

# On the Monomial Birational Maps of the Projective Space

## GÉRARD GONZALEZ-SPRINBERG<sup>1</sup> and IVAN PAN<sup>2</sup>

<sup>1</sup>Institut Fourier, Université de Grenoble I UMR 5582 CNRS-UJF 38402 Saint-Martin-d'Hères, France <sup>2</sup>Instituto de Matemática, Universidade Federal do Rio Grande do Sul, 91501-970 Porto Alegre, Brasil

Manuscript received on December 18, 2002; accepted for publication on April 30, 2003; presented by Aron Simis

#### ABSTRACT

We describe the group structure of monomial Cremona transformations. It follows that every element of this group is a product of quadratic monomial transformations, and geometric descriptions in terms of fans.

Key words: monomial birational maps, toric varieties.

## 1. INTRODUCTION

The best known birational map of  $\mathbb{P}^r$  is maybe the *standard Cremona transformation*  $S_r: \mathbb{P}^r \longrightarrow \mathbb{P}^r$  defined by  $S_r = (X_0^{-1}: \cdots: X_r^{-1})$ . For r=2 there is a geometric description in the classic references on this subject; moreover a Max Noether's famous theorem shows that every birational map of  $\mathbb{P}^2$  is a composition of automorphisms and  $S_2$ , and then that every Cremona transformation is a composition of *quadratic* ones. If r>2 there is no analogous to Noether's theorem (Hudson 1927, Katz 1992, Pan 1999). In this note we consider birational maps generalizing  $S_r$ . Our approach is based in a toric point of view and the property that  $S_r$  stabilizes the open set  $X_0 \cdots X_r \neq 0$ . We consider birational maps of  $\mathbb{P}^r$  with this property. In the second paragraph of (Russo and Simis 2001) the birationality of these maps is characterized in terms of certain syzygies as an application of a more general criterion; see also (Simis and Villarreal 2002): compare their Proposition 1.1 with our Proposition 3.1.

Mathematics Subject Classification: 14E07, 14M25.

Correspondence to: Ivan Pan

E-mail: pan@mat.ufrgs.br / gonsprin@ujf-grenoble.fr

## 2. MONOMIAL CREMONA TRANSFORMATIONS

Let N be a rank r free  $\mathbb{Z}$ -module,  $\mathbb{K}$  an algebraically closed field,  $\mathbb{K}^* := \mathbb{K} \setminus \{0\}$ . Let  $\mathbb{T} = \mathbb{T}_N := N \otimes_{\mathbb{Z}} \mathbb{K}^*$  be the algebraic torus associated to N over  $\mathbb{K}$ . The action of  $\mathbb{T}$  on itself induces a natural inclusion  $\iota$  of the torus in the group  $\mathrm{Aut}(\mathbb{T})$  of the algebraic variety's automorphisms of  $\mathbb{T}$ ; we consider the subgroup  $G_{\mathbb{T}}$  of  $\mathrm{Aut}(\mathbb{T})$  given by the algebraic group's automorphisms of  $\mathbb{T}$ . Note that, with an appropriate choice of a basis of N, we may identify  $\mathbb{T}$  to  $(\mathbb{K}^*)^r$ . Since  $\mathbb{K}$  is algebraically closed an automorphism in  $\mathrm{Aut}(\mathbb{T})$  may be written in the form  $F = (F_1, \ldots, F_r)$  with

$$F_i = \lambda_i x_1^{a_{i1}} \cdots x_r^{a_{ir}} \tag{1}$$

where  $\lambda_i \in \mathbb{K}^*$ ,  $a_{ij} \in \mathbb{Z}$ ,  $1 \le i, j \le r$ .

LEMMA 2.1. There is a split exact sequence of groups

$$1 \to \mathbb{T} \stackrel{\iota}{\to} \operatorname{Aut}(\mathbb{T}) \stackrel{\phi}{\to} G_{\mathbb{T}} \to 1$$

where  $\phi(F) := \iota(F(1_{\mathbb{T}})^{-1}) \circ F$ , for  $F \in \operatorname{Aut}(\mathbb{T})$ .

PROOF. One has  $\phi(F) \in G_{\mathbb{T}}$ , because  $\lambda_i^{-1} F_i$  is a character of the torus,  $1 \le i \le r$ ; it is easily seen that the sequence is exact. The inclusion of  $G_{\mathbb{T}}$  in  $\operatorname{Aut}(\mathbb{T})$  is a section of  $\phi$ , hence the sequence splits.

In the following we consider the projective space  $\mathbb{P}^r$  over  $\mathbb{K}$  as a toric variety, e.g., a compactification of the torus  $\mathbb{T} = \mathbb{T}_N$  associated to a complete regular fan of  $N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}$  with (r+1) cones of dimension 1. (One standard reference for toric varieties is (Oda 1988)).

DEFINITION 2.2. A monomial Cremona transformation is a birational map  $F : \mathbb{P}^r \dashrightarrow \mathbb{P}^r$  defined on  $\mathbb{T}$  and such that  $F(\mathbb{T}) \subset \mathbb{T}$ .

We note  $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^r)$  the group of these transformations; there is a natural isomorphism  $\mathrm{Bir}_{\mathbb{T}}(\mathbb{P}^r) \stackrel{\sim}{\longrightarrow} \mathrm{Aut}(\mathbb{T})$ .

PROPOSITION 2.3. Fixing a basis of the lattice N, there exists an isomorphism

$$\operatorname{Bir}_{\mathbb{T}}(\mathbb{P}^r) \simeq \mathbb{T} \rtimes \operatorname{GL}_r(\mathbb{Z}).$$

PROOF. It follows from Lemma 2.1. Given  $F \in \operatorname{Bir}_{\mathbb{T}}(\mathbb{P}^r)$ , we associate  $(F(1_{\mathbb{T}}), (a_{ij}))$ , where  $(a_{ij})$  is the  $r \times r$ -matrix corresponding to  $\phi(F) \in G_{\mathbb{T}}$  via the equality (1) and the isomorphism  $G_{\mathbb{T}} \xrightarrow{\sim} \operatorname{GL}_r(\mathbb{Z})$  induced by the choice of the basis of N.

Remark 2.4. The action of  $GL_r(\mathbb{Z})$  on  $\mathbb{T}$  corresponding to the semidirect product in the proposition is

$$(a_{ij})\cdot(\lambda_1,\ldots,\lambda_r)=\bigg(\prod_j\lambda_j^{a_{1j}},\ldots,\prod_j\lambda_j^{a_{rj}}\bigg).$$

COROLLARY 2.5. The center of  $Bir_{\mathbb{T}}(\mathbb{P}^r)$  is trivial. The center of the subgroup  $\{F: F(1_{\mathbb{T}}) = 1_{\mathbb{T}}\}$  is  $\langle S_r \rangle$ .

As a consequence of Proposition 2.3 we obtain an analogous to the M. Noether's theorem (on the generators of the Cremona group of  $\mathbb{P}^2$ ) in arbitrary dimension for the monomial case.

THEOREM 2.6. The group  $Bir_{\mathbb{T}}(\mathbb{P}^r)$  is generated by a quadratic transformation and the linear monomial automorphisms. Consequently, every birational map is a product of quadratic transformations.

PROOF. It follows from Proposition 2.3 and the fact that  $GL_r(\mathbb{Z})$  is generated by a transvection and two permutations (Coxeter and Moser 1957, Trott 1962) which induce respectively a quadratic transformation and two linear maps in  $Bir_{\mathbb{T}}(\mathbb{P}^r)$ .

#### 3. DEGREES AND MATRICES

Let  $X_0, \ldots, X_r$  be homogeneous coordinates in  $\mathbb{P}^r$ ,  $x_i := X_i/X_0$ ,  $1 \le i \le r$ , the affine coordinates corresponding to the canonical basis of  $N = \mathbb{Z}^r$ . Every non-constant monomial *rational* map  $F : \mathbb{P}^r \longrightarrow \mathbb{P}^r$  satisfying  $F(1_{\mathbb{T}}) = 1_{\mathbb{T}}$  may be written uniquely in the form

$$F = (X_0^{\alpha_{00}} \cdots X_r^{\alpha_{0r}} : \cdots : X_0^{\alpha_{r0}} \cdots X_r^{\alpha_{rr}})$$
(2)

such that:

- (i)  $\alpha(e_j) \in \partial C_+$ ,  $0 \le j \le r$ ,
- (ii)  $\alpha(e) = \deg(F)e$ , where  $\alpha = \alpha_F = (\alpha_{ij}), e_j, 0 \le j \le r$ , is the canonical basis of  $\mathbb{Z}^{r+1}$ ,  $e = \sum_j e_j$ ,  $C_+$  is the cone  $\sum_j \mathbb{R}_{\ge 0} e_j$  and  $\partial C_+$  its boundary. The positive integer  $\deg(F)$  is the degree of F, e.g.  $\deg(F) = \sum_j \alpha_{ij}$ ,  $\forall i$ .

The map F is *birational* if and only if there exists an integer  $(r+1) \times (r+1)$ - matrix  $\beta$ , satisfying (i), (ii), and  $v \in \mathbb{Z}^{r+1}$  such that:

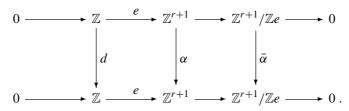
(iii)  $\alpha\beta = Id + e^{t}v$ .

In the affine open set  $X_0 \cdots X_r \neq 0$  one may write F as

$$F = (1: x_1^{a_{11}} \cdots x_r^{a_{1r}}: \cdots : x_1^{a_{r1}} \cdots x_r^{a_{rr}}).$$
(3)

The matrix  $A := (a_{ij})$  is the matrix of exponents of (1).

PROPOSITION 3.1. Let  $F \in \operatorname{Bir}_{\mathbb{T}}(\mathbb{P}^r)$  be such that  $F(1_{\mathbb{T}}) = 1_{\mathbb{T}}$ ,  $\alpha = \alpha_F$ ,  $d = \deg(F)$ . There exists a unique isomorphism  $\bar{\alpha}$  such that the following diagram is commutative and with exact lines.



One has  $d = |\det(\alpha)|$  and A is the matrix of  $\bar{\alpha}$  in the basis  $\bar{e_1}, \ldots, \bar{e_r}$ .

PROOF. It follows from properties (ii) and (iii) of  $\alpha$  and the form in which  $\alpha$  is written in the basis  $e, e_1, \ldots, e_r$  because (2) and (3) imply  $a_{ij} = \alpha_{ij} - \alpha_{0j}$ ,  $1 \le i, j \le r$ .

Remark 3.2.

- (a)  $\mathbb{Z}^{r+1}/\alpha(\mathbb{Z}^{r+1})$  is cyclic of order d.
- (b) We obtain  $\alpha$  from A as follows:

$$\alpha_{0j} = -\min(0, a_{ij}, 1 \le i \le r), \text{ for } 1 \le j \le r;$$
 $\alpha_{00} = \max(0, \sum_{j} a_{ij}, 1 \le i \le r);$ 
 $\alpha_{ij} = a_{ij} + \alpha_{0j}, \text{ for } 1 \le i, j \le r;$ 
 $\alpha_{i0} = \alpha_{00} - \sum_{j} a_{ij}, \text{ for } 1 \le i \le r.$ 

EXAMPLE 3.3. Let A be the matrix whose lines are (1,0,0), (a,1,0), (0,a,1),  $a \ge 1$ . Then

$$F_A = (X_0^{a+1} : X_0^a X_1 : X_1^a X_2 : X_2^a X_3)$$

and

$$F_{A^{-1}} = (X_0^{a^2 - a + 1} X_1^a X_2^a : X_0^{a^2 - a} X_1^{a + 1} X_2^a : X_0^{a^2} X_2^{a + 1} : X_1^{a^2 + a} X_3).$$

EXAMPLE 3.4. Finite subgroup of  $Bir_{\mathbb{T}}(\mathbb{P}^r)$  are obtained by this method. For example the Weyl's group of type  $W(D_r)$ ,  $r \geq 3$ , and order  $2^{r-1}r!$ , as a Cremona subgroup may be represented as the subgroup generated by an involution of degree 3

$$F_{\alpha_r} = (X_0 X_{r-1} X_r : X_1 X_{r-1} X_r : \dots : X_{r-2} X_{r-1} X_r : X_0^2 X_{r-1} : X_0^2 X_r)$$

and the linear automorphisms  $F_{\alpha_i}$  permuting  $X_i$  with  $X_{i+1}$ ,  $1 \le i < r$ . In an analogous form one obtains a representation of the group  $W(B_r)$ ,  $r \ge 2$ , of order  $2^r r!$ , generated by an involution of degree 2

$$F_{\beta_r} = (X_0 X_r : X_1 X_r : \cdots : X_{r-1} X_r : X_0^2)$$

and the preceding r-1 permutations.

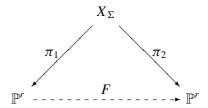
## 4. GEOMETRIC DESCRIPTION, EXAMPLES

Let  $F \in \operatorname{Bir}_{\mathbb{T}}(\mathbb{P}^r)$ ; we denote by  $B_F$  its base-scheme. By composing F with an automorphism induced by an element of the torus we may assume that  $F(1_{\mathbb{T}}) = 1_{\mathbb{T}}$ . e.g.  $F = (1_{\mathbb{T}}, A) \in \mathbb{T} \times \operatorname{GL}_r(\mathbb{Z})$ , we note  $F_A = F$ ; the base-scheme is not changed.

The union of the fundamental hyperplanes  $\mathbb{P}^r \setminus \mathbb{T}$  contains the base-points set.

Let  $e_1, \ldots, e_r$  the canonical basis for  $N = \mathbb{Z}^r$ ,  $e := e_1 + \cdots + e_r$ ,  $\Delta$  the fan associated to the faces of the simplex  $\delta = [e_1, \ldots, e_r, -e]$  and  $A(\Delta)$  the same with respect to the simplex  $A(\delta)$ . We consider  $\mathbb{P}^r$  defined by  $\Delta$ ; let  $\Sigma$  be a fan that is a common subdivision for  $\Delta$  and  $A(\Delta)$ .

Proposition 4.1. One has a commutative diagram



where  $X_{\Sigma}$  is the toric variety associated to  $\Sigma$ ,  $\pi_1$  is equivariant and  $\pi_2$  is equivariant relatively to the automorphism of the torus induced by A.

PROOF. From (3), il follows that F induce  $F^* : \mathbb{K}[\mathbb{T}] \to \mathbb{K}[\mathbb{T}]$  which corresponds, by duality, to the lattice's automorphism  $A : N \to N$ .

EXAMPLE 4.2. Let  $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$  be,  $F_A = (X_0^2, X_0 X_1, X_1 X_2)$ . One obtains a (regular) fan  $\Sigma$  that is a subdivision of  $\Delta$  and  $A(\Delta)$  by blowing-up the closed orbits associated to the cones  $\langle e_1, e_2 \rangle$  and  $\langle e_1, -e \rangle$  and then  $\langle -e_2, -e \rangle$ .

EXAMPLE 4.3. Let A = -Id, then  $F_A = S_r$ . One obtains a (regular) fan  $\Sigma$  by the elementary subdivisions of  $\Delta$  (resp. of  $-\Delta$ ), successively, of the cones of decreasing dimensions from r to 2. For example, if r = 3, let  $\Sigma_0$  (resp.  $\Sigma_0'$ ) be the fan obtained by the elementary subdivisions of the 4 maximal cones of  $\Delta$  (resp.  $-\Delta$ ), and  $\Sigma_1$  (resp.  $\Sigma_1'$ ), and the following subdivisions corresponding to the 6 cones of dimension 2 of  $\Delta$  (resp.  $-\Delta$ ). One has  $\Sigma = \Sigma_1 = \Sigma_1'$ . The toric variety  $V_{\Sigma_0}$  (resp.  $V_{\Sigma_0'}$ ) is the blowing-up of  $\mathbb{P}^3$  in 4 points and  $V_{\Sigma_1}$  (resp.  $V_{\Sigma_1'}$ ) is the blowing-up of these in the strict transforms of the 6 lines. The induced birational map  $V_{\Sigma_0} \longrightarrow V_{\Sigma_0'}$  is composition of 6 flops, e.g., corresponding to small resolutions of singularities of type an affine cone over a smooth quadric, associated to the 6 faces of the convex polyhedron  $P_{\Delta} := Conv(\Delta \cup (-\Delta))$ .

On the other hand, the (normalized) blowing-up of the base-scheme B is the toric variety associated to the fan  $\Sigma(B)$  defined by the faces of the polyhedron  $P(B) = Conv(P_{\Delta} \cup C)$ , where C is the set of sums of the vertices of a diagonal of each 2-face of  $P_{\Delta}$ . The toric variety  $V_{\Sigma(B)}$  has 12 singular points of type an affine cone over a smooth quadric. Finally, the fan  $\Sigma$  is a regular subdivision of  $\Sigma(B)$  and the induced morphism  $V_{\Sigma} \to V_{\Sigma(B)}$  is a minimal resolution of  $V_{\Sigma(B)}$ .

## **ACKNOWLEDGMENTS**

The second author was a visiting Professor at Institut Fourier and also partially supported by the Cooperação Franco-Brasileira.

## **RESUMO**

Descrevemos a estrutura do grupo das transformações de Cremona monomiais. Concluímos que todo elemento deste grupo é um produto de aplicações monomiais quadráticas e damos descrições geométricas em termos de leques.

Palavras-chave: aplicações birracionais monomiais, variedades tóricas.

## REFERENCES

COXETER H AND MOSER W. 1957. Generators and relations for Discrete Groups. Springer-Verlag.

HUDSON HP. 1927. Cremona Transformations in Plane and Space. Cambridge at the University Press.

KATZ S. 1992. Recent works on Cremona Transformations. Research Notes in Math. 2: 109-119, Sundance 90.

ODA T. 1988. Convex Bodies and Algebraic Geometry. Springer-Verlag.

PAN I. 1999. Une remarque sur la génération du groupe de Cremona. Bol Soc Bras Mat 30: 95-98.

Russo F and Simis A. 2001. On birational maps and Jacobian matrices. Compositio Math 126: 335-358.

SIMIS A AND VILLARREAL R. 2002. Constraints for the Normality of Monomial Subrings and birationality, to appear in Proc. of the A.M.S.

TROTT S. 1962. A pair of generators for the unimodular group. Can Math Bull 5: 245-252.