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MATHEMATICAL SCIENCES

The Transmuted Marshall-Olkin Extended Lomax Distribution

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Abstract: The transmuted family of distributions has been receiving increased attention over the last few years. In this paper, we generalize the Marshall-Olkin extended Lomax distribution using the quadratic rank transmutation map to obtain the transmuted Marshall-Olkin extended Lomax distribution. Several properties of the new distribution are discussed including the hazard rate function, ordinary and incomplete moments, characteristic function and order statistics. We provide an estimation procedure by the maximum likelihood method and a simulation study to assess the performance of the new distribution. We prove empirically the flexibility of the new model by means of an application to a real data set. It is superior to other three and four parameter lifetime distributions.

Key words: generalized distribution, lifetime analysis, Lomax distribution, Marshall-Olkin extended, transmuted family.

INTRODUCTION

Non-negative random variables are used to model a wide variety of applications in survival analysis, demography, reliability, actuarial study and other areas. For this reason, there is a growing interest in constructing new distributions with positive real support to model lifetime data in several fields. One of the most useful methods to generate new distributions is the integral transform of existing distributions, usually referred to as generalized *G* classes (Tahir & Nadarajah 2015). The principal reason for this is the ability of these generalized distributions to be more flexible than the baseline *G* distribution and therefore provide better fits to skewed data (Pescim et al. 2010). The second reason is the powerful computational facilities available in several analytical platforms, which facilitate handling and computing complex mathematical expressions.

Some of the best known generalized *G* classes of distributions are: the Marshall-Olkin extended (MOE) family (Marshall & Olkin 1997), exponentiated-generated (exp-G) families (Cordeiro et al. 2013, Gupta et al. 1998), beta-generated (beta-G) family (Eugene et al. 2002), Kumaraswamy-generated (Kw-G) family (Cordeiro & Castro 2011), gamma-generated (gamma-G) families (Nadarajah et al. 2015, Ristić & Balakrishnan 2012, Zografos & Balakrishnan 2009), McDonald-generated (Mc-G) family (Alexander et al. 2012) and *T–X* family (Alzaatreh et al. 2013). A detailed compilation of these families can be found in Tahir & Nadarajah (2015).

Shaw & Buckley (2009) pioneered an interesting method by adding a new parameter to an existing distribution that would offer more distributional flexibility. They used the quadratic rank transmutation map (QRTM) to generate a flexible family. The generated class called the *transmuted extended family* includes as a special case the baseline distribution and gives more flexibility to model various types of data. General results for this family and new models are discussed in Bourguignon et al. (2016).

In this paper, we adopt the transmuted generated (T-G) family to define a new distribution called the *transmuted Marshall-Olkin extended Lomax* (TMOELx) distribution by taking the Marshall-Olkin extended Lomax distribution (Ghitany et al. 2007) as the baseline G model. We obtain the TMOELx density function as a linear combination of *exponentiated-Lomax* (exp-Lx) densities. Given that the new distribution has positive real support, our objective is to define a flexible distribution for lifetime applications. Also, we present explicit expressions for the quantile function (qf), moments, characteristic function and order statistics. In addition, we consider a study of the maximum likelihood estimates of the model for complete samples and a simulation study to verify the performance of these estimates. Finally, we consider an application of the TMOELx distribution and compare it with others distributions based on some goodness-of-fit statistics.

THE NEW DISTRIBUTION

The cumulative distribution function (cdf) of the *Lomax distribution*, say Lx(β, γ), also known as the Pareto distribution of the second kind, is

$$R(x) = 1 - (1+\beta x)^{-\gamma},\ x > 0,\ \beta, \gamma > 0, \tag{1}$$

where β and γ are, respectively, the scale and shape parameters.

The *Marshall-Olkin extended Lomax* (MOELx) distribution (Ghitany et al. 2007) is obtained by taking the Lomax distribution (1) as the baseline model in the MOE family. Its cdf has the form

$$G(x) = 1 - \frac{\alpha}{(1+\beta x)^{\gamma} - \bar{\alpha}}, \quad x > 0,\ \alpha, \beta, \gamma > 0,\ \bar{\alpha} = 1 - \alpha. \tag{2}$$

The cdf and probability density function (pdf) of the T-G family are, respectively,

$$T(x) = T(x; \xi, \lambda) = (1+\lambda)G(x;\xi) - \lambda G(x;\xi)^2, \quad \lambda \in [-1, 1] \tag{3}$$

and

$$t(x) = t(x;\xi,\lambda) = [1 + \lambda - 2\lambda G(x;\xi)]g(x;\xi),\ x \in \mathcal{D} \subseteq \mathbb{R}, \tag{4}$$

where $G(x;\xi)$ and $g(x;\xi)$ are, respectively, the baseline cdf and pdf, and ξ is the parameter vector of the baseline distribution. For $\lambda = 0$, it reduces to the baseline model. In Bourguignon et al. (2016), the authors proved that the pdf (4) can be expressed as a linear combination of exp-G densities.

Based on the T-G family and the MOELx distribution, we propose a new four-parameter distribution so-called the TMOELx distribution. By inserting (2) as the baseline distribution in equation (3), the cdf of the TMOELx distribution (for $x > 0$) can be expressed as

$$F(x) = \frac{[(1+\beta x)^{\gamma} - 1][(1+\beta x)^{\gamma} + \alpha\lambda - \bar{\alpha}]}{[(1+\beta x)^{\gamma} - \bar{\alpha}]^2}, \tag{5}$$

whereas its pdf is

$$f(x) = \frac{\alpha\beta\gamma\,(1+\beta x)^{\gamma-1}\{[1-(1+\beta x)^{\gamma}](\lambda-1)+\alpha(\lambda+1)\}}{[(1+\beta x)^{\gamma}-\bar{\alpha}]^{3}}, \tag{6}$$

where $\alpha > 0,\ \beta > 0,\ \gamma > 0$ and $\lambda \in [-1, 1]$. Hereafter, a random variable X having pdf (6) will be denoted by $X \sim \mathrm{TMOELx}(\alpha, \beta, \gamma, \lambda)$.

In lifetime analysis, a useful function is the hazard rate function (hrf) $h(x)$. So, the hrf of X is given by

$$h(x) = \frac{\beta\gamma(1+\beta x)^{\gamma-1}\{(\lambda-1)[(1+\beta x)^{\gamma}-1]-\alpha(1+\lambda)\}}{[(1+\beta x)^{\gamma}-\bar{\alpha}]\{(\lambda-1)[(1+\beta x)^{\gamma}-1]-\alpha\}}. \tag{7}$$

For selected values of the parameters α, β, γ and λ, some sub-models of the TMOELx distribution published in the literature are listed in Table I.

Table I. Some TMOELx sub-models. MOELx: Marshall-Olkin extended Lomax, TLx: Transmuted Lomax, Lx: Lomax.

α	β	γ	λ	Model	Reference
α	β	γ	0	MOELx (α, β, γ)	Ghitany et al. (2007)
1	β	γ	λ	TLx (β, γ, λ)	Ashour & Eltehiwy (2013)
1	β	γ	0	Lx (β, γ)	Lomax (1954)

SHAPES OF THE DENSITY AND HAZARD RATE FUNCTIONS

The shapes of the pdf (6) can be described analytically by examining the roots of the equation $f'(x) = 0$ and analyzing its limits when $x \to 0$ or $x \to \infty$. Since $f(x)$ is the pdf of a continuous random variable, then $\lim_{x\to\infty} f(x) = 0$. Further, we have

$$\lim_{x\to 0} f(x) = \frac{\beta\gamma(\lambda+1)}{\alpha}$$

and, therefore, $\lim_{x\to 0} f(x) = 0$ if and only if $\lambda = -1$. Some plots of the TMOELx pdf, for different parameter values, are displayed in Figure 1. These plots reveal that the pdf of X can be strictly decreasing or unimodal with mode $x = x_0$ at

$$x_0 = \frac{1}{\beta}\left[-1 + \left(\frac{-1+\alpha+\lambda+2\alpha\gamma\lambda-\{\alpha^2\lambda^2+4\alpha\gamma\lambda\,(-1+\alpha+\lambda)+\gamma^2[2\alpha\,(\lambda-1)+(\lambda-1)^2+\alpha^2(3\lambda^2+1)]\}^{1/2}}{(1+\gamma)(\lambda-1)}\right)^{1/\gamma}\right].$$

Further, we obtain the conditions of the behavior of the TMOELx density in terms of the parameters. In fact, from (4), the density $t(x)$ is decreasing when $t'(x) = g'(x)[1+\lambda-2\lambda\,G(x)] - 2\lambda\,g^2(x) < 0$ for all $x > 0$, where $G(x)$ and $g(x)$ are, respectively, the cdf and pdf of the MOELx distribution. So, $t(x)$ is decreasing when

$$\frac{g'(x)}{g(x)}[1+\lambda-2\lambda\,G(x)] < 2\lambda\,g(x), \quad \forall x > 0.$$

After some algebraic manipulation and considering $x \to 0^+$, the above inequality is equivalent to

$$\alpha + 2\,\gamma - \alpha\gamma + \alpha\lambda + 4\,\gamma\lambda - \alpha\gamma\lambda > 0.$$

Therefore, $t(x)$ is unimodal if and only if

$$\alpha + 2\,\gamma - \alpha\gamma + \alpha\lambda + 4\,\gamma\lambda - \alpha\gamma\lambda < 0.$$

Thus, the parameters α, γ and λ control the shapes of the pdf, while β is the scale parameter. The shape parameters allow extensive control on the right tail, providing more heavy (light) tails when α increases (decreases) and γ and λ decrease (increase).

The corresponding hrf can have shapes such as decreasing and unimodal as shown in Figure 2. Thus, the new distribution can be appropriate for different applications in lifetime analysis. For the conditions of the behavior of the hrf $h(x)$, note that $h'(x) = \frac{t'(x)[1-T(x)]+t^2(x)}{[1-T(x)]^2}$ and, therefore, $h(x)$ is decreasing if and only if

$$t'(x)[1 - T(x)] + t^2(x) < 0, \quad \forall x.$$

From (4) and considering $x \to 0^+$, the above inequality is equivalent to

$$\alpha(\gamma - 1)(\lambda + 1) + \gamma(\lambda^2 - 2\,\lambda - 1) < 0.$$

Then, $h(x)$ is unimodal if and only if

$$\alpha(\gamma - 1)(\lambda + 1) + \gamma(\lambda^2 - 2\,\lambda - 1) > 0.$$

Thus, the parameters α, γ and λ control the shapes of the hrf of X.

USEFUL EXPANSIONS

We can obtain a power series for the cdf of the TMOEL distribution from eqs. (1), (2) and (3) using the generalized binomial expansion (see appendix A)

$$F(x) = \sum_{k=0}^{\infty} p_k\, H_{k+1}(x), \tag{8}$$

where $H_{k+1}(x) = R^{k+1}(x)$ is the exp-Lx cdf with power parameter $k + 1$. The coefficients are

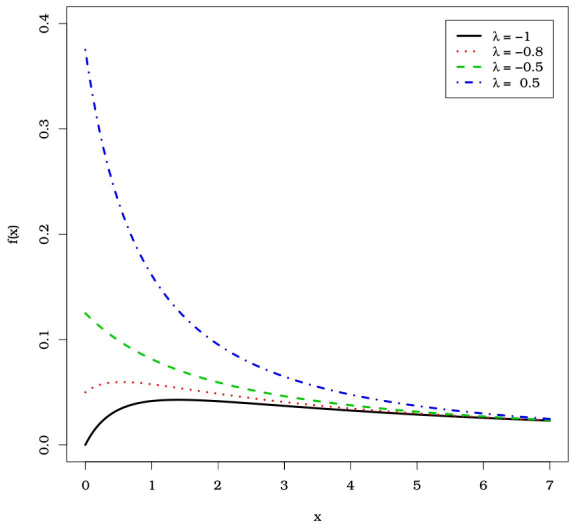
$$p_k = \begin{cases} (1+\lambda)\, b_k - \lambda\, c_k, & \text{if } 0 < \alpha \leq 1/2, \\ a_k, & \text{if } \alpha > 1/2, \end{cases}$$

and

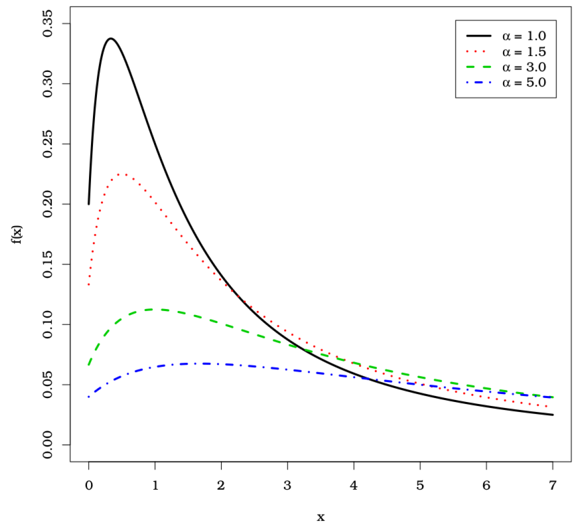
$$a_k = \frac{[(1+\lambda)(\alpha - 1) - k\lambda]}{\alpha^2} \left(\frac{\alpha - 1}{\alpha}\right)^{k-1}, \; k = 0, 1, 2, \ldots$$

$$b_k = \sum_{j=k}^{\infty} \binom{j}{k} (1-\alpha)^j, \; k = 0, 1, 2, \ldots$$

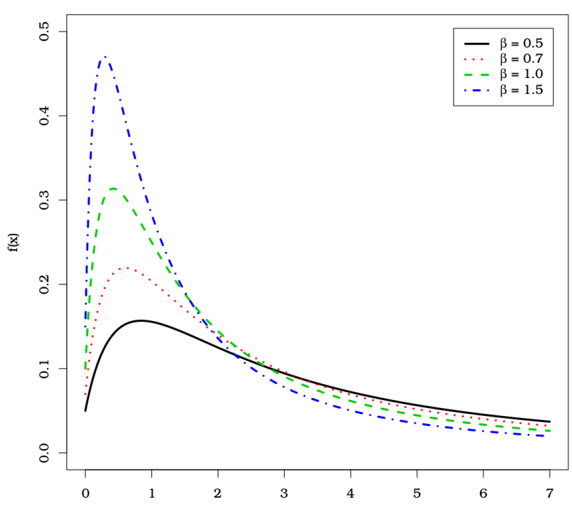
$$c_0 = b_0^2 \text{ and } c_m = \frac{1}{m\, b_0} \sum_{k=1}^{m} (3k - m)\, b_k\, c_{m-k}, \; m = 1, 2, \ldots$$



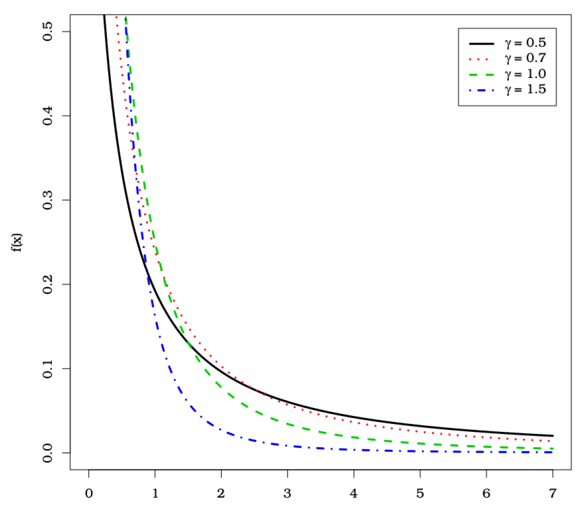
(a) $\alpha = 2.0,\ \beta = 1.0$ and $\gamma = 0.5$



(b) $\beta = \gamma = 1.0$ and $\lambda = -0.8$

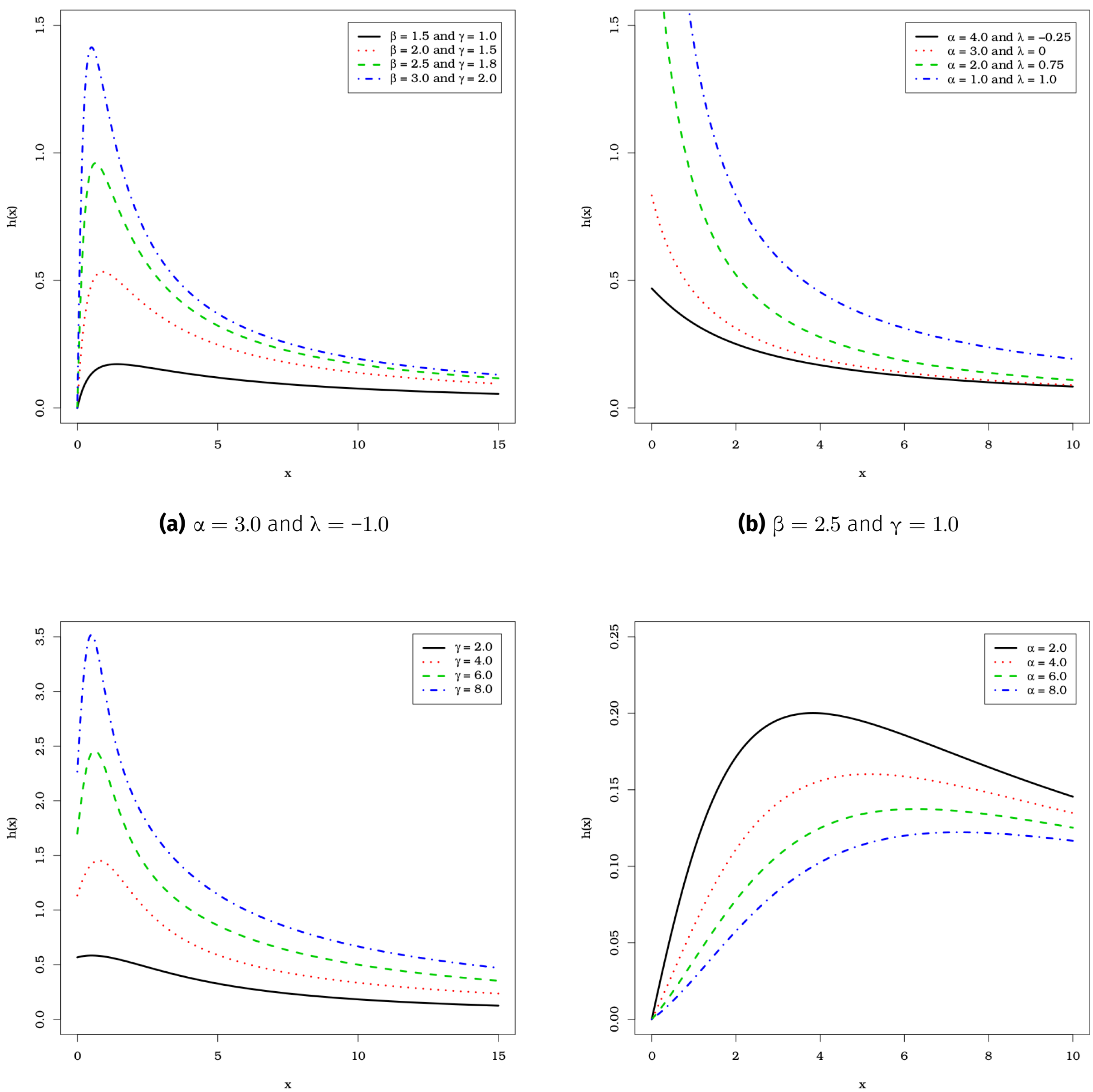


(c) $\alpha = \gamma = 1.0$ and $\lambda = -0.9$



(d) $\alpha = \beta = 4.0$ and $\lambda = 0.9$

Figure 1. Plots of the pdf (6) for selected parameters.



(a) $\alpha = 3.0$ and $\lambda = -1.0$

(b) $\beta = 2.5$ and $\gamma = 1.0$

(c) $\alpha = 3.0,\ \beta = 0.5$ and $\lambda = 0.7$

(d) $\beta = 0.5,\ \gamma = 2.0$ and $\lambda = -1$

Figure 2. Plots of the hrf (7) for selected parameters.

Differentiating (8) gives

$$f(x)=\sum_{k=0}^{\infty} p_k\, h_{k+1}(x), \tag{9}$$

where $h_{k+1}(x)=\frac{d}{dx}H_{k+1}(x)$ is the exp-Lx density with power parameter $k+1$.

Equation (9) reveals that the pdf of X can be expressed as a linear combination of exp-Lx densities. Thus, some structural properties of the TMOELx distribution can be determined from those of the exp-Lx distribution (Salem 2014).

QUANTILE FUNCTION

Since the cdf $F(x)$ given in (5) is continuous and strictly increasing, the qf of X is $Q(u)=F^{-1}(u)$, for $0<u<1$. From Bourguignon et al. (2016), we obtain the qf of the TMOELx distribution as

$$Q(u;\xi,\lambda)=\begin{cases} Q_G\left(\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda u}}{2\lambda};\xi\right), & \text{if } \lambda\neq 0,\\ Q_G(u;\xi), & \text{if } \lambda=0,\end{cases} \tag{10}$$

where $\xi=(\alpha,\beta,\gamma)^\top$ is the parameter vector and $Q_G(u;\xi)$ is the qf of the MOELx distribution

$$Q_G(u;\xi)=\beta^{-1}\left[\left(1-\frac{\alpha u}{u-1}\right)^{1/\gamma}-1\right].$$

Using (10), we can generate random numbers from the TMOELx distribution as follows. If $U\sim\mathcal{U}(0,1)$, then

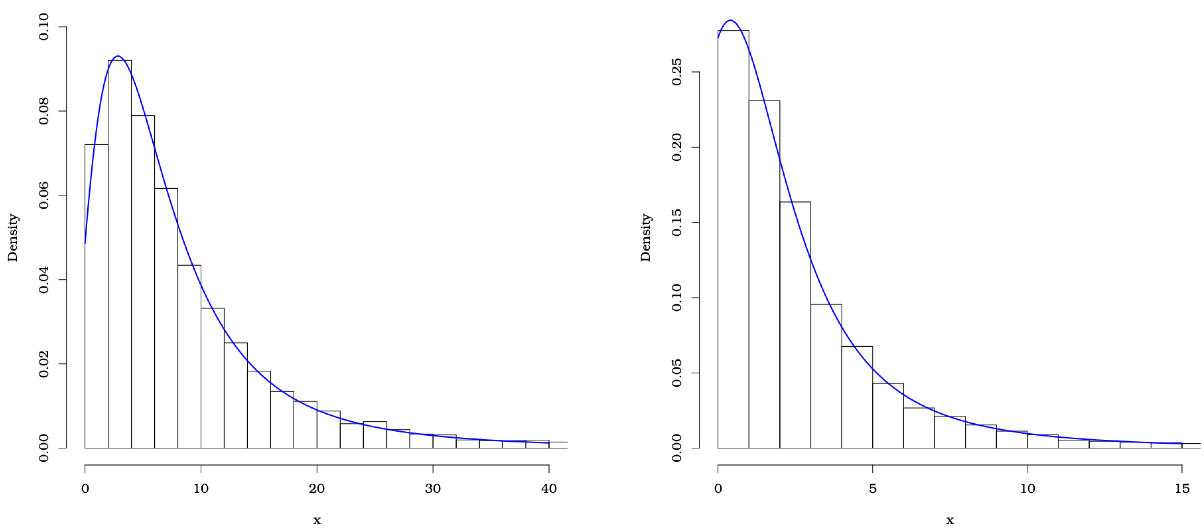
$$Q(U;\alpha,\beta,\gamma,\lambda)\sim \mathrm{TMOELx}\,(\alpha,\beta,\gamma,\lambda).$$

Another alternative to generate random numbers from the TMOELx distribution can be based on random extrema in transmuted distributions, which are given in Kozubowski & Podgórski (2016) (Proposition 2.1). Let X_1 and X_2 be i.i.d. random variables from the MOELx distribution and let N_p be an integer-valued random variable such as $N_p-1\sim\mathrm{Bernoulli(p)}$, $0\leq p\leq 1$. Further, suppose that N_p and X_i, $i=1,2$, are independent random variables. Then, an observation y from the TMOELx distribution can be generated in the following way:

1. Generate u_1 and u_2 independently from the uniform distribution $\mathcal{U}(0,1)$.
2. Calculate $x_i=Q_G(u_i)$, $i=1,2$.
3. If $\lambda\in[-1,0]$, define $p=-\lambda\in[0,1]$ and generate n_p-1 from the $\mathrm{Bernoulli}(p)$ distribution.
4. Then, obtain $y=\vee_{i=1}^{n_p}x_i$.
5. If $\lambda\in[0,1]$, define $p=\lambda$ and generate n_p-1 from the $\mathrm{Bernoulli}(p)$ distribution.
6. Finally, obtain $y=\wedge_{i=1}^{n_p}x_i$.

For $\lambda = 0$ in equation (3), we have $n_p = 1$ almost surely and the steps 4 and 6 are satisfied simultaneously with their right-hand-sides reducing to x_1. Further, in the extreme cases $\lambda = \pm 1$, we have $n_p = 2$ almost surely and the steps 4 and 6 are reduced to $y = \max(x_1, x_2)$ and $y = \min(x_1, x_2)$, respectively.

In Figure 3, we compare the exact TMOELx densities and histograms from two simulated data sets for selected parameters which show the consistent of the simulated values from the above algorithm with the TMOELx distribution. We simulate the data using the R software (version 3.2.3).



(a) $\alpha = 2.5,\ \beta = 0.15,\ \gamma = 2.7,\ \lambda = -0.7$ **(b)** $\alpha = 5.5,\ \beta = 0.5, \gamma = 2.5,\ \lambda = 0.2$

Figure 3. Plots of the exact TMOELx densities and histograms of the simulated data for some parameter values.

Skewness and kurtosis

Useful skewness and kurtosis measures are given by $\alpha_3 = \mu_3/\varsigma^3$ and $\alpha_4 = \mu_4/\varsigma^4$, respectively, where μ_j is the j-th central moment and ς is the standard deviation.

For some distributions in the T-G family, it could be difficult to find the third and fourth moments. Alternative measures for the skewness and kurtosis based on quantiles are sometimes more appropriate. The measure of skewness S of Bowley and the measure of kurtosis K of Moors are given by

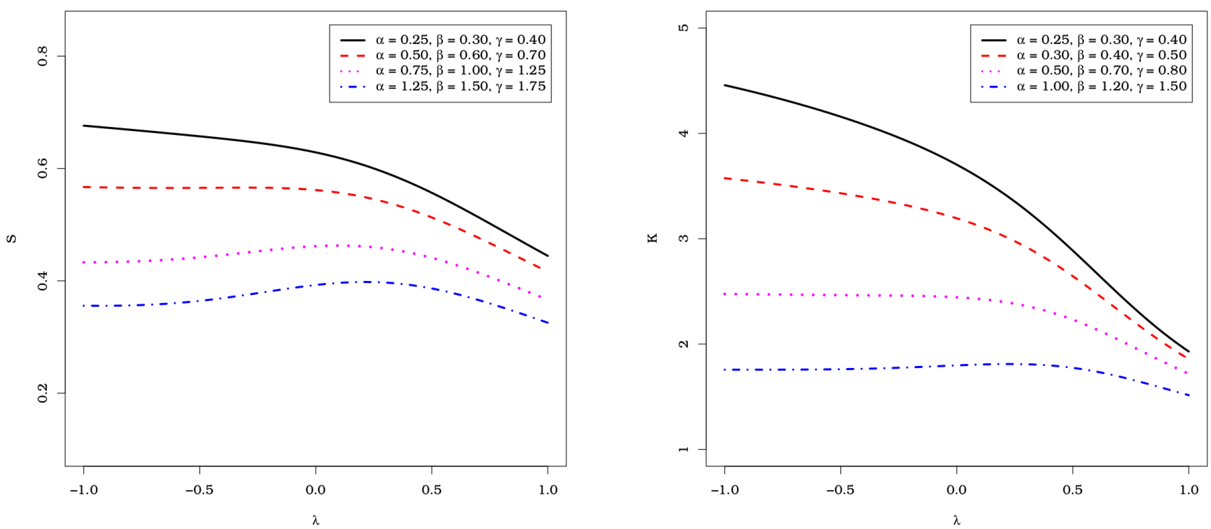
$$S = \frac{Q(6/8) + Q(2/8) - 2\,Q(4/8)}{Q(6/8) - Q(2/8)} \tag{11}$$

and

$$K = \frac{Q(7/8) - Q(5/8) + Q(3/8) - Q(1/8)}{Q(6/8) - Q(2/8)}, \tag{12}$$

respectively, where $Q(\cdot)$ is given by (10). These measures exist even for distributions without moments.

The plots in Figure 4 display the skewness (11) and kurtosis (12) as functions of λ for some parameter values. They reveal that the skewness and kurtosis of X decrease rapidly when λ converges to one.



(a) Skewness **(b)** Kurtosis

Figure 4. Plots of the skewness and kurtosis for selected parameters.

MOMENTS AND CHARACTERISTIC FUNCTION

Moments are important in any statistical analysis. For example, some characteristics of a distribution can be described using measures such as the mean, variance, skewness and kurtosis, which are determined from the first four ordinary moments.

For $r \in \mathbb{N}$, let $\mu'_r = E(X^r)$ be the r-th ordinary moment of X. From equation (9), we can express μ'_r as a linear combination of the r-th ordinary moments of exp-Lx random variables. In fact, for $r < \gamma$,

$$\mu'_r = \sum_{k=0}^{\infty} p_k \, E(Y_k^r), \tag{13}$$

where $Y_k \sim \mathrm{exp\text{-}Lx}(k+1, \beta, \gamma)$.

The r-th ordinary moment of Y_k is given by Salem (2014) (for $r < \gamma$) as

$$E(Y_k^r) = \frac{(k+1)}{\beta^r} \sum_{j=0}^{r} (-1)^j \binom{r}{j} B\left(1 - \frac{1}{\gamma}(r-j), k+1\right), \tag{14}$$

where $B(a,b) = \Gamma(a)\,\Gamma(b)/\Gamma(a+b)$ is the beta function and $\Gamma(\cdot)$ is the gamma function. Inserting (14) in equation (13) gives (for $r < \gamma$)

$$\mu'_r = \sum_{k=0}^{\infty}\sum_{j=0}^{r} b_r(k,j)\, B\left(1 - \frac{1}{\gamma}(r-j), k+1\right), \qquad (15)$$

where $b_r(k,j) = (-1)^j \binom{r}{j} \frac{(k+1)}{\beta^r}\, p_k$. From equation (15), we note that $\mu'_r < \infty$ for $r < \gamma$, a condition that also holds for the Lomax distribution.

Numerical results

In this section, we use the expansions for the r-th ordinary moment of X to compare the numerical results (for some parameter values) for the mean, variance, skewness and kurtosis of the TMOELx distribution obtained by the methods of truncation, numerical integration and Monte Carlo simulation ($50,000$ replications). For reasons of simplicity, we consider $\alpha > 1/2$ since, in this case, the expansions for the moments are easier to obtain.

For the truncation method, the r-th truncated ordinary moment of X follows from (15) as

$$\mu'_{r,N} = \sum_{k=0}^{N}\sum_{j=0}^{r} b_r(k,j)\, B\left(1 - \frac{1}{\gamma}(r-j), k+1\right), \quad r < \gamma,\ N \in \mathbb{N}.$$

For the truncation and Monte Carlo methods, we use the `Ox` plataform (version 7.10, see Doornik (2007)). For the numerical integration method, we adopt algorithms in the `Mathematica` software for recursively subdivide the integration region.

The results for the three methods are given in Table II. For the truncation method, we consider the values $N = 5,\ 10$ and 20. We note that the values for the truncation method are more accurate when N increases in agreement with $\mu'_r = \lim_{N\to\infty} \mu'_{r,N}$. Further, these values can be considered sufficiently precise when $N = 20$ compared with those values from numerical integration method. For the Monte Carlo method, the results are less accurate when we consider large-order moments.

The script (in `Ox` language) for calculating the values corresponding to truncation and Monte Carlo methods in Table II is given in Appendix B.

Incomplete moments

The r-th incomplete moment of X is determined from (9) as

$$m_X^{(r)}(z) = \int_0^z x^r f(x)\, dx = \sum_{k=0}^{\infty} p_k\, m_{Y_k}^{(r)}(z),$$

where

$$m_{Y_k}^{(r)}(z) = \int_0^z x^r\, h_{k+1}(x)\, dx = \beta\,\gamma\,(k+1) \int_0^z x^r\, [1-(1+\beta x)^{-\gamma}]^k\, (1+\beta x)^{-\gamma-1}\, dx$$

is the r-th incomplete moment of Y_k.

Table II. Results for the mean, variance, skewness and kurtosis for some parameter values.

Parameter	Method	N	mean	variance	skewness	kurtosis
α=0.75	Truncation	5	0.7356	0.7407	5.1369	134.01
β=0.5		10	0.6960	0.6916	5.2047	138.12
γ=4.5		20	0.6963	0.6923	5.2040	138.06
λ=-0.75	Num. int.		0.6963	0.6923	5.2040	138.07
	Monte Carlo		0.6949	0.7069	5.7817	99.360
α=1.5	Truncation	5	0.3861	0.1688	3.8940	55.140
β=1.0		10	0.3933	0.1715	3.8878	54.934
γ=5.0		20	0.3934	0.1715	3.8877	54.930
λ=-0.5	Num. int.		0.3934	0.1715	3.8877	54.930
	Monte Carlo		0.3927	0.1735	4.3059	56.020
α=3.0	Truncation	5	0.1621	0.0344	3.1759	32.507
β=2.0		10	0.2060	0.0402	3.0934	31.455
γ=5.5		20	0.2174	0.0420	3.0889	31.254
λ=-0.25	Num. int.		0.2178	0.0421	3.0887	31.238
	Monte Carlo		0.2175	0.0424	3.3848	35.354
α=5.0	Truncation	5	0.0956	0.0117	2.8857	24.900
β=3.0		10	0.1239	0.0137	2.7908	23.949
γ=6.0		20	0.1314	0.0143	2.7898	23.838
λ=0.25	Num. int.		0.1304	0.0142	2.7891	23.886
	Monte Carlo		0.1302	0.0142	3.0287	27.900
α=7.0	Truncation	5	0.0587	0.0043	2.4756	17.285
β=4.0		10	0.0820	0.0051	2.3795	16.737
γ=7.0		20	0.0888	0.0053	2.4155	16.961
λ=0.5	Num. int.		0.0828	0.0048	2.3729	17.031
	Monte Carlo		0.0827	0.0048	2.5454	20.037

Using the binomial expansion gives

$$[1-(1+\beta x)^{-\gamma}]^k = \sum_{j=0}^{k}(-1)^j \binom{k}{j}(1+\beta x)^{-j\gamma}.$$

By replacing this expansion in $m_{Y_k}^{(r)}(z)$ and interchanging $\sum_{k=0}^{\infty}\sum_{j=0}^{k}$ by $\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}$, we have

$$m_X^{(r)}(z) = \sum_{j=0}^{\infty} e_j\, m_{V_j}^{(r)}(z),$$

where $V_j \sim \mathrm{Lx}(\beta, (j+1)\gamma)$, $e_j = \sum_{k=j}^{\infty} (-1)^j \binom{k}{j} \frac{(k+1)}{(j+1)}\, p_k$,

$$m_{V_j}^{(r)}(z) = \frac{(j+1)\beta\,\gamma\, z^{r+1}\,{}_2F_1(r+1, 1+(j+1)\gamma; r+2; -\beta z)}{r+1},$$

and ${}_pF_q\big(a_1, \ldots, a_p; b_1, \ldots, b_q; x\big)$ is the hypergeometric function (Gradshteyn & Ryzhik 2007).

Characteristic function

The generating and characteristic functions are useful tools, since they can be used for computing the moments and cumulants of a distribution. For the Lomax distribution, the generating function is defined only for $t \leq 0$. Consequently, the TMOELx generating function is also defined only for $t \leq 0$. However, the characteristic function (chf) of a distribution exists for all $t \in \mathbb{R}$. The chf of X is

$$\varphi_X(t) = E(\mathrm{e}^{itX}) = \int_0^{\infty} \mathrm{e}^{itx} f(x)\, dx,$$

where $i = \sqrt{-1}$ and $t \in \mathbb{R}$.

Using equation (9), we obtain

$$\varphi_X(t) = \sum_{k=0}^{\infty} p_k\, \varphi_{Y_k}(t).$$

Further, using equation (16), we can write

$$\varphi_X(t) = \sum_{j=0}^{\infty} e_j\, \varphi_{V_j}(t), \tag{16}$$

where

$$\varphi_{V_j}(t) = (j+1)\,\gamma\, \mathrm{e}^{-it/\beta} (-it)^{(j+1)\gamma} \beta^{-(j+1)\gamma}\, \Gamma\left(-(j+1)\gamma, -\frac{it}{\beta}\right) \tag{17}$$

and $\Gamma(s,z) = \int_z^{\infty} t^{s-1}\, \mathrm{e}^{-t}\, dt$ is the upper incomplete gamma function, which, by analytic continuation, is defined for almost all combinations of complex s and z. Replacing (17) in equation (16) and setting

$$d_j(t) = (j+1)\,\gamma\, e_j\, \mathrm{e}^{-i\,t/\beta} (-i\,t)^{(j+1)\gamma} \beta^{-(j+1)\gamma},$$

we obtain

$$\varphi_X(t) = \sum_{j=0}^{\infty} d_j(t)\, \Gamma\left(-(j+1)\,\gamma, -\frac{i\,t}{\beta}\right). \tag{18}$$

Equation (18) is the main result of this section.

ORDER STATISTICS

Let $X_1,\dots,X_n$ be a random sample of size n from a distribution $F(x)$. Then, for $1\le m\le n$, the pdf of the m-th order statistic, $X_{(m)}$, can be expressed as (Severini 2005)

$$f_{(m)}(x)=M\,F(x)^{m-1}(1-F(x))^{n-m}f(x)=M\,f(x)\sum_{j=0}^{n-m}(-1)^j\binom{n-m}{j}F(x)^{m+j-1},$$

where $M=n!/[(m-1)!\,(n-m)!]$.

Based on (8) and using the expansion for a power series raised to positive integer powers (Gradshteyn & Ryzhik 2007), we have

$$F(x)^{m+j-1}=\left(\sum_{k=0}^{\infty}p_k\,H_{k+1}(x)\right)^{m+j-1}=\sum_{k=0}^{\infty}c_{m+j-1,k}\,H_{k+1}(x),$$

where $c_{m+j-1,0}=p_0^{m+j-1}$ and, for $i\ge 1$,

$$c_{m+j-1,i}=\frac{1}{i\,p_0}\sum_{s=1}^{i}[(m+j)s-i]\,p_s\,c_{m+j-1,i-s}.$$

Therefore, we obtain

$$f_{(m)}(x)=Mf(x)\sum_{j=0}^{n-m}\sum_{k=0}^{\infty}(-1)^j\binom{n-m}{j}c_{m+j-1,k}\,H_{k+1}(x),$$

Replacing $f(x)$ by the expansion (9) in the last equation and, after some algebraic manipulation, we can writte

$$f_{(m)}(x)=M\sum_{j=0}^{n-m}\sum_{k,\ell=0}^{\infty}(-1)^j\binom{n-m}{j}\frac{(\ell+1)}{(\ell+k+2)}\,c_{m+j-1,k}\,p_\ell\,h_{\ell+k+2}(x).$$

We note that $f_{(m)}(x)$ in the above equation is a linear combination of exp-Lx densities. So, some properties of the TMOELx order statistics can be easily obtained from those of the exp-Lx.

MAXIMUM LIKELIHOOD ESTIMATION

In this section, we consider the estimation of the parameters of the TMOELx distribution by the maximum likelihood method. Let $x=(x_1,\dots,x_n)^\top$ be a sample of size n from $X\sim\mathrm{TMOELx}(\alpha,\beta,\gamma,\lambda)$ and $\vartheta=(\alpha,\beta,\gamma,\lambda)^\top$ be the parameter vector. The log-likelihood for ϑ, denoted by $\ell(\vartheta)$, is given by

$$\begin{aligned}\ell(\vartheta)&=n[\log(\alpha)+\log(\beta)+\log(\gamma)]+(\gamma-1)\sum_{i=1}^{n}\log(1+\beta x_i)+\sum_{i=1}^{n}\log\{[1-(1+\beta x_i)^\gamma](\lambda-1)\\&+\alpha(\lambda+1)\}-3\sum_{i=1}^{n}\log[(1+\beta x_i)^\gamma-\bar{\alpha}].\end{aligned}\qquad(19)$$

The MLE $\hat{\vartheta}$ of ϑ can be obtained by maximizing (19) directly by using the `SAS` (`PROC NLMIXED`), `R` (`optim` and `MaxLik` functions) and `Ox` program (`MaxBFGS` sub-routine).

Alternatively, the components of the score vector $U_\vartheta = (U_\alpha, U_\beta, U_\gamma, U_\lambda)^\top$ are

$$U_\alpha = \frac{\partial \ell(\vartheta)}{\partial \alpha} = n\alpha^{-1} - 3\sum_{i=1}^{n}(\alpha - 1 + u_i^\gamma)^{-1} + (\lambda+1)\sum_{i=1}^{n}[\alpha(\lambda+1) + (\lambda-1)(1-u_i^\gamma)]^{-1},$$

$$U_\beta = \frac{\partial \ell(\vartheta)}{\partial \beta} = n\beta^{-1} + \gamma(\gamma-1)\sum_{i=1}^{n} x_i u_i^{-1} - 3\gamma\sum_{i=1}^{n} x_i u_i^{\gamma-1}(\alpha - 1 + u_i^\gamma)^{-1}$$
$$- \gamma(\lambda-1)\sum_{i=1}^{n} x_i u_i^{\gamma-1}[\alpha(\lambda+1) + (\lambda-1)(1-u_i^\gamma)]^{-1},$$

$$U_\gamma = \frac{\partial \ell(\vartheta)}{\partial \gamma} = n\gamma^{-1} + (\gamma-1)\sum_{i=1}^{n}(\log u_i) + \sum_{i=1}^{n}(\log u_i^\gamma) - 3\sum_{i=1}^{n} u_i^\gamma(\log u_i)(\alpha - 1 + u_i^\gamma)^{-1}$$
$$- (\lambda-1)\sum_{i=1}^{n} u_i^\gamma(\log u_i)[\alpha(\lambda+1) + (\lambda-1)(1-u_i^\gamma)]^{-1},$$

$$U_\lambda = \frac{\partial \ell(\vartheta)}{\partial \lambda} = \sum_{i=1}^{n}(1 + \alpha - u_i^\gamma)[\alpha(\lambda+1) + (\lambda-1)(1-u_i^\gamma)]^{-1},$$

where $u_i = 1 + \beta x_i$.

The MLE $\hat{\vartheta}$ can also be determined by solving the nonlinear equations $U_\alpha = U_\beta = U_\gamma = U_\lambda = 0$ simultaneously. In this case, these equations should be evaluated numerically using Newton-Raphson algorithms.

Under general regularity conditions, we have $(\hat{\vartheta} - \vartheta) \overset{a}{\sim} N_4(0, K(\vartheta)^{-1})$, where $K(\vartheta)$ is the 4×4 expected information matrix and $\overset{a}{\sim}$ denotes asymptotic distribution. For large n, $K(\vartheta)$ can be approximated by the observed information matrix. This normal approximation for the MLE $\hat{\vartheta}$ can be used for determining approximate confidence intervals and for testing hypotheses on the parameters α, β, γ and λ.

Suppose that the parameter vector is partitioned as $\vartheta = (\psi_1^\top, \psi_2^\top)^\top$, where $\dim(\psi_1) + \dim(\psi_2) = \dim(\vartheta)$. The likelihood ratio (LR) statistic for testing the null hypothesis $\mathcal{H}_0 : \psi_1 = \psi_1^{(0)}$ against the alternative hypothesis $\mathcal{H}_1 : \psi_1 \neq \psi_1^{(0)}$ is given by $LR = 2\{\ell(\hat{\vartheta}) - \ell(\tilde{\vartheta})\}$, where $\hat{\vartheta} = (\hat{\psi}_1^\top, \hat{\psi}_2^\top)^\top$, $\tilde{\vartheta} = (\psi_1^{(0)\top}, \tilde{\psi}_2^\top)^\top$, $\hat{\psi}_i$ and $\tilde{\psi}_i$ are the MLEs under the alternative and null hypotheses, respectively, and $\psi_1^{(0)}$ is a specified parameter vector. Based on the first-order asymptotic theory, we know that $LR \overset{a}{\sim} \chi_k^2$ (chi-square distribution with k degrees of freedom), where $k = \dim(\psi_1)$. Therefore, to the significance level ν, we reject $\mathcal{H}_0$ if $LR > \chi^2_{(1-\nu,k)}$, where $\chi^2_{(1-\nu,k)}$ is the quantile $1-\nu$ of the χ_k^2. Thus, we can test sub-models of the TMOELx distribution and analyze how significant are the parameters tested for modeling a given data set.

SIMULATION STUDY

In this section, we perform a Monte Carlo simulation experiment in order to evaluate the behavior of the MLE $\hat{\vartheta} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\lambda})$ and estimate the relative biases and mean squared errors (MSEs) for sample sizes $n = 100, 200$ and 250.

We consider $10,000$ Monte Carlo replications and use the BFGS method in the `Ox` plataform (version 7.10, `MaxBFGS` function) to maximize the log-likelihood function (19). We set the parameter values $\beta = 0.25$, $\gamma = 0.3$ and vary α and λ. Some computational aspects related to the simulation study are detailed in Appendix C.

The results, given in Table III, reveal generally that the relative biases and MSE values decrease when n increases. The minimum absolute values for the relative biases and MSEs are equal to 0.001 and 0.003, respectively, whereas the maximum absolute values for the relative bias and MSE are 1.632 and 4.467, respectively. Moreover, we note in Table III that the parameter λ was underestimated in most cases (negative relative biases).

Table III. Relative biases and MSE values for the MLEs $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\lambda})$ (β=0.25 and γ=0.3).

			relative bias				MSE			
λ	α	n	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$
-0.5	0.25	100	0.954	0.975	0.057	-0.086	0.687	0.932	0.012	0.142
		200	0.549	0.696	0.045	-0.049	0.144	0.337	0.007	0.124
		250	0.471	0.632	0.040	-0.049	0.111	0.301	0.006	0.117
	0.5	100	0.742	0.884	0.021	-0.158	2.155	1.025	0.007	0.153
		200	0.403	0.695	0.020	-0.073	0.521	0.290	0.004	0.139
		250	0.348	0.671	0.020	-0.059	0.225	0.279	0.003	0.133
0.25	0.25	100	1.245	1.632	0.098	-0.349	2.260	3.752	0.021	0.151
		200	0.495	0.751	0.063	-0.312	0.215	0.432	0.013	0.126
		250	0.382	0.570	0.049	-0.272	0.117	0.238	0.010	0.109
	0.5	100	0.802	1.122	0.043	-0.110	3.145	1.638	0.013	0.122
		200	0.376	0.528	0.001	0.062	1.097	0.452	0.008	0.119
		250	0.277	0.429	-0.011	0.076	0.301	0.144	0.007	0.122
0.5	0.25	100	0.931	1.131	0.144	-0.292	1.436	1.867	0.028	0.136
		200	0.379	0.462	0.087	-0.202	0.227	0.297	0.018	0.105
		250	0.308	0.385	0.062	-0.154	0.153	0.221	0.015	0.096
	0.5	100	0.772	1.057	0.101	-0.324	4.467	2.560	0.019	0.121
		200	0.237	0.348	0.040	-0.222	0.609	0.288	0.011	0.101
		250	0.144	0.249	0.022	-0.196	0.379	0.122	0.009	0.098

APPLICATION

In this section, we present two applications of the TMOELx distribution.

First application: We compare the TMOELx distribution with its sub-models: the TLx, MOELx and Lx distributions (see Table I). We use an uncensored data set corresponding to 128 intervals between the times where vehicles pass a point on a road (traffic data). The data are given in Jorgensen (2012).

Since the parameter λ in the TMOELx distribution is such that $|\lambda| \leq 1$, we employ the SQP method (`MaxSQP` function of the `Ox` language) to maximizing the log-likelihood function (19). For maximizing the log-likelihood for the sub-models, we employ the `R` software (R Core Team 2018), `AdequacyModel` package (Diniz Marinho et al. 2016). For checking the uniqueness of the solution to the score equations, we have perturbed the initial values, besides consider different methods: quasi-Newton methods (BFGS and Nelder-Mead) and heuristic methods (simulated annealing and particle swarm optimization). The MLEs of the model parameters of the TMOELx distribution and its sub-models are listed in Table IV.

We compare the fitted models by means of some goodness-of-fit statistics: Akaike Information Criterion (AIC) (Akaike 1974), Bayesian Information Criterion (BIC) (Schwarz 1978), Hannan-Quinn Information Criterion (HQIC) (Hannan & Quinn 1979), Cramér-von Mises Criterion (W*) and Anderson-Darling Criterion (A*) (Chen & Balakrishnan 1995). In general, small values of these statistics indicate better fits. We employ the `R` software (`AdequacyModel` package) to calculate these statistics. The goodness-of-fit values of the fitted distributions are listed in Table V.

Table IV. MLEs (standard errors).

Distribution	MLE			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\lambda}$
TMOELx	2,087	0,775	1,062	-0,99
	(0.497)	(0.247)	(0.141)	(0.160)
MOELx	118,067	7,962	1,227	-
	(179.861)	(10.360)	(0.095)	-
TLx	-	0,067	1,679	0,189
	-	(0.040)	(0.467)	(0.435)
Lx	-	0,083	1,610	-
	-	(0.031)	(0.392)	-

The values in Table V indicate that the TMOELx distribution presents the smallest values of the AIC, HQIC, W* and A* statistics among the fitted models. Therefore, according to these statistics, we can conclude that the TMOELx distribution gives the best fit to the current data.

To analyze how significant is the parameter λ of the TMOELx distribution in modeling these data, we use the LR statistic for testing the MOELx model against the TMOELx model, that is, we test $\mathcal{H}_0 : \lambda = 0$ against $\mathcal{H}_1 : \lambda \neq 0$. We obtain an approximate *p-value* of $0,0019$. Therefore, at the 5% significance

Table V. **Goodness-of-fit statistics (first application).**

Distribution	Statistic				
	AIC	BIC	HQIC	W*	A*
TMOELx	928,168	939,576	932,803	0,256	1,528
MOELx	933,116	941,672	936,593	0,334	2,029
TLx	935,166	943,722	938,642	0,361	2,213
Lx	933,298	939,002	935,616	0,360	2,207

level, the test rejects the null hypothesis, that is, we reject the MOELx model. Thus, we have evidence of the potential need for including the parameter λ to model these data.

Second application: In this case, we compare the TMOELx distribution with other non-nested models proposed in the literature: exponentiated Lomax (ELx) (Lemonte & Cordeiro 2013), exponentiated standard Lomax (EsLx) (Lemonte & Cordeiro 2013), transmuted Marshall-Olkin Fréchet (TMOFr) (AFIFY et al. 2015), beta Lomax (BLx) (Rajab et al. 2013) and Kumaraswamy Lomax (KwLx) (Shams 2013) distributions, whose densities are given in Appendix D. The uncensored data set refers to 213 times of successive failures of air conditioning system of airplanes available in Proschan (1963).

Since all considered models are non-nested, we compare them by using the statistics W* and A*, since the AIC, BIC and HQIC criterions are useful only for nested models. The goodness-of-fit values of the fitted distributions are listed in Table VI. We can note that the TMOELx distribution has the smallest values of the W* and A* among the fitted models. Therefore, the TMOELx model gives the best fit to the current data.

The plots of the estimated TMOELx, TMOFr and KwLx densities are displayed in Figure 5.

Table VI. **Goodness-of-fit statistics (second application).**

Distribution	Statistic	
	W*	A*
TMOELx	0,034	0,249
TMOFr	0,116	0,813
EsLx	0,698	4,487
ELx	0,051	0,370
BLx	0,401	2,662
KwLx	0,085	0,632

CONCLUSIONS

In this paper, we study a new four-parameter lifetime model, named the *transmuted Marshall–Olkin extended Lomax* (TMOELx) distribution, obtained from the transmuted-G (T-G) family (Shaw & Buckley

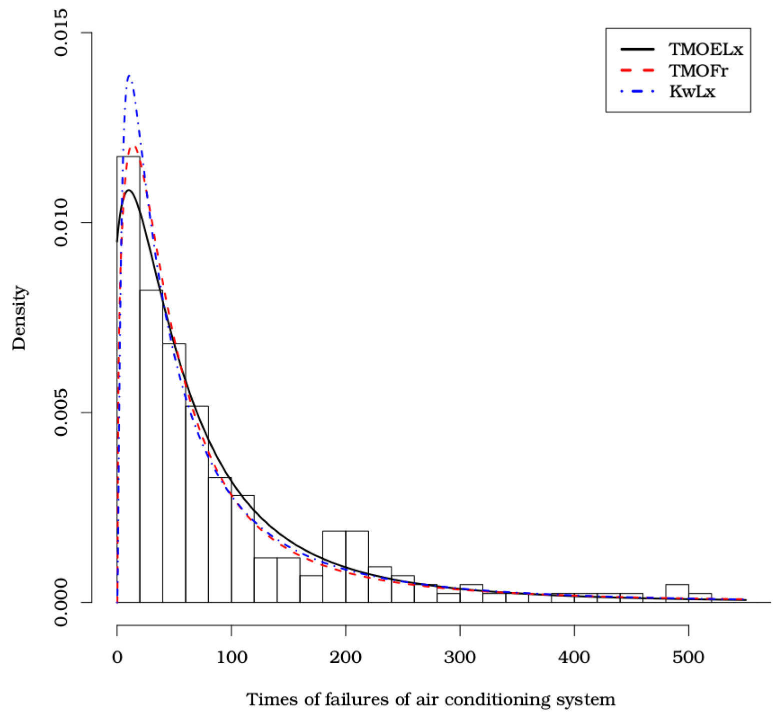


Figure 5. Comparison of the TMOELx, TMOFr and KwLx estimated densities.

2009) when the baseline model is the Marshall-Olkin extended Lomax (MOELx) distribution (Ghitany et al. 2007). We present some sub-models of the new distribution. We obtain simple expressions for the cumulative and density functions. We demonstrate that the TMOELx density can be expressed as a linear combination of exponentiated-Lomax densities and then some of its structural properties can be determined from those of these models. We obtain explicit expressions for the quantile function, ordinary and incomplete moments, characteristic function and order statistics. We determine the maximum likelihood estimates for complete samples and perform a Monte Carlo study to evaluate the behavior of these estimates in finite samples. We compare the performance of the new model with other distributions using classical goodness-of-fit statistics. The overall results confirm that the TMOELx model is very appropriate for lifetime applications.

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APPENDIX A

Here, we provide the mathematical development to derive eq. (8). The Lomax cdf (1) is given by

$$R(x) = 1 - (1+\beta x)^{-\gamma} = \frac{(1+\beta x)^{\gamma} - 1}{(1+\beta x)^{\gamma}},$$

and then $(1+\beta x)^{-\gamma} = 1 - R(x)$.

The MOEL cdf can be expressed in terms of $R(x)$ from eq. (2) as

$$\begin{aligned} G(x) &= 1 - \frac{\alpha}{(1+\beta x)^{\gamma} - (1-\alpha)} = 1 - \frac{\alpha(1+\beta x)^{-\gamma}}{[(1+\beta x)^{\gamma} - 1](1+\beta x)^{-\gamma} + \alpha(1+\beta x)^{-\gamma}} \\ &= 1 - \frac{\alpha(1-R(x))}{R(x) + \alpha(1-R(x))} = 1 - \frac{\alpha(1-R(x))}{(1-\alpha)R(x) + \alpha} = \frac{R(x)}{\alpha\left[\frac{1-\alpha}{\alpha} R(x) + 1\right]}. \end{aligned} \tag{20}$$

By considering the cdf (3) of the transmuted family, the TMOEL cdf (for $\lambda \in [-1,1]$) is

$$F(x) = (1+\lambda)\, G(x) - \lambda\, G^2(x) = \frac{(1+\lambda)\, R(x)}{\alpha\left[1 + \left(\frac{1-\alpha}{\alpha}\right) R(x)\right]} - \frac{\lambda\, R^2(x)}{\alpha^2\left[1 + \left(\frac{1-\alpha}{\alpha}\right) R(x)\right]^2}. \tag{21}$$

For $\alpha > 1/2$, we have $\left|\frac{1-\alpha}{\alpha}\right| < 1$. Thus, since $0 < R(x) < 1$ (for $x > 0$), using the generalized binomial expansion (Gradshteyn & Ryzhik 2007, p. 25, subsection 1.112), we have (for $\alpha > 1/2$)

$$\left[1 + \left(\frac{1-\alpha}{\alpha}\right) R(x)\right]^{-1} = \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{1-\alpha}{\alpha}\right)^{k-1} R^{k-1}(x) = \sum_{k=0}^{\infty} \left(\frac{\alpha-1}{\alpha}\right)^{k} R^{k}(x) \tag{22}$$

and

$$\left[1 + \left(\frac{1-\alpha}{\alpha}\right) R(x)\right]^{-2} = \sum_{k=1}^{\infty} (-1)^{k-1} k \left(\frac{1-\alpha}{\alpha}\right)^{k-1} R^{k-1}(x) = \sum_{k=1}^{\infty} k \left(\frac{\alpha-1}{\alpha}\right)^{k-1} R^{k-1}(x). \tag{23}$$

Inserting (22) and (23) in eq. (21) gives

$$\begin{aligned} F(x) &= \frac{(1+\lambda)}{\alpha} R(x) \sum_{k=0}^{\infty} \left(\frac{\alpha-1}{\alpha}\right)^{k} R^{k}(x) - \frac{\lambda}{\alpha^2} R^2(x) \sum_{k=1}^{\infty} k \left(\frac{\alpha-1}{\alpha}\right)^{k-1} R^{k-1}(x) \\ &= \frac{(1+\lambda)}{\alpha} R(x) \left[1 + \sum_{k=1}^{\infty} \left(\frac{\alpha-1}{\alpha}\right)^{k} R^{k}(x)\right] - \frac{\lambda}{\alpha^2} \sum_{k=1}^{\infty} k \left(\frac{\alpha-1}{\alpha}\right)^{k-1} R^{k+1}(x) \\ &= \frac{(1+\lambda)}{\alpha} R(x) + \sum_{k=1}^{\infty} \left[\frac{((1+\lambda)(\alpha-1) - k\lambda)}{\alpha^2} \left(\frac{\alpha-1}{\alpha}\right)^{k-1}\right] R^{k+1}(x) \\ &= \sum_{k=0}^{\infty} a_k\, R^{k+1}(x), \end{aligned} \tag{24}$$

where $a_k = \frac{[(1+\lambda)(\alpha-1)-k\lambda]}{\alpha^2}\left(\frac{\alpha-1}{\alpha}\right)^{k-1}$ for $k = 0, 1, 2, \ldots$ (with $a_0 = 0$ when $\alpha = 1$).

To obtain an expansion for $F(x)$ when $0 < \alpha \leq 1/2$, we can rewrite equation (20) as

$$G(x) = \frac{R(x)}{1 - (1-\alpha)(1-R(x))}. \tag{25}$$

Since $|(1-\alpha)(1-R(x))| < 1$, we consider the generalized binomial expansion (for $0 < \alpha \leq 1/2$)

$$[1 - (1-\alpha)(1-R(x))]^{-1} = \sum_{j=1}^{\infty} (1-\alpha)^{j-1} (1-R(x))^{j-1} = \sum_{j=0}^{\infty} (1-\alpha)^{j} (1-R(x))^{j}.$$

For $j \in \mathbb{N}$ fixed, expanding $(1-R(x))^j = \sum_{k=0}^{j}\binom{j}{k}R^k(x)$ gives

$$[1-(1-\alpha)(1-R(x))]^{-1} = \sum_{j=0}^{\infty}\sum_{k=0}^{j}\binom{j}{k}(1-\alpha)^j R^k(x) = \sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\binom{j}{k}(1-\alpha)^j R^k(x),$$

where the last equality follows by rearranging terms in the summations.

By replacing the above equation in eq. (25), we obtain (for $0 < \alpha \leq 1/2$)

$$G(x) = \sum_{k=0}^{\infty}\sum_{j=k}^{\infty}\binom{j}{k}(1-\alpha)^j R^{k+1}(x) = \sum_{k=0}^{\infty} b_k\, R^{k+1}(x),$$

where $b_k = \sum_{j=k}^{\infty}\binom{j}{k}(1-\alpha)^j$ (for $k = 0, 1, 2, \ldots$)

Inserting the last equation in (3) gives

$$F(x) \quad = \quad (1+\lambda)\, G(x) - \lambda\, G^2(x) = (1+\lambda)\sum_{k=0}^{\infty} b_k\, R^{k+1}(x) - \lambda\left[\sum_{k=0}^{\infty} b_k\, R^{k+1}(x)\right]^2$$

By using the expansion for a power series raised to positive integer powers (Gradshteyn & Ryzhik 2007, p. 17, section 0.314) we have

$$\left[\sum_{k=0}^{\infty} b_k\, R^{k+1}(x)\right]^2 = \sum_{k=0}^{\infty} c_k\, R^{k+1}(x),$$

where $c_0 = b_0^2$ and $c_m = \frac{1}{m\, b_0}\sum_{k=1}^{m}(3k-m)\, b_k\, c_{m-k}$, for $m \geq 1$. Thus, for $0 < \alpha \leq 1/2$, we can write

$$F(x) = (1+\lambda)\sum_{k=0}^{\infty} b_k\, R^{k+1}(x) - \lambda\sum_{k=0}^{\infty} c_k\, R^{k+1}(x) = \sum_{k=0}^{\infty}[(1+\lambda)\, b_k - \lambda\, c_k]\, R^{k+1}(x). \tag{26}$$

Finally, from equations (24) and (26), the cdf $F(x)$ (for all $\alpha > 0$) can be expressed as

$$F(x) = \sum_{k=0}^{\infty} p_k\, H_{k+1}(x),$$

where $H_{k+1}(x) = R^{k+1}(x)$ is the exp-Lx cdf with power parameter $k+1$ and

$$p_k = \begin{cases} (1+\lambda)\, b_k - \lambda\, c_k, & \text{if } 0 < \alpha \leq 1/2, \\ a_k, & \text{if } \alpha > 1/2. \end{cases}$$

APPENDIX B

The routine (in `Ox` language) for calculating the values of the mean, variance, skewness and kurtosis of the TMOEL distribution in Table II is given below:

```
#include <oxstd.oxh>
#include <oxfloat.oxh>
#include <oxprob.h>
decl NREP = 50000; /* number of replicas */
decl x;  /* sample of size N */
decl alpha = 7.0; \\values for alpha: 0.75, 1.5, 3.0, 5.0, 7.0
decl beta = 4.0;  \\values for beta: 0.5, 1.0, 2.0, 3.0, 4.0
decl gamma = 7.0; \\values for gamma: 4.5, 5.0, 5.5, 6.0, 7.0
decl lambda = 0.5; \\values for lambda: -0.75, -0.5, -0.25, 0.25, 0.5
decl K=20; /* truncation, K = 5, 10, 20 */
 // generating sample
```

```
sampleTMOEL(const N, const vP)
{ decl u   = ranu(N, 1);
  decl u_0 = (1 + vP[3] - sqrt ((1 + vP[3])^2 - 4*vP[3].*u))./(2*vP[3]);
  decl sample  = (vP[1]^(-1))*(((1 - (vP[0].*u_0)./(u_0 - 1)).^(1/vP[2]) - 1));
  return sample; }
// function for calculate the r-th truncated ordinary moment
rmoment_TMOEL(const k, const r)
{ decl i,j, m, m_r, a_i;
  m= zeros(k+1,r+1);
  for(i = 0; i <= k; i++)
   { a_i = (((alpha-1)*(1+lambda)-i*lambda)/alpha^2) *((alpha-1)/alpha)^(i-1);
     for(j=0; j <= r; j++)
       { m[i][j] = (-1)^j*binomial(r,j)*((i+1)/beta^r)*
         betafunc(M_INF ,1-(1/gamma)*(r-j),i+1)*a_i; }}
   m_r = sumc(sumr(m));
   return m_r;}
/************************************************************************
                      main program
************************************************************************/
main()
{ decl i, theta, moment , assim, curtose,  mean, var, mean_2, var_2,
        assim_2, curtose_2;
        ranseed("GM");
        ranseed({1111,1111});
        theta = alpha|beta|gamma|lambda;
 /***************** calculate moments using Monte Carlo *********************/
        // gerating sample x~TMOEL(alpha, beta, gamma, lambda)
        x = sampleTMOEL(NREP, theta);
        moment = moments(x, 4);
        skew = moment[3];
        kur = moment[4];
        mean = meanc(x);
        var = varc(x);
        \\mean, variance, skewness and kurtosis via Monte Carlo
        println(mean|var|skew|kur, "\n");
/************* calculate moments using expansions ***************************/
        mean_2 = rmoment_TMOEL(K,1);
        var_2 = rmoment_TMOEL(K,2) - mean_2^2;
        skew_2 = (rmoment_TMOEL(K,3)-3*rmoment_TMOEL(K,1)*rmoment_TMOEL(K,2) +
                  2*rmoment_TMOEL(K,1)^3)/(sqrt(var_2))^3;
        kur_2 = (rmoment_TMOEL(K,4)-4*rmoment_TMOEL(K,1)*rmoment_TMOEL(K,3)+
                    6*(rmoment_TMOEL(K,1)^2)*rmoment_TMOEL(K,2)-
                    3*rmoment_TMOEL(K,1)^4)/(var_2)^2;
        \\mean, variance, skewness and kurtosis via expansions
        println( mean_2~var_2~skew_2~kur_2|mean~var~skew~kur); }
```

APPENDIX C

In this appendix, we detail some computational aspects related to the section of simulation study . All Monte Carlo simulations are performed using scripts implemented in the `Ox` programming language. A free version of `Ox` is available in https://www.doornik.com.

For maximizing the log-likelihood function (19), we use the routine `MaxBFGS` implemented in `Ox`:

```
#import <maximize>
MaxBFGS(const func, const avP, const adFunc, const amInvHess, const fNumDer);
```

- `func`
  In: the function to be maximized with p parameters (the log-likelihood function in this case).
- `avP`
  In: matrix of order $p \times 1$ with the initial values.
  Out: matrix of order $p \times 1$ with the values maximizing `func`.
- `adFunc`
  In: address to object.
  Out: maximum value of `func`.
- `amHessian`
  In: 0 or the Hessian matrix addressed (we used 0).
- `fNumDer`
  In: 0, to use the first analytic derivatives or 1, to use the first numeric derivatives (we used 1).

The log-likelihood function (19) is implemented by using the `Ox` language as follows:

```
// log-likelihood function
flogTMOEL(const vP, const adFunc, const avSore, const amHess)
{ adFunc[0] = n*(log (vP[0]) + log (vP[1]) + log (vP[2])) + (vP[2] - 1) *
              sumc(log (1 + vP[1]*x)) + sumc(log ((1 - (1 + vP[1]*x).^vP[2])*
              (vP[3] - 1) + vP[0]*(vP[3] + 1))) -3 *
              sumc(log ((1 + vP[1]*x).^vP[2] - 1 + vP[0]));
/* checks whether any element of adFunc[0] is NaN or infinity */
 if(isnan(adFunc[0]) || isdotinf(adFunc[0]) )
 return 0;
 else
 return 1; }
```

Finally, the function (19) is maximized by using the routine:

```
MaxBFGS(flogTMOEL, &theta_0, &dfunc, 0, 1);
```

General comments:

- the initial values (ϑ_0) used for maximizing the log-likelihood function were obtained from the true value of the parameter ϑ by adding a small arbitrary constant $\delta < 1$, that is, $\vartheta_0 = \vartheta + \delta$;
- For $10,000$ Monte Carlo replications, the convergence rate, in almost all scenarios considered, was greater than 85%. For $\beta > 0.5$ or $n < 100$, the BFGS method exhibits poor convergence. Therefore, it is not recommended to work with samples smaller than 100.

APPENDIX D

The model densities used for comparison with the TMOELx distribution are given below:

The TMOFr pdf is

$$f(x) = \frac{\alpha\beta\sigma^{\beta}x^{-(\beta+1)}\mathrm{e}^{-\left(\frac{s}{x}\right)^{b}}}{\left[\alpha + (1-\alpha)\mathrm{e}^{-\left(\frac{s}{x}\right)^{b}}\right]^{2}}\left[1 + \lambda - \frac{2\lambda \mathrm{e}^{-\left(\frac{s}{x}\right)^{b}}}{\alpha + (1-\alpha)\mathrm{e}^{-\left(\frac{s}{x}\right)^{b}}}\right],\ x > 0,$$

where β, α, ς are positive parameters and $|\lambda| \leq 1$.

The ELx pdf is

$$f(x) = \frac{\alpha\beta^{\alpha}(\beta + x)^{-(\alpha+1)}}{B(a, 1)}\left[1 - \left(\frac{\beta}{\beta + x}\right)^{\alpha}\right]^{a-1},\ x > 0,$$

where $B(a, b)$ is the beta function and α, β and a are positive parameters.

The EsLx pdf is

$$f(x) = \frac{\alpha(1+x)^{-(\alpha+1)}}{B(a,1)} \left[1 - \left(\frac{1}{1+x}\right)^{\alpha}\right]^{a-1}, \; x > 0.$$

where α and a are positive parameters.

The BLx pdf is

$$f(x) = \frac{\alpha}{\lambda B(a,b)} \left\{1 - \left[1 + \left(\frac{x}{\lambda}\right)\right]^{-\alpha}\right\}^{a-1} \left[1 + \left(\frac{x}{\lambda}\right)\right]^{-(\alpha b+1)}, \; x > 0,$$

where α, λ, a and b are positive parameters.

Finally, the KwLx pdf is

$$f(x) = ab\frac{\alpha}{\lambda}\left(1 + \frac{x}{\lambda}\right)^{-(\alpha+1)} \left[1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}\right]^{a-1} \left\{1 - \left[1 - \left(1 + \frac{x}{\lambda}\right)^{-\alpha}\right]^{a}\right\}^{b-1}, \; x > 0,$$

where α, λ, a and b are positive parameters.

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