assumed by $E(s)$. Therefore, the surfaces here are complete, and the $s-$curves which are geodesics have arc-length (not the parameter s, but $z = \int_{s_0}^{s} \sqrt{E(s)}ds$) that extends from $-\infty$ to $+\infty$. Also, the global function $E(s)$ cannot approach zero and cannot become large either for any given $H \neq 0$ constant. Similarly, the global function $J(s)(> 0)$ is bounded below by the positive constant

$$\frac{2AH^2}{B + \sqrt{B^2 - 4A^2H^2}} \quad (A > 0 \text{ when } J(s) > 0)$$

and above by the positive constant

$$\frac{2AH^2}{B - \sqrt{B^2 - 4A^2H^2}}$$

These constants are sharp, since they are assumed by $J(s)$. This means, $J(s) = \dfrac{a - c}{2} > 0$ ($a > c$ are the principal curvatures) cannot approach zero and cannot become large either for any given $H \neq 0$ constant.

In conclusion, the work of this part provides a new intrinsic characterization of all surfaces in E^3 invariant under a one-parameter group of isometries of E^3 and with constant mean curvature. This new exposition is original in the sense that it makes use of the global function J and it is based on the general theory of section 1. Moreover, several old and new facts about these surfaces are easily drawn.

Remark. Another approach to find the solution of the differential equation (35) when $H \neq 0$ constant, is to use the results in Do Carmo & Dajczer 1982. We find and invert the parameter $\sigma = \sigma(s)$ (page 433) in terms of elliptic integrals and then plug it into (3.9) (page 430). Then we have

$$J(\sigma) = \frac{A}{E(\sigma)} = \frac{A}{U^2(s(\sigma))}$$

Note that our (s, t) parameters here are the (σ, t) in Do Carmo & Dajczer 1982 and

$$E(\sigma) = U^2(s(\sigma))$$

REMARK. In the limiting case $B = 2|AH|$ we get that

$$E = |\frac{A}{H}| \text{ and } J = \pm H \neq 0 \quad \text{constants}$$

So, either $a \neq 0$ is constant and $c = 0$ or $a = 0$ and $c \neq 0$ constant. In this case we get the right circular cylinders described as helicoidal surfaces, which are the only flat helicoidal surfaces with constant mean curvature, see Do Carmo & Dajczer 1982 and Hitt & Roussos 1991.

B) We can write (24) more general as

$$H'' = 2ABe^{-\frac{H}{A}}(H^2 - A^2)\ ,\ \ A \neq 0\ ,\ \ B > 0 \quad \text{constants.}$$

We have the trivial solutions $H = \pm A$ constants. Then $H = \pm J = \pm A \neq 0$ constant and the surface is a right circular cylinder, a result expected *a-priori*, for surfaces with J constant.

We can reduce the order of this differential equation by one by making the standard transformation

$$\frac{dH}{ds} = y(H)$$

Then,

$$\frac{d^2H}{ds^2} = \frac{dy}{dH} \cdot \frac{dH}{ds} = \frac{dy}{dH} \cdot y = \frac{d}{dH}(\frac{y^2}{2})$$

After the computation we find

$$\frac{dH}{ds} = y = \pm 2\sqrt{-BA^2e^{-\frac{H}{A}}(H^2 + 2AH - 2A^2) + C}, \quad C \ \text{constant}$$

We see that this is non-trivial to integrate.

C) Similarly, in equation (32) we may have $J \neq 0$ constant and C a constant such that $CJ > 0$. We can write equation (32) as

$$\sin(2\psi)\cos(2\psi)\psi'' - 2(\psi')^2 = CJ[(\ln(\cot\psi) + a)^2 - 1] \cdot \sin(2\psi), \quad 0 < \psi < \frac{\pi}{2}$$

We observe that $\psi = 0\,,\ \frac{\pi}{2}$ are limiting constant solutions, i.e., the first side of the equation becomes zero for $\psi = 0\,,\ \frac{\pi}{2}$ and the second side of the equation has limit equal to zero, as $\psi \longrightarrow 0$ or $\frac{\pi}{2}$.

Again as we did in (B), we can reduce the order by one, if we use the transformation

$$\frac{d\psi}{ds} = \phi(\psi) \Longrightarrow \frac{d^2\psi}{ds^2} = \frac{d\phi}{d\psi}\frac{d\psi}{ds} = \frac{d\phi}{d\psi} \cdot \phi = \frac{d}{d\psi}(\frac{\phi^2}{2})$$

and carry out the computation in the equation.

4. THE FLAT HELICOIDAL SURFACES WITH NON-CONSTANT MEAN CURVATURE

In this section we study the flat helicoidal surfaces with non-constant mean curvature. Since H is non-constant, $K = 0$ and there are no umbilic points these surfaces must be tangential developable surfaces of curves in E^3 (see Klingenberg 1977, page 59, 3.7.9 Theorem and 3.7.10 Proposition). We will show that these curves are precisely the circular helices. By the facts proven in section 2, part (E), these surfaces cannot be isothermic. First, we need to start with:

A) *Some Preliminary Facts About the Tangential Developable Surfaces in E^3.*

A tangential developable in E^3 can be “naturally" expressed as

$$X(u, v) = \varsigma(u) + ve_1(u)$$

where $\varsigma(u)$ is a curve in E^3, paramaterized by its arclength u, $e_1(u) = \dot{\varsigma}(u)$ and $v > 0$ ($v < 0$ gives the second sheet of the tangential developable). (See Roussos 1999b, Soyuçok 1995) We can call (u, v) the *usual parameters* of the tangential developable surface. Let $e_2(u)$ be the principal normal of $\varsigma(u)$. Then $\dot{e_1}(u) = k(u)e_2(u)$ where $k(u) \geq 0$ is the curvature of $\varsigma(u)$. Then we have

$$X_u = e_1(u) + vk(u)e_2(u)$$
$$X_v = e_1(u)$$

So, the tangent plane of $X(u, v)$ is spanned by e_1 and e_2 as long as $v > 0$ and $k(u) > 0$. (Even though the vectors e_1, e_2 are originally defined along $\varsigma(u)$, their parallel translations along the geodesic straight lines that foliate the whole tangential developable surface make up a global frame field over the whole surface.) The first fundamental form in the coordinates (u, v) is given by:

$$I = (1 + v^2k^2(u))du^2 + 2dudv + dv^2$$

Now, we write

$$dX = \omega_1 e_1 + \omega_2 e_2 = (e_1 + vke_2)du + e_1 dv = (du + dv)e_1 + (vkdu)e_2$$

So, we have

$$\omega_1 = du + dv$$
$$\omega_2 = vk(u)du$$
$$\omega_{12} = k(u)du = \frac{1}{v}\,\omega_2$$

We orient the surface by the binormal of $\varsigma(u)$ $e_3 = e_1 \times e_2$. We let $\tau(u)$ be the torsion of $\varsigma(u)$. Then by the formulas of Frenet-Serret we find:

$$\omega_{13} =< de_1, e_3 >= 0 = 0 \cdot \omega_1$$
$$\omega_{23} =< de_2, e_3 >= \tau(u)du = \frac{\tau(u)}{vk(u)}\omega_2$$

This shows that $\{e_1, e_2\}$ is the principal frame with $\{\omega_1, \omega_2\}$ corresponding principal coframe and corresponding principal curvatures

$$0 , \quad \frac{\tau(u)}{vk(u)}$$

We observe that if we have $\tau(u) = 0$ at some u then the whole straight line $l(v) := \varsigma(u) + ve_1(u)$ would consist of nonisolated umbilic points. (In particular the umbilic points at which both principal curvatures are zero are called *planar points*.) (For more information about these surfaces see Eisenhart 1909, where minimal curves and isotropic developables are discussed.)

From the previous exposition we easily get that the *principal coordinates* of $X(u, v)$ are x, y such that

$$X(x, y) = \varsigma(y) + (x - y)e_1(y)$$

$$\left\{ \begin{array}{l} u = y \\ v = x - y \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} x = u + v \\ y = u \end{array} \right\}$$

In these coordinates the first fundamental form is

$$I = dx^2 + (x - y)^2 k^2(y) dy^2$$

and the second fundamental form is

$$II = (x - y)k(y)\tau(y)dy^2$$

B) *Tangential Developables of Circular Helices.*

In this part we are going to prove that the flat helicoidal surfaces of non-constant mean curvature are exactly the tangential developables of the circular helices. The flat helicoidal surfaces of constant mean curvature are the circular cylinders, which can also be considered as surfaces of revolution (and Delaunay Surfaces, since the mean curvature is constant).

Since the flat surfaces are the cylinders, cones, the tangential developables and smooth darnings of pieces of theirs (see Klingenberg 1977 for instance) we see that apart from the circular cylinders, a flat helicoidal surface must be a tangential developable.

We consider the tangential developable of a non-plane curve $C(u)$ in E^3

$$X(u, v) = C(u) + v \cdot e_1(u) \ , \ \ v > 0 \ \ (\text{or } v < 0)$$

which we assume to be helicoidal. u is the arclength parameter of $C(u)$ and $e_1(u) = \dot{C}(u)$. The curvature $k = k(u)$, and torsion $\tau = \tau(u)$ of $C(u)$ are not zero. We plan to show that k, τ are constants (non-zero), which is just as proving that $C(u)$ is a circular helix (see Millman & Parker 1977). Notice that here, the parametrization (u, v) is not the natural one, that is, the one of section

2, but what we have earlier called usual parametrization. $X(u, v)$ is foliated by the circular helices of the helicoidal motion. We pick one of these helices, $h(l) = X(u(l)\ ,\ v(l))$, parametrized by its arclength l. We have $h(l) = C(u(l)) + v(l) \cdot e_1(u(l))$.

We let e_2 be the principal normal of $C(u)$. $e_3 = e_1 \times e_2$ is the binormal of $C(u)$. In the previous part we saw that $e_1\ ,\ e_2$ are the principal directions of $X(u\ ,\ v)$ and e_3 is its normal vector. The frame e_1, e_2, e_3 is the Serret-Frenet frame of $C(u)$, and the Serret-Frenet formulas for $C(u)$ are:

$$\frac{d}{du}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

(here the speed of $C(u)$ is 1, since u is the arclength parameter).

We put $h_1(l) = h'(l)$ and we find:

$$\begin{aligned} h_1(l) &= u'(l) \cdot e_1(u(l)) + v'(l) \cdot e_1(u(l)) + v(l)k(u(l))u'(l) \cdot e_2(u(l)) \\ &= [u'(l) + v'(l)] \cdot e_1(u(l)) + v(l)k(u(l))u'(l) \cdot e_2(u(l)) \end{aligned}$$

Since a helicoidal motion is a rigid motion of E^3, for all helicoidal surfaces, we have that the angle ψ , as defined in section 2, is constant along each of the helices of the helicoidal motion, but not necessarily all are the surface, unless the mean curvature of the surface is constant. This then gives

$$\begin{array}{lrcl} (\alpha) & u'(l) + v'(l) = h_1(l) \cdot e_1 = c_1 & \text{constant along } h(l) \\ (\beta) & k(u(l))u'(l)v(l) = h_1(l) \cdot e_2 = c_2 & \text{constant along } h(l) \end{array}$$

We now let $(h_1(l), h_2(l), h_3(l))$ be the Darboux frame of $h(l)$ with respect to $X(u, v)$. Then $h_3 = e_3$. Also, the following general formulas hold (see Spivak 1979, Volume 3, Chapter 4)

$$\frac{d}{dl}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix} = \begin{pmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ -k_n & -\tau_g & 0 \end{pmatrix}\begin{pmatrix} h_1 \\ h_2 \\ h_3 \end{pmatrix}$$

The helices of the helicoidal motion in a helicoidal surface other than the circular cylinder are not: geodesics, principal curves and asymptotic curves. (Use Chapter 4, Volume 3 in Spivak 1979 and the first and second fundamental forms of a helicoidal surface in the natural coordinates as described earlier or as may be found in Baikoussis & Koufogiorgos 1997, 1998, Do Carmo & Dajczer 1982). Therefore, as before for ψ, we have that $k_g\ ,\ k_n\ ,\ \tau_g$ are all non-zero constants (along each individual helix). k_g , k_n are the geodesic and normal curvature respectively, and τ_g is the geodesic torsion of $h(l)$.

We have

$$\frac{d}{dl}h_1(l) = k_g h_2(l) + k_n h_3(l)$$

Also, since (α) , (β) are true we compute that

$$\frac{d}{dl}h_1(l) = [u'(l) + v'(l)]k(u(l))u'(l) \cdot e_2(u(l)) - \\ v(l)k^2(u(l))(u'(l))^2 \cdot e_1(u(l)) + v(l)k(u(l))\tau(u(l))(u'(l))^2 \cdot e_3(u(l))$$

We know

$$\langle h_2(l)\ ,\ e_1\rangle = \text{ constant}$$
$$\langle h_2(l)\ ,\ e_2\rangle = \text{ constant}$$

So we get

$$\begin{array}{lll} (\gamma) & [u'(l) + v'(l)]k(u(l))u'(l) = c_3 & \text{constant} \\ (\delta) & k^2(u(l))(u'(l))^2 v(l) = c_4 & \text{constant} \end{array}$$

We observe that $u(l)$ cannot be constant (otherwise $h(l)$ would be a straight line and not a helix), so $u'(l) \neq 0$, and $v(l) \neq 0$. Then from (β) and (δ) we have that $c_2 \neq 0$ and $c_4 \neq 0$. Since c_2 is constant then $v(l)$ is never zero. This means that the helix $h(l)$ never intersects the curve $C(u)$. Now from (β) and (δ) we have

$$k(u(l))u'(l) = \frac{c_4}{c_2} \quad \text{non-zero constant}$$
$$v(l) = \frac{c_2^2}{c_4} \quad \text{non-zero constant}$$

Then $v'(l) = 0$, so that from (a) we get

$$u'(l) = c_1 \text{ (non-zero constant)}$$

Hence

$$k(u(l)) = \frac{c_4}{c_2 c_1} \quad \text{(non-zero constant)}$$

Also

$$k(u(l))(u'(l))^2 = \frac{c_4 c_1}{c_2}$$

or from (γ) we get $c_3 \neq 0$ and

$$k(u(l))(u'(l))^2 = c_3$$

Next

$$\frac{d}{dl}(h_1(l)) \cdot h_3 = \frac{d}{dl}(h_1(l)) \cdot e_3$$

from which we have

$$k_n = v(l)k(u(l))\tau(u(l))(u'(l))^2$$

Thus

$$\tau(u(l)) = \frac{k_n}{v(l)k(u(l))(u'(l))^2}$$

or

$$\tau(u(l)) = \frac{k_n}{c_1 c_2} \quad \text{non-zero constant.}$$

The above computations show that k , τ are non-zero constants. Thus, $C(u)$ is a circular helix, proving our assertion.

Now, we examine the tangential developable surface of a circular helix more closely. We consider $C(u)$ a circular helix with u the arclength and

$$X(u, v) = C(u) + ve_1(u) \ , \qquad v > 0 \quad (\text{or } v < 0)$$

the corresponding tangential developable. $e_1(u)$ is the unit tangent vector of $C(u)$ and let $k > 0$, $\tau \neq 0$ constants be the curvature and torsion of $C(u)$ respectively. Assume L to be the axis of $C(u)$ with direction vector D. Then D is a fixed unit vector and from the theory of helices we know that

$$D \cdot e_1(u) = \frac{\tau}{\sqrt{\tau^2 + k^2}} \quad \text{constant}$$

Hence, by taking derivative we get

$$D \cdot k \cdot e_2(u) = 0 \Leftrightarrow D \cdot e_2(u) = 0 \quad (k \neq 0)$$

(For more information about the theory of helices, see Millman & Parker 1977, sections 2.3, 2.4 and 2.5.)

Now, for any $v \neq 0$ fixed we consider the curve $r(u) = C(u) + ve_1(u)$. Then the tangent vector of $r(u)$ is

$$r'(u) = e_1(u) + vke_2(u)$$

This has length

$$|r'(u)| = \sqrt{1 + v^2k^2} > 0 \quad \text{constant}$$

So, the unit tangent vector of $r(u)$ is

$$T(u) = \frac{e_1(u) + vke_2(u)}{\sqrt{1 + v^2k^2}}$$

Hence,

$$D \cdot T(u) = \frac{\tau}{\sqrt{1+v^2k^2}\sqrt{\tau^2+k^2}} \quad \text{constant}$$

Therefore, $r(u)$ is a helix with axial direction the same with the initial helix $C(u)$. The fact that $r(u)$ is a circular helix follows from the computation of its curvature and torsion, which both turn out to be respectively the constants.

$$k_r = \frac{k \cdot \sqrt{1+v^2(k^2+\tau^2)}}{\sqrt{1+v^2k^2}}$$
$$\tau_r = \frac{\tau}{1+v^2k^2}$$

So, $r(u)$ is a circular helix with axis parallel to L. Now, L is, in fact, the axis of $r(u)$ because if $0 < R$ constant is the distance of any point of $C(u)$ from L then the distance of any point of $r(u)$ from L is easily computed to be

$$\sqrt{R^2 + v^2\frac{\tau^2}{k^2+\tau^2}}$$

which is constant for any given $v \neq 0$ fixed. Therefore, all $r(u)$'s are coaxial helices with $C(u)$ and the tangential developable surface $X(u, v)$ is a helicoidal surface.

C) Here, we study some consequences of the (s, t) natural coordinates for the flat helicoidal surfaces with non-constant mean curvature. Depending on the orientation we have

$$J = \pm H$$
$$K = 0 \Leftrightarrow (\ln E)'' = 0 \Leftrightarrow E(s) = Ae^{Bs} \ , \ A > 0 \ , \ B \neq 0 \text{ constants}$$

($B = 0$, for the cylinders). The constant A can be geometrically set at 1. So, equations (19) in section (2) become

$$2\psi' = -\sin 2\psi(\frac{\pm J'}{J})$$
$$\frac{J'}{J} + B = \cos 2\psi(\frac{\pm J'}{J}) \ , \ 0 \neq B \text{ constant}$$

From the first one we get that

$$\psi = \text{ arc } \cot(\gamma J^{\pm 1}) \ , \ \gamma \neq 0 \text{ constant}$$

and then the second equation gives

$$(\gamma^2 J^{\pm 2} + 1)J' + BJ(\gamma^2 J^{\pm 2} + 1) = \pm(\gamma^2 J^{\pm 2} - 1)J'$$

or

$$(+)\quad 2J' + BJ(\gamma^2 J^2 + 1) = 0$$
$$(-)\quad 2\gamma^2 J' + BJ(\gamma^2 + J^2) = 0$$

Both equations $(+)$, $(-)$ are integrated elementarily and we find

$$(+)\quad J(s) = \frac{\pm 1}{\sqrt{\delta e^{Bs} - \gamma^2}}$$
$$(-)\quad J(s) = \frac{\pm\gamma}{\sqrt{\delta e^{Bs} - 1}}$$

Essentially both answers are the same. $\delta > 0$ is a new constant and γ , B are non-zero constants found before. So, we have found expressions of $J(s)$, $H(s)$, $\psi(s)$, $E(s)$, $K(s) = 0$ up to some constants, in the natural parameters. (We remark that equations (26) could be used to solve the above equations a bit faster.)

We are going to find the characterization of these surfaces in their natural parameters (s, t) and find the relation of the (s, t) parameters with the usual parameters (u, v).

To make the computations simpler we will impose some normalization and the other cases are variations of the one we discuss next. We eliminate the constant $A > 0$ by replacing s by $s - \dfrac{\ln A}{B}$ so that we have

$$E(s) = e^{Bs}\ , \qquad B \neq 0 \quad \text{constant}$$

We consider the case

$$0 < J = H = \frac{\tau}{2vk}$$

So, we must have $\dfrac{\tau}{v} > 0$. Then by the previous formulae we have

$$\psi = \text{ arc }\cot(\gamma J)$$
$$J^2(s) = \frac{1}{\delta e^{Bs} - \gamma^2}$$
$$e^{Bs} = \frac{1}{\delta}\left(\frac{4k^2}{\tau^2}v^2 + \gamma^2\right)$$

Since we are allowed to approach the initial helix $C(u)$ by letting $v \to 0$ and/or $s \to 0$ we get that it must be $\delta = \gamma^2$ and therefore

$$e^{Bs} = \frac{4k^2}{\gamma^2\tau^2}v^2 + 1 > 1 \Rightarrow Bs > 0 \quad (v^2 \neq 0)$$
$$J^2(s) = \frac{1}{\gamma^2(e^{Bs} - 1)}$$

Hence we consider $B > 0$ and $s > 0$ (or $B < 0$ and $s < 0$) and we get

$$s = \frac{1}{B} \ln\left(\frac{4k^2}{\gamma^2\tau^2} v^2 + 1\right)$$

Set for convenience $R = \dfrac{4k^2}{\gamma^2\tau^2}$ constant. We have

$$E(s) = e^{Bs} = Rv^2 + 1s = \frac{1}{B}\ln(Rv^2+1)$$

If now $t = t(u, v)$ we take

$$I = E(s)(ds^2 + dt^2) = (Rv^2+1)\left[\frac{1}{B^2}\left(\frac{2Rv}{Rv^2+1}\right)^2 dv^2 + t_u^2 du^2 + 2t_u t_v du dv + t_v^2 dv^2\right]$$

So, by comparison with $I = (k^2v^2+1)du^2 + 2dudv + dv^2$ we get

$$\begin{aligned} \frac{1}{B^2}\cdot\frac{4R^2v^2}{Rv^2+1} + (Rv^2+1)t_v^2 &= 1 \\ t_u^2 &= \frac{k^2v^2+1}{Rv^2+1} \\ t_u t_v &= \frac{1}{Rv^2+1} \end{aligned}$$

These relations imply $R = k^2$ and $B = 2\sqrt{R} = 2k > 0$ and consequently

$$\gamma^2 = \frac{4}{\tau^2} \text{ and } J^2(s) = \frac{\tau^2}{4(e^{2ks}-1)} = H^2(s).$$

Then,

$$\begin{aligned} s &= \frac{1}{2k}\ln(k^2v^2+1) \\ t &= \pm u \pm \frac{1}{k}\arctan(kv) \end{aligned}$$

So, in the natural parameters (s, t) we have that the first fundamental form is

$$I = e^{2ks}(ds^2 + dt^2)\ ,\ \ s > 0$$

and the relation between (s, t) and (u, v) is

$$\left\{\begin{array}{lcl} s &=& \dfrac{1}{2k}\ln(k^2v^2+1) > 0 \\ t &=& \pm(u + \dfrac{1}{k}\arctan(kv)) \end{array}\right\} \Leftrightarrow \left\{\begin{array}{lcl} v &=& \dfrac{\pm\sqrt{e^{2ks}-1}}{k} \\ u &=& \pm\left(t - \dfrac{1}{k}\arctan\sqrt{e^{2ks}-1}\right) \end{array}\right\}$$

Now, the second fundamental form is given by

$$II = Lds^2 + 2Mdsdt + Ndt^2$$

such that, after direct computation and using that $\psi = \arctan(\gamma J)$ with $\gamma = \frac{2}{|\tau|}$ and the expressions for $E(s)$ and $J(s)$,

$$L = E(H + J\cos 2\psi) = EJ(1+\cos 2\psi) = 2EJ\cos^2\psi = \frac{|\tau|}{\sqrt{e^{2ks}-1}}$$

$$M = -EJ\sin 2\psi = -|\tau|$$

$$N = E(H - J\cos 2\psi) = EJ(1-\cos 2\psi) = 2EJ\sin^2\psi = |\tau|\sqrt{e^{2ks}-1}$$

(We observe that the pitch of the helicoidal motion is $|\tau|$, as it is the case for unit speed helices. $C(u)$ is a unit speed helix because we have assumed that u is the arclength parameter. We have analogous results for $\gamma = -\frac{2}{|\tau|}$, etc.)

So, we have expressed all the fundamental quantities of the flat helicoidal surfaces in the natural parameters (s, t).

There are two expressions for t in terms of u and v of opposite sign. By keeping the orientation the same in both cases the mapping from the surface to itself $(s, t) \longrightarrow (s, -t)$ is an isometry that preserves the mean curvature

$$H(s) = \frac{|\tau|}{2\sqrt{e^{2ks}-1}}\ , \qquad s > 0$$

and it is not trivial, because the new coefficient M will be $+|\tau| \neq 0$ and therefore different from the old one $-|\tau|$. (See also Roussos 1999b.)

We also observe that

$$II = \frac{|\tau|}{\sqrt{e^{2ks}-1}}\left(ds - \sqrt{e^{2ks}-1}\,dt\right)^2$$

Therefore, a vector with direction

$$\frac{dt}{ds} = \frac{1}{\sqrt{e^{2ks}-1}}$$

is an asymptotic vector. Solving this differential equation we get

$$t + c = \frac{1}{k}\arctan\sqrt{e^{2ks}-1}\ , \qquad s > 0\ , \qquad \text{c = constant}$$

This is exactly the equation of the ruling straight lines in the coordinates (s, t) of this tangential developable, which are asymptotic lines as well as lines of curvature with principal curvature zero and geodesics.

Finally, since the relation of the principal coordinates (x, y) with the usual ones (u, v) was found earlier to be $x = u + v$ and $y = u$ we immediately obtain the relation of the principal coordinates (x, y) with the natural coordinates (s, t).

5. HELICOIDAL SURFACES WITH NON-ZERO CONSTANT GAUSSIAN CURVATURE AND SURFACES OF REVOLUTION WITH CONSTANT GAUSSIAN CURVATURE

A) Assume that for a helicoidal surface the Gaussian curvature $K \neq 0$ is constant. (The case $K = 0$ was thoroughly examined in section 4.) Then by the Gauss Equation we have

$$\frac{1}{-2E} \cdot (\ln E)'' = K \Leftrightarrow (\ln E)'' = -2KE$$

This is easily integrated, like equation (36) in section 3. We find that:
If $K > 0$ then

$$E(s) = \frac{\sigma^2}{K \cos h^2(\sigma s)}\ , \qquad \sigma \neq 0 \text{ constant}$$

If $K < 0$ then we have the following three solutions:

$$E(s) = \frac{1}{-Ks^2}\ , \qquad s \neq 0$$

$$E(s) = \frac{\sigma^2}{-K \sin^2(\sigma s)}\ , \qquad s \neq 0\ , \qquad \sigma \neq 0 \text{ constant}$$

$$E(s) = \frac{\sigma^2}{-K \sin h^2(\sigma s)}\ , \qquad s \neq 0\ ,\ \sigma \neq 0 \text{ constant}$$

Now, from $H^2 - J^2 = K$ we have that

$$H = \pm\sqrt{J^2 + K}$$

We assume that $J > 0$ (analogous work if $J < 0$). So, the first equation of (26) gives

$$\pm\frac{dJ}{\sqrt{J^2 + K}} = [\ln(\cot\psi)]'\ , \quad 0 < \psi < \frac{\pi}{2}$$

For any $K \neq 0$ constant this gives

$$\cot\psi = \gamma(J + \sqrt{J^2 + K})^{\pm 1}\ ,\ \gamma > 0 \text{ constant}$$

Then from the second equation of (26) we get

$$J \sin 2\psi = \frac{C}{E} \quad \text{or,} \quad \frac{2\gamma J(J + \sqrt{J^2 + K})^{\pm 1}}{1 + \gamma^2(J + \sqrt{J^2 + K})^{\pm 2}} = \frac{C}{E}$$

Thus, with the $(+)$ we have

$$\frac{2\gamma(J^2 + J\sqrt{J^2 + K})}{1 + \gamma^2 K + 2\gamma^2(J^2 + J\sqrt{J^2 + K})} = \frac{C}{E}$$

and with the $(-)$

$$\frac{2\gamma(J^2 + J\sqrt{J^2 + K})}{2(J^2 + J\sqrt{J^2 + K}) + \gamma^2 + K} = \frac{C}{E}$$

Now, for each of the $E(s)$ found earlier, we solve for $J^2 + J\sqrt{J^2+K}$ and then we find $J = J(s)$. From $E(s)$ and $J(s)$ we find $H(s)$ and $\psi(s)$ by their formulas just reported. Then in the usual way we find the coefficients of the second fundamental form $L(s)$, $M(s)$, $N(s)$ of this helicoidal surface. Hence, all fundamental quantities are explicitly discovered in terms of s. As a matter of fact: With the $(+)$ we find

$$J^2 = \frac{C^2(1+\gamma^2 K)^2}{4\gamma(E-\gamma C)(\gamma K E + C)}$$

and with the $(-)$

$$J^2 = \frac{C^2(K+\gamma^2)^2}{4\gamma(\gamma E - C)(K E + \gamma C)}$$

The rest of the computations proceed as usual.

B) The formulas found for $E(s)$ when $K > 0$ or $K < 0$ constant are unchanged if the surface is a surface of revolution instead of a helicoidal surface. Also, for $K = 0$ we get $E(s) = Ae^{Bs}$, $A > 0$, B are constants (the same as in section 4). In this case $\psi = 0$, mod $\frac{\pi}{2}$, so by equation (20) we get (again assume $J > 0$ and we have $H = \pm\sqrt{J^2+K}$)

$$[\ln(JE)]' = \pm\frac{dJ}{\sqrt{J^2+K}} = [\ln(J+\sqrt{J^2+K})^{\pm 1}]'$$

So

$$JE = \gamma(J+\sqrt{J^2+K})^{\pm 1}, \qquad \gamma > 0 \quad \text{constant}$$

We solve this for J. When $K \neq 0$ constant we find

$$(+) \quad J = \frac{\gamma}{\sqrt{E}}\sqrt{\frac{K}{E-2\gamma}}$$

$$(-) \quad J = \frac{\gamma}{\sqrt{E}\sqrt{EK+2\gamma}}$$

When $K = 0$ we find that either $E = A$ and $J(s) = \pm H(s)$ is anything or $E = Ae^{Bs}$, $B \neq 0$ and $J = \pm H = \gamma e^{-\frac{B}{2}s}$, $\gamma > 0$ constant. (In the first alternative we have the planes, the right circular cylinders and all generalized cylinders. In the second alternative we have the right circular cones.)

So, we have found $E(s)$, $J(s)$, $H(s)$ for all surfaces of revolution with constant Gaussian curvature (zero, or non-zero) and then

$$L = E(s)(H(s)+J(s)), \; M = 0, \; N(s) = E(s)(H(s)-J(s))$$

Therefore, we have a complete intrinsic characterization of these surfaces. Notice that the generalized cylinders are included too. They may be considered as surfaces of revolution with axis at infinity. For the generalized cylinders we may allow $J(s)$ to be zero at some points and therefore along the generators containing these points.

6. HELICOIDAL SURFACES AND SURFACES OF REVOLUTION WITH RATIO OF PRINCIPAL CURVATURES CONSTANT

A) The Helicoidal surfaces with ratio of principal curvatures constant were studied in Baikoussis & Koufogiorgos 1997, because of their special properties. They were characterized implicitely by means of a first order differential equation. Here we apply the previous theory and we are going to characterize them again by a first order differential equation.

When the principal curvatures $a > c$ satisfy

$$a = \lambda c \ , \quad \lambda \text{ constant and } c \neq 0 \ , \qquad \text{we also assume that}$$
$$\lambda \neq 0 \quad \text{(case of non-flat helicoidal surfaces)}$$
$$\lambda \neq 1 \quad \text{(there are no umbilic points)}$$
$$\lambda \neq -1 \quad \text{(case of non-minimal helicoidal surfaces)}$$

Then $J = \dfrac{\lambda - 1}{2} c > 0$ and $H = \dfrac{\lambda + 1}{2} c$. Therefore

$$H = \mu J \ , \quad \mu = \frac{\lambda + 1}{\lambda - 1} \text{ constant} \ , \ \mu \neq -1, 0, 1, \pm\infty$$

Hence, the first equation of (26) gives

$$\cot \psi = \gamma J^{\mu} \ , \ \gamma > 0 \text{ constant and } \ 0 < \psi < \frac{\pi}{2}$$

and the second equation of (26)

$$E = \frac{C(1 + \gamma^2 J^{2\mu})}{2\gamma J^{\mu+1}} \ , \ (C > 0 \text{ constant})$$

The Gaussian curvature is

$$K = H^2 - J^2 = (\mu^2 - 1) J^2$$

Therefore the Gauss equation becomes

$$\left[\ln \left(\frac{1 + \gamma^2 J^{2\mu}}{J^{\mu+1}} \right)\right]'' = (1 - \mu^2) \frac{C}{\gamma} \cdot \frac{1 + \gamma^2 J^{2\mu}}{J^{\mu-1}}$$

The solution(s) of this equation will determine $J = J(s)$ and then $E(s)$, $\psi(s)$, $H(s)$, $L(s)$, $M(s)$, $N(s)$, in the way we have already seen several times before. To solve this equation for $J > 0$ when $\mu \neq \pm 1, 0$ is very hard. We can expand it to

$$\frac{\gamma^2[(1 + \gamma J^{2\mu})(J^{2\mu})'' - [(J^{2\mu})']^2]}{(1 + \gamma^2 J^{2\mu})^2} - (\mu + 1) \frac{J J'' - (J')^2}{J^2} = (1 - \mu^2) \frac{C}{\gamma} \cdot \frac{(1 + \gamma^2 J^{2\mu})^2}{J^{\mu-1}}$$

After some trivial simplification we can reduce the order by one by making the standard transformation

$$J' = \frac{dJ}{ds} = y(J)$$

Then

$$J'' = \frac{d^2J}{ds^2} = \frac{dy}{dJ} \cdot \frac{dJ}{ds} = y \cdot \frac{dy}{dJ} = \frac{d}{dJ}(\frac{y^2}{2})$$

The resulting first order differential equation characterizes these surfaces implicitely. Its complete solution seems to be illusive. However, one may want to try some specific combinations of μ, γ, C lest he comes up with an equation easy to solve. (Say $\mu = \pm\frac{1}{2}$, $\gamma = 1$, $C = \frac{2}{3}$, etc.) Finding J by solving this equation, we find L, M, N, that is, all fundamental quantities of the surface.

B) The same problem for the surfaces of revolution is much easier because $\psi = 0$, mod $\frac{\pi}{2}$ and the equations can be explicitly integrated. (Also see Baikoussis & Koufogiorgos 1997 and Kühnel 1981, Baikoussis & Koufogiorgos 1997.)

Again $H = \mu J$, $K = (\mu^2 - 1)J^2$. Equation (20) gives

$$[\ln(JE)]' = (\ln J^{\pm\mu})'$$

and so

$$E = \gamma J^{\pm\mu-1} , \qquad \gamma > 0 \text{ constant}$$

Then the Gauss equation becomes

$$(\ln J)'' = -2\gamma(\pm\mu + 1)J^{\pm\mu+1}$$

This is an equation for $J > 0$ of the form

$$(\ln J)'' = -2AJ^{\alpha} \ , \ A, \ \alpha \text{ constants and } A \cdot \alpha \neq 0$$

It can be solved by making the transformation

$$J = \frac{1}{(\sqrt{|A\alpha|}f)^{\frac{2}{\alpha}}} \ , \qquad f > 0 \text{ function}$$

$$ff'' - (f')^2 = \begin{cases} +1, & \text{if } \ A\alpha > 0 \\ -1, & \text{if } \ A\alpha < 0 \end{cases}$$

The solutions of these new equations have been found in previous sections and are:

when $+1$, one solution

$$f(s) = \frac{\cosh(ks + c)}{k} \,, \qquad k \neq 0 \,,\ c \text{ constants}$$

when -1 , three solutions

$$f(s) = \frac{\sigma s + c}{\sigma} \,, \qquad \sigma \neq 0 \,,\ c \text{ constants}$$

$$f(s) = \frac{\sin(\sigma s + c)}{\sigma} \,, \qquad \sigma \neq 0 \,,\ c \text{ constants}$$

$$f(s) = \frac{\sin h(\sigma s + c)}{\sigma} \,, \qquad \sigma \neq 0 \,, c \text{ constants}$$

In all these solutions s extends in the maximal intervals so that the corresponding solutions stay positive.

Hence, in any case $J(s)$ is explicitly determined and then we get $H(s)$, $K(s)$, $E(s)$, $L(s)$, $M = 0$, $N(s)$ explicitly. So, we obtain an explicit determination of all fundamental quantities of the surface. We observe that in the first case $(+1)$ the resulting surface could be complete, since $-\infty < s < +\infty$. In the second case (-1) the surfaces are not complete, since s is not allowed to run from $-\infty$ to $+\infty$. In the former case the completeness of the surfaces depends on the behavior of $E(s)$ at $\pm\infty$. But in the latter case the surfaces are not complete for sure.

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