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An elementary proof of MinVol($\mathbb{R}^n$) $= 0$ for $n \geq 3$

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ABSTRACT

In this paper, we give an elementary proof of the result that the minimal volumes of $\mathbb{R}^3$ and $\mathbb{R}^4$ are zero. The approach is to construct a sequence of explicit complete metrics on them such that the sectional curvatures are bounded in absolute value by 1 and the volumes tend to zero. As a direct consequence, we get that MinVol $(\mathbb{R}^n) = 0$ for $n \geq 3$.

Key words: minimal volume, smooth gluing, bounded geometry.

1 INTRODUCTION

The definition of minimal volume of a C^∞ manifold M (without boundary) was first introduced by Gromov (Gromov 1982). Denote $\mathcal{G}(M)$ the set of all complete smooth Riemannian metrics on M such that the corresponding sectional curvatures are bounded in absolute value by 1. We say that (M, g) has bounded geometry (Cheeger and Gromov 1985) if its metric belongs to $\mathcal{G}(M)$. The minimal volume of M is a geometric invariant which is defined as

$$\mathrm{MinVol}(M) := \inf\left\{\mathrm{Vol}(M, g) \mid g \in \mathcal{G}(M)\right\}. \tag{1}$$

For closed surfaces M, by Gauss-Bonnet formula, it's easy to see that

$$\mathrm{MinVol}(M) = 2\pi\,|\chi(M)|\,.$$

Thus the minimal volume of a closed surface is actually a topological invariant. For the two dimensional plane, Gromov (Gromov 1982) obtained the following estimate

$$\mathrm{MinVol}(\mathbb{R}^2) \leq (2 + 2\sqrt{2})\pi.$$

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Bavard and Pansu proved that this is an equality (Bavard and Pansu 1986). B.H. Bowditch (Bowditch 1993) gave a different proof by using spherical isoperimetric inequality. Gromov had shown that $\mathrm{MinVol}(\mathbb{R}^n) = 0$ without going into details in (Gromov 1982, Appendix 2). Cheeger and Gromov (Cheeger and Gromov 1985, Example 1.6) showed that $\mathbb{R}^n (n \geq 4)$ has a similar solid torus decomposition as $\mathbb{R}^3$. Thus $\mathbb{R}^n$ admits a family of metrics such that the sectional curvatures are bounded and the volumes tend to zero. In this paper, we give a detailed proof of the result in another direct way by constructing the explicit metrics which are different from those in (Cheeger and Gromov 1985, Example 1.6) on the higher dimensional Euclidean spaces.

We state a few results about minimal volume. As stated in (Paternain and Petean 2003), the minimal volume does depend on the smooth structure of the manifold (also see (Bessières 1998)). J. Cheeger and M. Gromov introduced in (Cheeger and Gromov 1986, Gromov 1982) the concepts of F-structure and T-structure and obtained some results about F-structure and minimal volume. They proved that if M admits a polarized F-structure then the minimal volume of M vanishes. Notice that there is a little difference between the original definition of T-structure given in (Gromov 1982) and the later definition given in (Cheeger and Gromov 1986, Paternain and Petean 2003). The graph manifold is a 3-manifold which admit a polarized T-structure. So graph manifold is a special T-manifold. Thus the minimal volume of graph manifold is zero. Furthermore, T. Soma proved in (Soma 1981) that the connected sum of two graph manifold is still a graph manifold. In (Gromov 1982) Gromov pointed out that this result holds for odd dimensional manifolds with T-structures. Paternain and Petean proved in (Paternain and Petean 2003) that the result also holds for the family of manifolds which admit general T-structures and for any dimension greater than 2. The minimal volume is closely related to the collapsing theory in Riemannian geometry. Cheeger, Fukaya and Gromov have developed collapsing theory, and they obtained many important results (Cheeger and Gromov 1986, 1990, Fukaya 1990).

The organization of this paper is as follows: In Section 2, we discuss how to realize smooth gluing of metrics on 2-dimensional surfaces and to construct metrics on Y-pieces (Buser 1992) by a simple method. This method is intuitive even without using uniformization theorem. The "Y-piece" (also called "pair of pants") means a compact topological surface obtained from a 2 dimensional sphere by cutting away the interior of 3 disjoint closed topological disks. In Section 3, we give an explicit construction of a sequence of complete metrics on $\mathbb{R}^3$ with bounded curvatures such that the corresponding volumes tend to zero. We just take product metric on Y-piece$\times S^1$ and equip the metric on each piece. So we didn't change the topology of $\mathbb{R}^3$. We also apply the similar construction to $\mathbb{R}^4$. As an immediate corollary, we have $\mathrm{MinVol}(\mathbb{R}^n) = 0$, for $n \geq 3$.

2 CONSTRUCTION OF METRICS ON Y-PIECES

Our goal is going to construct metrics on Y-piece with uniformly bounded curvatures which are independent of the lengthes of the boundary. In order to realize smooth gluing of metrics, we simply require that the metrics on a small tubular neighborhood of the boundary of such Y-piece are product metrics.

To do this, it is sufficient to construct metrics on a disk. A simple way is to glue $S^2 \setminus D^2$ with a cylinder $S^1 \times I$ along the circle (boundary). By viewing $S^2 \setminus D^2$ and cylinder as surfaces of revolution, what remains to do is to consider smooth gluing of the images of functions and calculate the Gauss curvature of surface of revolution. But here the radius of S^1 must be very small (e.g. ε) for our purpose. Hence, to maintain the curvatures of the surface in $[-1, +1]$, we must insert a good surface. Here we choose a part of pseudosphere. Then, we prove that the surface after gluing is still smooth, the curvature is uniformly bounded (i.e. independent of ε), and the volume changes a little.

We first use the mollifier to construct a cut-off function (Lemma 2.1). Using this lemma, we can glue smoothly two functions which are tangent at a point (Lemma 2.2). By a result of Dan Henry, we construct a better cut-off function (see Lemma 2.5).

These results are used to construct metrics on disk D^2 such that the metrics have bounded curvatures and the metrics when restricted in a small neighborhood of ∂D^2 are product metrics. See Lemma 2.6 and 2.8.

LEMMA 2.1. *There is a smooth function $\phi(x) \in [0, b]$, $b > 0$ on $\mathbb{R}$, such that*

$$\phi(x) = \begin{cases} b, & x \leq 0, \\ 0, & x \geq a. \end{cases} \tag{2}$$

and

$$\left|\phi'(x)\right| \leq \frac{2b}{a}. \tag{3}$$

PROOF. Let

$$\widetilde{\phi}_\delta(x) = \begin{cases} b, & x \leq \delta, \\ -\dfrac{b}{a-2\delta}x + \dfrac{b(a-\delta)}{a-2\delta}, & \delta < x < a-\delta, \\ 0, & x \geq a-\delta, \end{cases}$$

where $0 < \delta \leq \frac{a}{4}$. Set

$$\phi_\delta(x) = \int_{\mathbb{R}} j_\delta(x-t)\widetilde{\phi}_\delta(t)dt = \int_{\mathbb{R}} \frac{1}{\delta} j\left(\frac{x-t}{\delta}\right)\widetilde{\phi}_\delta(t)dt,$$

where $j(x)$ is the mollifier function defined on $\mathbb{R}$ by

$$j(x) = \begin{cases} \dfrac{1}{A} e^{\frac{1}{x^2-1}}, & |x| < 1, \\ 0, & |x| \geq 1, \end{cases}$$

and A is equal to $\int_{-1}^{1} e^{\frac{1}{x^2-1}} dx$, so that $\int_{\mathbb{R}} j(x)dx = 1$. For any $\delta > 0$, let

$$j_\delta(x) = \frac{1}{\delta} j\left(\frac{x}{\delta}\right).$$

Then by calculation, we get

$$\left|\phi_\delta'(x)\right| \leq \frac{b}{a-2\delta} \leq \frac{2b}{a}.$$

Hence, $\phi_\delta(x)$ (simply denoted by $\phi(x)$) is the required function.

Moreover, we have

$$\left|\phi_\delta''(x)\right| \leq \frac{b}{A(a-2\delta)\delta e} \leq \frac{b}{(a-2\delta)\delta} \leq \frac{2b}{a\delta}. \tag{4}$$

since $Ae > 1$. □

LEMMA 2.2. *Suppose $f(x)$, $g(x)$ are two smooth functions defined on $\mathbb{R}$ and satisfy*

$$f(c) = g(c) \quad \text{and} \quad f'(c) = g'(c)$$

at some point $x = c \in \mathbb{R}$. Given any $\delta > 0$, there is a smooth function $h_\delta(x)$ on $\mathbb{R}$ such that

$$h_\delta|_{(-\infty,\, c-2\delta]} = f,\ h_\delta|_{[c+2\delta,\, +\infty)} = g.$$

PROOF. By the proof of Lemma 2.1, there is a smooth function $\varphi_\delta(x) \in [0, 1]$ such that

$$\varphi_\delta(x) = \begin{cases} 1, & x \in (-\infty, c - 2\delta], \\ 0, & x \in [c + 2\delta, +\infty). \end{cases}$$

Let

$$h_\delta(x) = f(x)\varphi_\delta(x) + g(x)(1 - \varphi_\delta(x)). \tag{5}$$

Then h_δ is the required function. □

Still consider functions as in the above lemma. Denote

$$\widetilde{h}(x) = \begin{cases} f(x), & x \leq c, \\ g(x), & x > c. \end{cases}$$

Let Σ_{h_δ}, $\Sigma_{\widetilde{h}}$ be the surfaces in $\mathbb{R}^3$ generated by the rotation around the x-axis of the graphs of the functions h_δ and $\widetilde{h}$ respectively.

Assume that $f(x), g(x) \geq \sigma > 0$ for all $x \in [c - 1, c + 1]$, and let $\delta \in (0, 1/2)$ be small enough. σ is independent of δ. Let

$$M_2 = \sup_{x\in[c-1,\, c+1]} |f''(x)|, \qquad N_2 = \sup_{x\in[c-1,\, c+1]} |g''(x)|.$$

Then we have the following lemma.

LEMMA 2.3. *There is a constant $C(\sigma, M_2, N_2) > 0$, such that*

$$\left|K_{\Sigma_{h_\delta}}\right| \leq C(\sigma, M_2, N_2), \tag{6}$$

where $K_{\Sigma_{h_\delta}}$ is the Gauss curvature of the surface Σ_{h_δ}. Moreover, we have

$$\left|\mathrm{Vol}(\Sigma_{h_\delta}) - \mathrm{Vol}(\Sigma_{\widetilde{h}})\right| \to 0, \quad \text{as } \delta \to 0, \tag{7}$$

PROOF. By equation (5),

$$h_\delta(x) \geq \sigma.$$

For $x \in [c - 2\delta, c + 2\delta]$, by Taylor expansion formula and equations (3)–(4) in Lemma 2.1, we have

$$\begin{aligned} |h_\delta{}''| &= |f''\varphi_\delta + g''(1 - \varphi_\delta) + 2(f' - g')\varphi_\delta' + (f - g)\varphi_\delta''| \\ &\leq |f''| + |g''| + 2|f''(\xi_1) - g''(\xi_2)|2\delta \cdot \frac{1}{2\delta} + \frac{1}{2}|f''(\xi_3) - g''(\xi_4)|(2\delta)^2 \cdot \frac{1}{2\delta^2} \\ &\leq 4(M_2 + N_2), \end{aligned}$$

where $\xi_i \in (c - 2\delta, c + 2\delta),\ i = 1, 2, 3, 4$. Hence,

$$\left|K_{\Sigma_{h_\delta}}\right| \leq \frac{|h_\delta{}''|}{h_\delta} \leq \frac{4(M+N)}{\sigma} = C(\sigma, M_2, N_2).$$

By assumption, $C(\sigma, M, N)$ is independent of δ.

Since $h_\delta \to \widetilde{h}$ as $\delta \to 0$, we have

$$\left|\mathrm{Vol}(\Sigma_{h_\delta}) - \mathrm{Vol}(\Sigma_{\widetilde{h}})\right| \to 0, \quad \text{as } \delta \to 0.$$

In fact, if $x \leq b$,

$$|h_\delta - f| = |(g - f)\psi| \to 0, \quad \text{as } \delta \to 0;$$

and if $x \geq b$,

$$|h_\delta - f| = |(f - g)\varphi| \to 0, \quad \text{as } \delta \to 0. \qquad \square$$

REMARK 2.4. In Lemma 2.2, if we suppose $f''(c) = g''(c)$ additionally, and we set

$$M_3 = \sup_{x\in[c-1,\, c+1]} \left|f'''(x)\right|, \qquad N_3 = \sup_{x\in[c-1,\, c+1]} \left|g'''(x)\right|,$$

by a similar argument as in the proof of Lemma 2.3, we get $\left|h_\delta'''\right| \leq 14(M_3 + N_3)$.

LEMMA 2.5. *There is a smooth function $\phi(x)(0 \leq \phi(x) \leq b)$ defined on $\mathbb{R}$, such that the following conditions are satisfied.*

1. $\phi(x) = b$, *for* $x \leq 0$, $\phi(x) = 0$ *for* $x \geq a$;
2. $\phi^{(k)}(0) = \phi^{(k)}(a) = 0$, *for any* $k \in \mathbb{Z}^+$;
3. $\max\limits_{i=1,2,3} \left[\max\limits_{x\in[a,b]} \left|\phi^{(i)}(x)\right|\right] \leq 15\big(2^3(3!)^2 b + \max\left\{2^3(3!)^2 b, 3b/a\right\}\big)$.

PROOF. Let

$$\phi_1(x) = \begin{cases} b, & x \leq 0, \\ b\left(1 - e^{-\frac{1}{x}}\right), & 0 < x \leq \eta a; \end{cases} \tag{8}$$

$$\phi_2(x) = \begin{cases} be^{\frac{1}{x-a}}, & a - \eta a \leq x < a, \\ 0, & x \geq a. \end{cases} \tag{9}$$

Here we select $0 < \eta < 1$ such that $\phi_1(\eta a) > \phi_2(a - \eta a)$.

By a result of Dan Henry (Henry 1994) (which is available on the net at the address: http://www.ime.usp.br/map/dhenry/danhenry/main.htm) we have

$$\max_{i=1,2,3}\left[\max_{x\in[a,b]} \left|\phi_j^{(i)}(x)\right|\right] \leq 2^3(3!)^2 b, \quad j = 1, 2.$$

Let $p_1(x)$, $p_2(x)$ be two polynomial functions as in Figure 1 which join $\phi_1(x)$, $\phi_2(x)$ to a line l C^2-smoothly respectively.

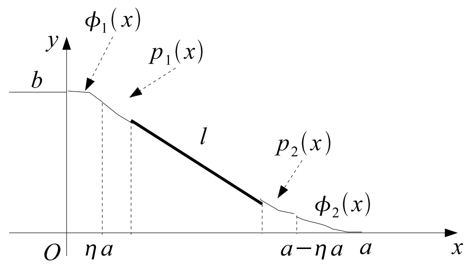

Figure 1

Let

$$C_j^{(i)} = \max_{x\in[a/3,2a/3]} \left|p_j^{(i)}(x)\right|, \quad \text{and} \quad C_j(a,b) := \max\left\{C_j^{(1)}, C_j^{(2)}, C_j^{(3)}\right\}, \quad j=1,2.$$

It is clear that $C_j(a,b)$ is in a small neighborhood of $\max\left\{2^3(3!)^2 b, b/((1-2\eta)a)\right\}$. Therefore by Remark 2.4, we have

$$\max_{i=1,2,3}\left[\max_{x\in[a,b]}\left|\phi^{(i)}(x)\right|\right] \leq 15\big(2^3(3!)^2 b + \max\left\{2^3(3!)^2 b, 3b/a\right\}\big). \qquad \square$$

Let us recall some properties of the pseudosphere. Suppose the parametric equations of pseudosphere are

$$\begin{cases} x &= \pm\big(\ln\tan\frac{t}{2} + \cos t\big), \\ y &= \sin t\cos\theta, \\ z &= \sin t\sin\theta, \end{cases} \tag{10}$$

where $t \in [\frac{\pi}{2}, \pi)$, $\theta \in [0, 2\pi]$. We only consider the part of $x \geq 0$, and we have

$$x = \cosh^{-1}\left(\frac{1}{y}\right) - \sqrt{1-y^2}.$$

A direct calculation implies that

$$\frac{dx}{dt} = \frac{\cos^2 t}{\sin t}, \quad \frac{dx}{dy} = -\frac{\sqrt{1-y^2}}{y}, \quad y'_x = \tan t, \quad y''_x = \frac{\sin t}{\cos^4 t}. \tag{11}$$

LEMMA 2.6. *For any given* $0 < \varepsilon << 1$ *and* $0 < \delta << 1$, *there is a smooth function* $h_{\varepsilon,\delta}(x)$ *on* $[0, +\infty)$ *such that*

1. *For* $x \in [0, x_{2\varepsilon} - \sqrt{1-4\varepsilon^2}]$, *the parametric equation of* $h_{\varepsilon,\delta}(x)$ *is*

$$\begin{cases} x &= \ln\big(\tan\frac{t}{2} + \cos t\big), \\ y &= \sin t, \end{cases}$$

where $t \in \left[\frac{\pi}{2}, t_\varepsilon\right]$, t_ε *satisfies the equation* $\ln\big(\tan\frac{t_\varepsilon}{2} + \cos t_\varepsilon\big) = x_{2\varepsilon} - \sqrt{1-4\varepsilon^2}$;

2. *For* $x \in \big[x_{2\varepsilon} + \sqrt{1-4\varepsilon^2}, +\infty\big)$, $h_{\varepsilon,\delta}(x) = \varepsilon$.

Here

$$x_{2\varepsilon} = \ln \frac{1 + (1-2\varepsilon)\sqrt{1-4\varepsilon^2}}{2\varepsilon}.$$

PROOF. Suppose $f(x)$ is a smooth function which is defined as

$$\begin{cases} x = \ln\left(\tan\frac{t}{2} + \cos t\right), \\ y = \sin t, \end{cases}$$

where $t \in \left[\frac{\pi}{2}, \pi\right)$.

It is easy to check that point

$$A = \left(\ln \frac{1 + (1-2\varepsilon)\sqrt{1-4\varepsilon^2}}{2\varepsilon},\ 2\varepsilon\right)$$

belongs to the image of function $f(x)$. Suppose line AB is tangent to $f(x)$ at point A (Fig. 2). B is the intersection point of line AB and x-axis. AC is perpendicular to x-axis. D, E are the midpoints of AB and BC respectively. DF is parallel to x-axis. And $x_F = x_B$.

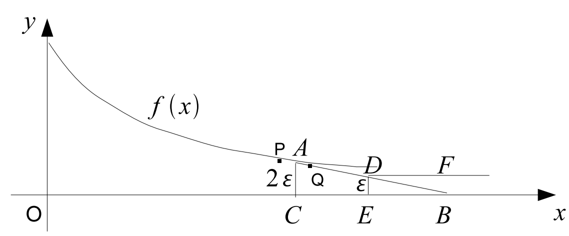

Figure 2

It's easy to see that

$$|AB| = 1.$$

Thus, we have

$$|BC| = \sqrt{1-4\varepsilon^2}, \qquad |DE| = \varepsilon, \qquad |CE| = \frac{1}{2}\sqrt{1-4\varepsilon^2}.$$

and

$$x_C = x_{2\varepsilon} := \ln \frac{1 + (1-2\varepsilon)\sqrt{1-4\varepsilon^2}}{2\varepsilon},$$

$$x_E = x_C + |CE| = x_{2\varepsilon} + \frac{1}{2}\sqrt{1-4\varepsilon^2},$$

$$x_B = x_C + |BC| = x_{2\varepsilon} + \sqrt{1-4\varepsilon^2}.$$

Define $\widetilde{g}(x)$ as follows

$$\widetilde{g}(x) = \begin{cases} -\dfrac{2\varepsilon}{\sqrt{1-4\varepsilon^2}}x + \dfrac{2\varepsilon}{\sqrt{1-4\varepsilon^2}}(x_{2\varepsilon} + \sqrt{1-4\varepsilon^2}), & x \in [0, x_E], \\ \varepsilon, & x \in [x_E, +\infty). \end{cases}$$

That is the broken line $\widetilde{ADF}$.

We smooth it by the method in Lemma 2.1 and denote the result smooth function by $g(x)$. Choose

$$\delta_1 = \frac{1}{4}|CE| = \frac{1}{8}\sqrt{1-4\varepsilon^2}.$$

By inequality (4), we have the estimate:

$$\left|g''(x)\right| \leq \frac{2\varepsilon}{\frac{1}{2}\sqrt{1-4\varepsilon^2}\cdot\frac{1}{8}\sqrt{1-4\varepsilon^2}} = \frac{32\varepsilon}{1-4\varepsilon^2}.$$

Next, we observe that $f(x_{2\varepsilon}) = g(x_{2\varepsilon})$ and $f'(x_{2\varepsilon}) = g'(x_{2\varepsilon})$. According to Lemma 2.2, for any $0 < \delta < \frac{1}{4}|CE|$, there is a smooth function $k_\delta(x)$ on $\left[0, x_{2\varepsilon} + \frac{1}{2}|CE|\right]$, such that

$$k_\delta(x)\big|_{[0,x_{2\varepsilon}-2\delta]} = f(x), \quad k_\delta(x)\big|_{\left[x_{2\varepsilon}+2\delta,x_{2\varepsilon}+\frac{1}{2}|CE|\right]} = g(x).$$

Choose point $P = (x_P, y_P) \in \operatorname{Im} f(x)$ and $Q = (x_Q, y_Q) \in \operatorname{Im} g(x)$ such that

$$2\varepsilon < y_P \leq 4\varepsilon, \qquad x_{2\varepsilon} < x_Q \leq x_{2\varepsilon} + \frac{1}{4}|CE|.$$

According to Lemma 2.3, we have

$$\left|k''_\delta(x)\right| \leq 4(M+N),$$

where

$$M := \sup_{x\in[x_P,x_Q]}\left|f''(x)\right|, \qquad N := \sup_{x\in[x_P,x_Q]}\left|g''(x)\right| = 0.$$

Since $f''(x) = \dfrac{\sin t}{\cos^4 t}$ (see (11)), $f''(x_{2\varepsilon}) = \dfrac{2\varepsilon}{(1-4\varepsilon^2)^2}$. Because $\dfrac{x}{(1-x^2)^2}$ is a increasing function about x for $x \in (0,1)$,

$$M = \left|f''(x_P)\right| \leq \frac{4\varepsilon}{\left(1-16\varepsilon^2\right)^2}.$$

Then, we get a global smooth function $h_{\varepsilon,\delta}(x)$ on $[0,+\infty)$ such that

$$\begin{cases} h_{\varepsilon,\delta}(x)\big|_{[0,x_{2\varepsilon}-\sqrt{1-4\varepsilon^2}]} = f(x), \\ h_{\varepsilon,\delta}(x)\big|_{[x_{2\varepsilon}+\sqrt{1-4\varepsilon^2},+\infty)} = \varepsilon. \end{cases}$$

since

$$k_\delta(x) = g(x), \qquad \text{for} \quad x \in \left(x_{2\varepsilon} + \frac{1}{4}|CE|, x_{2\varepsilon} + \frac{3}{4}|CE|\right).$$ □

REMARK 2.7. Let $\Sigma_{h_{\varepsilon,\delta}}$ be the surface in $\mathbb{R}^3$ generated by the rotation around the x-axis of the graph of the function $h_{\varepsilon,\delta}$. Then we have the estimate about the sectional curvature:

1. For $x \in \left[0, x_{2\varepsilon} - \sqrt{1-4\varepsilon^2}\right]$,

$$\left|K_{\Sigma_{h_{\varepsilon,\delta}}}\right| \leq \frac{\left|g''(x)\right|}{g(x)} \leq \frac{\frac{32\varepsilon}{1-4\varepsilon^2}}{\varepsilon} = \frac{32}{1-4\varepsilon^2}.$$

2. For $x \in \left[x_{2\varepsilon} + \sqrt{1-4\varepsilon^2}, +\infty\right)$,

$$\left|K_{\Sigma_{h_{\varepsilon,\delta}}}\right| \leq \frac{4(M+N)}{\varepsilon} = \frac{4M}{\varepsilon} \leq \frac{4 \cdot \frac{4\varepsilon}{(1-16\varepsilon^2)^2}}{\varepsilon} = \frac{16}{(1-16\varepsilon^2)^2}.$$

So, we can choose a constant C which is independent of ε and δ, such that

$$\left|K_{\Sigma_{h_{\varepsilon,\delta}}}\right| \leq C.$$

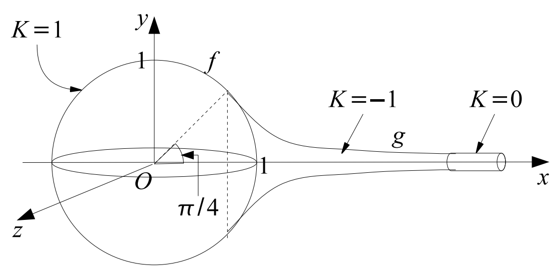

Figure 3

LEMMA 2.8. *For any given $0 < \varepsilon << 1$ and $0 < \delta << 1$, there is a smooth surface $\Sigma_{\varepsilon,\delta}$ which is generated by rotation of the graph of a smooth function $h_{\varepsilon,\delta}(x)$ satisfying the following conditions along x-axis.*

1. $h_{\varepsilon,\delta}(x) = \sqrt{1-x^2}$, *for* $x \in \left[-1, \frac{\sqrt{2}}{2} - \delta\right]$.

2. *For* $x \in \left[\frac{\sqrt{2}}{2} + \delta, x_{2\varepsilon} - \sqrt{1-4\varepsilon^2} + \ln\left(1 + \frac{\sqrt{2}}{2}\right)\right]$, $h_{\varepsilon,\delta}$ *has following parametric equation:*

$$\begin{cases} x = \ln\left(\tan\frac{t}{2} + \cos t\right) + \ln\left(1 + \frac{\sqrt{2}}{2}\right), \\ y = \sin t, \end{cases}$$

   where $t \in \left(\frac{3\pi}{4} + \delta, t_\varepsilon\right)$.

3. $h_{\varepsilon,\delta}(x) = \varepsilon$, *for* $x \in \left[x_{2\varepsilon} + \sqrt{1-4\varepsilon^2} + \ln\left(1+\frac{\sqrt{2}}{2}\right), x_{2\varepsilon} + \sqrt{1-4\varepsilon^2} + \ln\left(1+\frac{\sqrt{2}}{2}\right) + 1\right]$.

$x_{2\varepsilon}$ and t_ε are the same as in Lemma 2.6. Moreover, the sectional curvature of $\Sigma_{\varepsilon,\delta}$ is bounded by a constant C which is independent of ε and δ.

PROOF. By Lemma 2.2, 2.3, 2.6 and Remark 2.7. □

REMARK 2.9. If we let δ (which used in Lemma 2.2–2.3) be small enough in Lemma 2.8, the area of the smooth surface above is less than $(2 + 2\sqrt{2})\pi + 1$. If we add two ends to the surface, we will get the required Y-piece as showed in Figure 4.

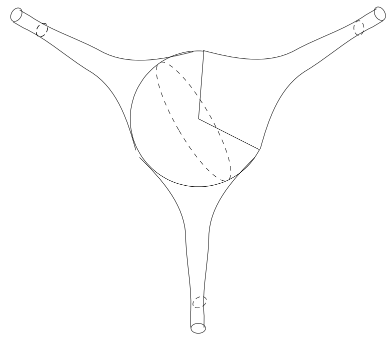

Figure 4

3 MINIMAL VOLUME OF $\mathbb{R}^3$

By Lemma 2.8 and torus decomposition of $\mathbb{R}^3$ in (Cheeger and Gromov 1985), we can construct a sequence of complete smooth metrics on $\mathbb{R}^3$ with bounded curvatures and the corresponding volumes tend to zero. So the minimal volume of $\mathbb{R}^3$ is zero. The completeness of every metric we constructed is followed from Proposition 3.1. Note that the diameters of the Y-pieces we constructed have a positive lower bound. By Hopf-Rinow theorem (Chavel 1993), $(\mathbb{R}^3, g_\varepsilon)$ is also geodesic complete.

PROPOSITION 3.1. *Let $A_1 \subset A_2 \subset \cdots A_n \subset \cdots$ be a sequence of compact metric spaces. Suppose d_n is the distance function on A_n, and $d_{n+1}|_{A_n} = d_n$ for all n. Define function $d : A = \cup_{n=1}^{\infty} A_n \to \mathbb{R}$ by $d(x, y) = d_m(x, y)$, where m is the positive integer such that $x, y \in A_m$. Then (A, d) is a metric space. If additionally we suppose that there is a sequence of concentric balls $\{B_n\}$ such that $B_n \subset A_n$ and* diameter$(B_n) \to \infty$ *as $n \to \infty$, then (A, d) is a complete metric space.*

Before the proof of Proposition 3.2, it maybe useful to keep Figure 5 in mind which describes the construction of metrics on $\mathbb{R}^3$.

As in (Cheeger and Gromov 1985, Example 1.4), we decompose $\mathbb{R}^3$ into a sequence of solid toruses,

$$C_1 \subset C_2 \subset \cdots \subset C_n \subset \cdots,$$

such that $\mathbb{R}^3 = \cup_{i=1}^{\infty} C_i$, where $C_i = D_i^2 \times S^1$, $\{D_i^2\}$ are closed disks. Every solid torus is contractible in the next. Let

$$C_1 = \Sigma_1^2 \times S^1 = D_1^2 \times S^1.$$

Let A_i denote the axis (i.e. $\{0\} \times S^1 \subset D_i^2 \times S^1$) of C_i. The tubular neighborhood of A_{i+1} denotes by

$$\Sigma_{2i+1}^2 \times S^1.$$

Then, we have

$$C_{i+1} \setminus C_i = \Sigma_{2i}^2 \times S^1 \cup \Sigma_{2i+1}^2 \times S^1,$$

where Σ_{2i}^2 is a surface with nonempty boundary which consists of three circles.

PROPOSITION 3.2. MinVol$(\mathbb{R}^3) = 0$.

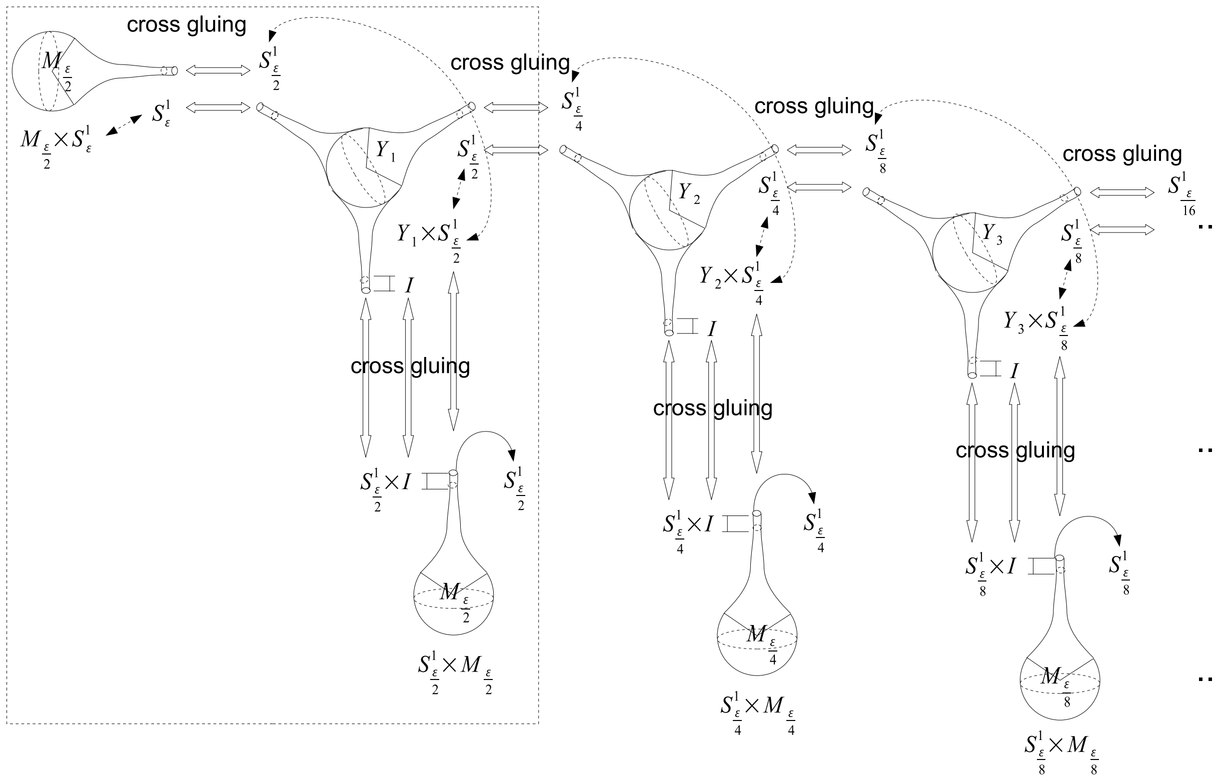

Fig. 5 – Metrics on $\mathbb{R}^3$.

PROOF. Our goal is to construct a sequence of complete smooth metrics g_ε on $\mathbb{R}^3$ such that

$$\mathrm{Vol}(\mathbb{R}^3, g_\varepsilon) \to 0, \qquad \text{as } \varepsilon \to 0,$$

and the corresponding curvatures are uniformly bounded by 1, i.e. $\left|K_{g_\varepsilon}\right| \leq 1$.

In Lemma 2.8, we have constructed a surface (disk) Σ with nonempty boundary (circle) $\partial\Sigma$ satisfying

$$\mathrm{Vol}(\Sigma, g_\Sigma) < +\infty, \quad \left|K_{g_\Sigma}\right| \leq C, \quad \mathrm{Length}(\partial\Sigma) = 2\pi\varepsilon, \tag{12}$$

and, if restricted on a tubular neighborhood of $\partial\Sigma$ in Σ, g_Σ is product metric.

As has been mentioned in Section 2, it is also easy to construct metric on Y-piece $\left(\Sigma^2_{2i}\right)$ satisfying the similar conditions as above. With such metric, Σ^2_{2i} looks like as in Figure 6.

Next, we give the explanation in details as follows.

For Σ^2_1, we assign it a metric $g_{1,\frac{\varepsilon}{2}}$ as in Lemma 2.8. Denote the geometric surface by $M_{\frac{\varepsilon}{2},\delta}$, or simply by $M_{\frac{\varepsilon}{2}}$, where the $\frac{\varepsilon}{2}$ means the length of $\partial M_{\frac{\varepsilon}{2}}$ is $\pi\varepsilon$ under the metric given. By Remark 2.9,

$$\mathrm{Vol}\left(M_{\frac{\varepsilon}{2}}\right) \leq \left(2 + 2\sqrt{2}\right)\pi + 1,$$

when δ is small enough.

For $\Sigma^2_{2i+1},\ i \geq 1$, define

$$M_{\frac{\varepsilon}{2^i}} := \left(\Sigma^2_{2i+1},\ g_{1,\frac{\varepsilon}{2^i}}\right), \quad \text{for } i \geq 1.$$

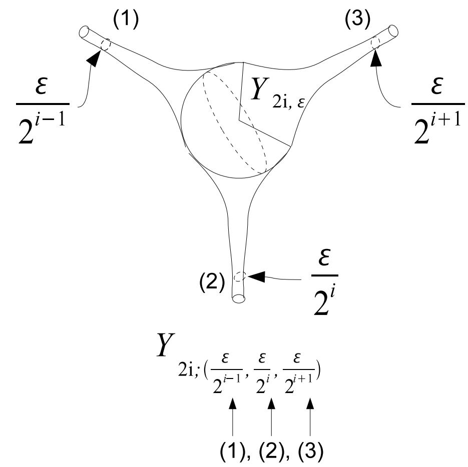

Fig. 6 – Metric on Y-piece.

We assign Σ^2_{2i} a metric to be a Y-piece (see Fig. 6), and denote it by

$$Y_i = Y_{2i,\varepsilon} = Y_{2i;\left(\frac{\varepsilon}{2^{i-1}},\frac{\varepsilon}{2^{i}},\frac{\varepsilon}{2^{i+1}}\right)},$$

which means the lengths of the components of the boundary $\partial Y_{2i,\varepsilon}$ are $\frac{2\pi\varepsilon}{2^{i-1}}$, $\frac{2\pi\varepsilon}{2^{i}}$, $\frac{2\pi\varepsilon}{2^{i+1}}$ respectively.

In summary, we use the following notations:

$$\begin{cases} M_0 = M_{\frac{\varepsilon}{2}} := \left(\Sigma^2_1,\ g_{1,\frac{\varepsilon}{2^1}}\right), & \\ M_i = M_{\frac{\varepsilon}{2^i}} := \left(\Sigma^2_{2i+1},\ g_{1,\frac{\varepsilon}{2^i}}\right), & \text{for } i \geq 1, \\ Y_i = Y_{2i;\left(\frac{\varepsilon}{2^{i-1}},\frac{\varepsilon}{2^{i}},\frac{\varepsilon}{2^{i+1}}\right)} := \left(\Sigma^2_{2i},\ g_{2;\left(\frac{\varepsilon}{2^{i-1}},\frac{\varepsilon}{2^{i}},\frac{\varepsilon}{2^{i+1}}\right)}\right), & \text{for } i \geq 1. \end{cases} \tag{13}$$

We simply write the last equation as

$$Y_i = Y_{2i,\varepsilon} := \left(\Sigma^2_{2i}, g_{2,\varepsilon}\right). \tag{14}$$

Then the process of constructing the metric g_ε on $\mathbb{R}^3$ is as follows.

According to the torus decomposition of $\mathbb{R}^3$, to get a global smooth metric on $\mathbb{R}^3$, we have to glue the metrics on the pieces

$$\begin{gathered} M_0 \times S^1_\varepsilon, \quad S^1_{\frac{\varepsilon}{2}} \times M_1, \quad S^1_{\frac{\varepsilon}{4}} \times M_2, \quad S^1_{\frac{\varepsilon}{8}} \times M_3, \dots \\ Y_1 \times S^1_{\frac{\varepsilon}{2}}, \quad Y_2 \times S^1_{\frac{\varepsilon}{4}}, \quad Y_3 \times S^1_{\frac{\varepsilon}{8}}, \dots \end{gathered} \tag{15}$$

along boundary circles or the product factors S^1 in certain pairs and certain order. The metrics on such product manifolds are chosen simply as product metrics. S^1_ε means that the circle has length $2\pi\varepsilon$ under the given metric $\varepsilon^2 d\theta^2$. The relation between the lengths of the boundary circles must be related to ε. The order and the relation of gluing is stated as follows.

Let $\partial Y_{i;(1)} \times I$, $\partial Y_{i;(2)} \times I$, and $\partial Y_{i;(3)} \times I$ denote the three *tubular neighborhoods* of the three boundary components of Y_i for $i \geq 1$.

First, the metrics on $\partial Y_{i;(3)} \times I \times S^1_{\frac{\varepsilon}{2^i}}$ and $\partial Y_{i+1;(1)} \times I \times S^1_{\frac{\varepsilon}{2^{i+1}}}$ are chosen to be the following product metrics:

$$\left(\frac{\varepsilon}{2^{i+1}}\right)^2 d\theta^2 + dt^2 + \left(\frac{\varepsilon}{2^i}\right)^2 d\alpha^2, \tag{16}$$

and

$$\left(\frac{\varepsilon}{2^i}\right)^2 d\theta^2 + dt^2 + \left(\frac{\varepsilon}{2^{i+1}}\right)^2 d\alpha^2. \tag{17}$$

The order of gluing here is that we direct glue the first term of (16) and the third term of (17); and direct glue the third term of (16) and the first term of (17).

Second, the metrics on $\partial Y_{i;(2)} \times I \times S^1_{\frac{\varepsilon}{2^i}}$ and $S^1_{\frac{\varepsilon}{2^i}} \times I \times \partial M_i$ are chosen to be

$$\left(\frac{\varepsilon}{2^i}\right)^2 d\theta^2 + dt^2 + \left(\frac{\varepsilon}{2^i}\right)^2 d\alpha^2, \tag{18}$$

and

$$\left(\frac{\varepsilon}{2^i}\right)^2 d\alpha^2 + dt^2 + \left(\frac{\varepsilon}{2^i}\right)^2 d\theta^2. \tag{19}$$

The order of gluing here is that we exchange the variables α and θ in the second (or first) metric, so that they can be glued directly with the first (or second) metric.

At last, what remains to do is to glue the metrics on $\partial M_0 \times I \times S^1_{\varepsilon}$ and $\partial Y_{1;(1)} \times I \times S^1_{\frac{\varepsilon}{2}}$.

Thus, we get a sequence of complete metrics $g_{\varepsilon,\delta}$ (simply denote by g_ε) on $\mathbb{R}^3$ such that

$$\left|K_{g_\varepsilon}\right| \leq C,$$

where C is independent of ε and δ; and we have

$$\mathrm{Vol}\left(Y_i \times S^1_{\frac{\varepsilon}{2^i}}\right) \leq 3(2+2\sqrt{2})\pi \cdot \frac{2\pi\varepsilon}{2^i}, \tag{20}$$

and

$$\mathrm{Vol}\left(S^1_{\frac{\varepsilon}{2^i}} \times M_i\right) \leq \frac{2\pi\varepsilon}{2^i}((2+2\sqrt{2})\pi + 1). \tag{21}$$

By scaling and the fact

$$K_{\lambda g} = \frac{1}{\lambda} K_g, \quad \mathrm{Vol}(M, \lambda g) = \lambda^{n/2}\mathrm{Vol}(M, g), \tag{22}$$

where $n = \dim M$, we get our metrics (still denote by g_ε) such that the curvature is uniformly bounded by 1, while the volumes of the surfaces are finite, since

$$\varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + \frac{\varepsilon}{8} + \cdots = 2\varepsilon. \tag{23}$$

Hence,

$$\mathrm{Vol}(\mathbb{R}^3, g_\varepsilon) \to 0, \text{ as } \varepsilon \to 0. \tag{24}$$

Therefore,

$$\mathrm{MinVol}(\mathbb{R}^3) = 0.$$

The proposition is proved. □

4 MINIMAL VOLUMES OF $\mathbb{R}^n$ FOR $n \geq 3$

PROPOSITION 4.1. $\mathrm{MinVol}(\mathbb{R}^4) = 0$.

PROOF. We still use the notations (13) and (14) in the proof of Proposition 3.2. The metric on Y_i (i.e. $Y_{2i,\varepsilon}$ or $Y_{2i;(\frac{\varepsilon}{2^{i-1}},\frac{\varepsilon}{2^i},\frac{\varepsilon}{2^{i+1}})}$) is denoted by $g_{2,\varepsilon}$ (means the second kind metric).

Let the metric on $Y_i \times S^1_{\frac{\varepsilon}{2^i}} \times \mathbb{R}$ be

$$\widetilde{g}_{2,\varepsilon} = g_{2,\varepsilon} + f^2(t) g_{S^1_{\frac{\varepsilon}{2^i}}} + dt^2, \quad i = 1, 2, 3, \ldots, \tag{25}$$

where

$$g_{S^1_{\frac{\varepsilon}{2^i}}} = \left(\frac{\varepsilon}{2^i}\right)^2 d\theta^2, \quad \theta \in [0, 2\pi], \quad t \in \mathbb{R}, \tag{26}$$

and $f(t)$ is defined by

$$f(t) = \begin{cases} 1 - e^{-\frac{24}{t^2}}, & t \neq 0, \\ 1, & t = 0. \end{cases} \tag{27}$$

It is easy to prove that $f(t)$ is a smooth function. Moreover, $f(t)$ satisfies

$$\begin{cases} \dfrac{|f''(t)|}{f(t)} \leq 1, \\ \dfrac{|f'(t)|}{f(t)} \leq \dfrac{\sqrt{3}}{6}, \\ \displaystyle\int_{-\infty}^{+\infty} f(t) dt = 4\sqrt{6\pi} < \infty, \\ f(t) > 0, \quad \text{for all } t \in \mathbb{R}. \end{cases} \tag{28}$$

Replace the metric $g_{1,\frac{\varepsilon}{2^i}}$ on Σ^2_{2i+1} (that is, the surface M_i, see (13)) by the metric $g_{1,\frac{\varepsilon(t)}{2^i}}$ and denote the surface $\left(\Sigma^2_{2i+1}, g_{1,\frac{\varepsilon(t)}{2^i}}\right)$ by $M_{\frac{\varepsilon(t)}{2^i}}$, where $\varepsilon(t)$ means that the length of the boundary circle is a function of t. If we want to realize the smooth gluing (i) as in Figure 7, that is to glue the metric on factor $S^1_{\frac{\varepsilon}{2^i}}$ and the metric on boundary $S^1_{\frac{\varepsilon(t)}{2^i}}$, the direct way is to require that the metric on a small tubular neighborhood of $\partial M_{\frac{\varepsilon(t)}{2^i}}$ is a product metric, and

$$\mathrm{Length}(\partial M_{\frac{\varepsilon(t)}{2^i}}) = 2\pi \frac{\varepsilon(t)}{2^i} = f(t) \cdot 2\pi \frac{\varepsilon}{2^i}.$$

Thus

$$\varepsilon(t) = f(t)\varepsilon.$$

But, in order to make the volume $\mathrm{Vol}\left(S^1_{\frac{\varepsilon}{2^i}} \times M_{\frac{\varepsilon(t)}{2^i}} \times \mathbb{R}\right)$ finite, the metric on $S^1_{\frac{\varepsilon}{2^i}} \times M_{\frac{f(t)\varepsilon}{2^i}} \times \mathbb{R}$ should has the following form

$$f^2(t)\left(\frac{\varepsilon}{2^i}\right)^2 d\theta^2 + g_{1,\frac{f(t)\varepsilon}{2^i}} + dt^2. \tag{29}$$

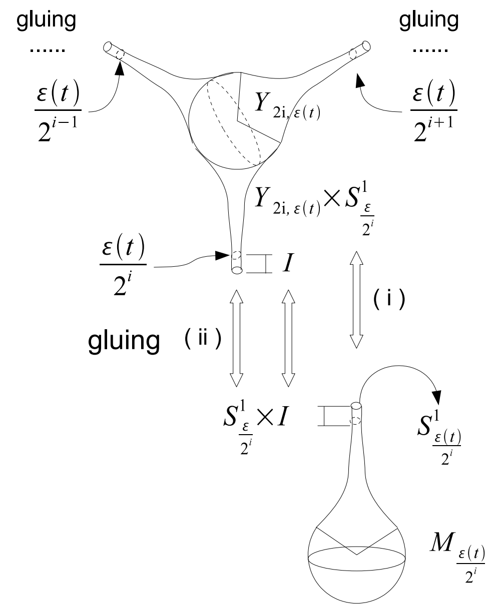

Fig. 7 – Gluing.

The gluing (ii) is similar to (i). Hence, the metric $g_{2,\varepsilon}$ on Y_i (i.e. $Y_{2i;(\frac{\varepsilon}{2^{i-1}},\frac{\varepsilon}{2^i},\frac{\varepsilon}{2^{i+1}})}$) should be changed to $g_{2,f(t)\varepsilon}$. That is,

$$Y_{2i;\left(\frac{\varepsilon}{2^{i-1}},\frac{\varepsilon}{2^i},\frac{\varepsilon}{2^{i+1}}\right)}$$

becomes to

$$Y_{2i;\left(\frac{f(t)\varepsilon}{2^{i-1}},\frac{f(t)\varepsilon}{2^i},\frac{f(t)\varepsilon}{2^{i+1}}\right)}.$$

To realize the gluing (i) and (ii) while keeping the curvature bounded, we first construct an ε related smooth function $G_\varepsilon(x,t)$ (simply denoted by $G(x,t)$, see (36)) which is used to construct the smooth surface $M_{(f(t)\varepsilon)/2^i}$. After that, we calculate the sectional curvatures and claim that the sectional curvatures are bounded. Then, by calculating the volumes, we complete the proof. For state clearly, we divided the proof into several steps:

Step 1: Construction of function $G(x,t)$. By Lemma 2.5, there is a C^∞ function $\phi(\bar{x})$ ($0 \leq \phi(\bar{x}) \leq 1$) on $[-1,+\infty)$ such that

$$\phi(\bar{x}) = \begin{cases} 1, & -1 \leq \bar{x} \leq 0, \\ 1 - e^{-\frac{1}{\bar{x}^2}}, & 0 < \bar{x} \leq \frac{1}{3}, \\ e^{-\frac{1}{(\bar{x}-1)^2}}, & \frac{2}{3} \leq \bar{x} < 1, \\ 0, & \bar{x} \geq 1. \end{cases} \tag{30}$$

There exists a constant $C > 0$, such that

$$\left|\phi'_{\bar{x}}\right|, \quad \left|\phi''_{\bar{x}}\right|, \quad \left|\phi'''_{\bar{x}}\right| \leq C. \tag{31}$$

In particular,

$$\begin{cases} \phi'_{\bar{x}} = -\dfrac{2}{\bar{x}^3} e^{-\frac{1}{\bar{x}^2}}, & \text{for } \bar{x} \in \left(0, \dfrac{1}{6}\right), \\ \phi'_{\bar{x}} = \dfrac{2}{(\bar{x}-1)^3} e^{-\frac{1}{(\bar{x}-1)^2}}, & \text{for } \bar{x} \in \left(\dfrac{5}{6}, 1\right). \end{cases}$$

Let

$$\varphi(x,t) = \begin{cases} 1, & -1 \le x \le 0, \\ \phi(f(t)x), & 0 < x < +\infty, \end{cases} \tag{32}$$

where $t \in \mathbb{R}$ and $\bar{x} = f(t)x$. Let

$$F(x,t) = \varepsilon(1 - f(t))h(x), \tag{33}$$

where $h(x)$ satisfies

$$\begin{cases} h(x) = 1, \quad \text{for } x \in [-1, 0], \\ \dfrac{|h'|}{h}, \ \dfrac{|h''|}{h} \le C, \\ \displaystyle\int_0^{+\infty} h(x)dx < \infty, \\ 0 < h(x) \le 1, \quad \text{for } \ x \in [-1, +\infty). \end{cases}$$

For simplicity, let

$$h(x) = \begin{cases} 1 - e^{-\frac{24}{x^2}}, & x > 0, \\ 1, & x \in [-1, 0]. \end{cases} \tag{34}$$

Then, we have

$$|h(x)|, \quad |h'(x)|, \quad |h''(x)| \le 1. \tag{35}$$

Then we define the function $G(x,t)$ by

$$G(x,t) = F(x,t)\varphi(x,t) + \varepsilon f(t), \qquad x \in [-1, +\infty), \ t \in \mathbb{R}. \tag{36}$$

Step 2: Calculation of the sectional curvature of surface $\Sigma_{G(x,t)}$. Now for every t, let $\Sigma_{G(x,t)}$ be the surface generated by rotation of the graph of the function $G(x,t)$. $\Sigma_{G(x,t)}$ is just a part of surface $M_{\frac{f(t)\varepsilon}{2^i}}$. It can also be used to construct the surface $Y_{2i;(\frac{f(t)\varepsilon}{2^{i-1}}, \frac{f(t)\varepsilon}{2^i}, \frac{f(t)\varepsilon}{2^{i+1}})}$.

On $S^1_{\frac{\varepsilon}{2^i}} \times \Sigma_{G(x,t)} \times \mathbb{R}$, the metric is given by

$$f^2(t)\left(\frac{\varepsilon}{2^i}\right)^2 d\theta^2 + \big(1 + (G'_x(x,t))^2\big)dx^2 + G^2(x,t)d\alpha^2 + dt^2. \tag{37}$$

where $\theta, \alpha \in [0, 2\pi]$. Let

$$\omega^1 = \frac{\varepsilon}{2^i} f(t)d\theta, \quad \omega^2 = \sqrt{1 + G'^2_x}dx, \quad \omega^3 = G(x,t)d\alpha, \quad \omega^4 = dt.$$

Then by the structure equations, we have

$$\begin{cases} \omega_1^2 = 0, & \omega_1^3 = 0, & \omega_1^4 = -\dfrac{\varepsilon}{2^i} f'(t) d\theta, \\ \omega_2^3 = \dfrac{G'_x}{\sqrt{1+G'^2_x}} d\alpha, & \omega_2^4 = -\dfrac{G'_x G''_{xt}}{\sqrt{1+G'^2_x}} dx, & \omega_3^4 = -G'_t d\alpha. \end{cases}$$

So the curvatures are

$$\begin{cases} K_{12} = -\dfrac{f'(t)}{f(t)} \dfrac{G'_x G''_{xt}}{1+G'^2_x}, \\ K_{13} = -\dfrac{f'(t)}{f(t)} \dfrac{G'_t}{G}, \\ K_{14} = -\dfrac{f''(t)}{f(t)}, \\ K_{23} = -\left(\dfrac{G''_{xx}}{G(1+G'^2_x)^2} + \dfrac{G'_x G'_t G''_{xt}}{G(1+G'^2_x)} \right), \\ K_{24} = -\dfrac{(G''_{xt})^2 + G'_x G'''_{xtt}(1+G'^2_x)}{(1+G'^2_x)^2}, \\ K_{34} = -\dfrac{G''_{tt}}{G}. \end{cases} \tag{38}$$

Step 3: We claim that:

$$\left| K_{ij} \right| \leq C, \quad \text{for } i, j = 1, 2, 3, 4. \tag{39}$$

PROOF. By Equation (32), we have

$$\begin{cases} \varphi'_x = \phi'_{\bar{x}} f(t), \\ \varphi'_t = \phi'_{\bar{x}} \dfrac{f'(t)}{f(t)} \bar{x}, \\ \varphi''_{xx} = \phi''_{\bar{x}} f^2(t), \\ \varphi''_{tt} = \phi''_{\bar{x}} \left(\dfrac{f'(t)}{f(t)} \right)^2 \bar{x}^2 + \phi'_{\bar{x}} \dfrac{f''(t)}{f(t)} \bar{x}, \\ \varphi''_{xt} = \phi''_{\bar{x}} f'(t) \bar{x} + \phi'_{\bar{x}} f'(t), \\ \varphi'''_{xtt} = \phi'''_{\bar{x}} \dfrac{(f'(t))^2}{f(t)} \bar{x}^2 + 2\phi''_{\bar{x}} \dfrac{(f'(t))^2}{f(t)} \bar{x} + \phi''_{\bar{x}} f''(t) \bar{x} + \phi'_{\bar{x}} f''(t). \end{cases} \tag{40}$$

By Equation (33), we have

$$\begin{cases} F'_x = \varepsilon(1 - f(t)) h'(x), & F'_t = -\varepsilon f'(t) h(x), \\ F''_{xx} = \varepsilon(1 - f(t)) h''(x), & F''_{tt} = -\varepsilon f''(t) h(x), \\ F''_{xt} = -\varepsilon f'(t) h'(x), & F'''_{xtt} = -\varepsilon f''(t) h'(x). \end{cases} \tag{41}$$

By Equation (36), we have

$$\begin{cases} G'_x = F'_x\varphi + F\varphi'_x, \\ G'_t = F'_t\varphi + F\varphi'_t + \varepsilon f'(t), \\ G''_{xx} = F''_{xx}\varphi + 2F'_x\varphi'_x + F\varphi''_{xx}, \\ G''_{tt} = F''_{tt}\varphi + 2F'_t\varphi'_t + F\varphi''_{tt} + \varepsilon f''(t), \\ G''_{xt} = F''_{xt}\varphi + F'_x\varphi'_t + F'_t\varphi'_x + F\varphi''_{xt}, \\ G'''_{xtt} = F'''_{xtt}\varphi + 2F''_{xt}\varphi'_t + F'_x\varphi''_{tt} + F''_{tt}\varphi'_x + 2F'_t\varphi''_{xt} + F\varphi'''_{xtt}. \end{cases} \tag{42}$$

Hence, by Equations (30)–(36), (40)–(42), we have

$$\left|G'_x\right| \leq \varepsilon + \varepsilon C, \qquad \left|G''_{xt}\right| \leq \varepsilon + 4C\varepsilon, \qquad \left|G'''_{xtt}\right| \leq \varepsilon + 14C\varepsilon. \tag{43}$$

$$\begin{aligned} \frac{\left|G''_{xx}\right|}{G} &\leq \frac{\left|h''\right|\varphi}{h\varphi} + \frac{2\left|h'\right|\left|\phi'_{\tilde{x}}\right| f(t)}{\frac{f}{1-f}} + \frac{h\left|\phi''_{\tilde{x}}\right| f^2(t)}{\frac{f}{1-f}}, \\ &\leq 1 + 3C. \end{aligned} \tag{44}$$

$$\begin{aligned} \frac{\left|G'_t\right|}{G} &\leq \frac{2\left|f'(t)\right| + (1-f(t))h(x)\left|\varphi'_t\right|}{(1-f(t))h(x)\varphi(x,t) + f(t)} \\ &\leq \frac{2\left|f'(t)\right|}{f(t)} + \frac{h(x)\left|\varphi'_t\right|}{h(x)\varphi(x,t) + \frac{f(t)}{1-f(t)}} \\ &\leq \frac{2\left|f'(t)\right|}{f(t)} + \frac{h(x)\left|\varphi'_t\right|}{h(x)\varphi(x,t) + f(t)} \\ &\leq 2 + \frac{C}{\varphi(x,t) + \frac{f(t)}{h(x)}}. \end{aligned} \tag{45}$$

Similarly, we have

$$\frac{\left|G''_{tt}\right|}{G} \leq 2 + 2C + \frac{2C}{\varphi(x,t) + \frac{f(t)}{h(x)}}. \tag{46}$$

Then,

$$\varphi(x,t) + \frac{f(t)}{h(x)} = \phi(f(t)x) + \frac{f(t)}{f(x)} \to 0$$

if and only if

$$\phi(f(t)x) \to 0, \quad \text{and} \quad \frac{f(t)}{f(x)} \to 0.$$

That is

$$f(t)x \to 1, \quad \text{and} \quad \frac{f(t)x}{f(x)x} \to 0.$$

This implies that $f(x)x \to +\infty$, and it is a contradiction since $f(x) < \frac{1}{x}$. Hence, there is a positive number $\sigma > 0$ which is independent of x, t and ε, such that

$$\varphi(x,t) + \frac{f(t)}{h(x)} > \sigma > 0.$$

Therefore,

$$\frac{|G'_t|}{G} < C, \quad \frac{|G''_{tt}|}{G} < C. \tag{47}$$

According to inequalities (43), (44) and (47), curvatures in (38) must be bounded. So the Claim is proved. □

Step 4: Conclusion. Therefore, for every piece considered above, we have constructed a metric on it. By scaling if necessary, we can make the metric satisfying the condition that the curvature is bounded by 1. Moreover, we have

$$\mathrm{Vol}\left(S^1_{\frac{\varepsilon}{2^i}} \times M_{\frac{f(t)\varepsilon}{2^i}} \times \mathbb{R}\right) \leq \frac{2\pi\varepsilon}{2^i} \cdot 4\sqrt{6\pi} \cdot \big((2+2\sqrt{2})\pi + 1\big), \tag{48}$$

and

$$\mathrm{Vol}\left(Y_{2i,f(t)\varepsilon} \times S^1_{\frac{\varepsilon}{2^i}} \times \mathbb{R}\right) \leq 3(2+2\sqrt{2})\pi \cdot \frac{2\pi\varepsilon}{2^i} \cdot 4\sqrt{6\pi}. \tag{49}$$

Hence, we have construct a sequence of complete metrics g_ε with bounded curvatures on $\mathbb{R}^3 \times \mathbb{R}$, and

$$\mathrm{Vol}\big(\mathbb{R}^3 \times \mathbb{R}, g_\varepsilon\big) \to 0, \quad \text{as } \varepsilon \to 0.$$

Therefore,

$$\mathrm{MinVol}\big(\mathbb{R}^4\big) = 0.$$ □

THEOREM 4.2. MinVol($\mathbb{R}^n$) = 0 for $n \geq 3$.

PROOF. Every positive integer n can be written in one of the forms:

$$3k, \quad 3k+1 = 3(k-1)+4, \quad 3k+2.$$

Then $\mathbb{R}^n$ ($n \geq 3$) can be written as

$$\underbrace{\mathbb{R}^3 \times \cdots \times \mathbb{R}^3}, \quad \text{or} \quad \underbrace{\mathbb{R}^3 \times \cdots \times \mathbb{R}^3} \times \mathbb{R}^4, \quad \text{or} \quad \underbrace{\mathbb{R}^3 \times \cdots \times \mathbb{R}^3} \times \mathbb{R}^2.$$

Take product metric on it, then MinVol($\mathbb{R}^n$) = 0 since MinVol($\mathbb{R}^3$) = MinVol($\mathbb{R}^4$) = 0 and MinVol($\mathbb{R}^2$) $= (2+2\sqrt{2})\pi$. □

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RESUMO

Neste artigo fornecemos uma demonstração elementar do resultado de que os volumes minimais de $\mathbb{R}^3$ e $\mathbb{R}^4$ são ambos iguais a zero. A abordagem consiste na construção de uma seqüência de métricas completas explícitas nesses espaços cujas curvaturas seccionais são limitadas em valor absoluto por 1 e os volumes tendem a zero. Como conseqüência direta, estabelecemos que $\mathrm{MinVol}(\mathbb{R}^n) = 0$ para $n \geq 3$.

Palavras-chave: volume mínimo, colagem diferenciável, geometria.

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