An Acad Bras Cienc (2021) 93(4): e20190301 DOI 10.1590/0001-3765202120190301
Anais da Academia Brasileira de Ciências | *Annals of the Brazilian Academy of Sciences*
Printed ISSN 0001-3765 I Online ISSN 1678-2690
www.scielo.br/aabc | www.fb.com/aabcjournal

MATHEMATICAL SCIENCES

Bayesian inference for the log-symmetric autoregressive conditional duration model

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Abstract: This paper adapts Hamiltonian Monte Carlo methods for application in log-symmetric autoregressive conditional duration models. These recent models are based on a class of log-symmetric distributions. In this class, it is possible to model both median and skewness of the duration time distribution. We use the Bayesian approach to estimate the model parameters of some log-symmetric autoregressive conditional duration models and evaluate their performance using a Monte Carlo simulation study. The usefulness of the estimation methodology is demonstrated by analyzing a high frequency financial data set from the German DAX of 2016.

Key words: ACD models, Bayesian inference, high frequency financial data, log-symmetric distributions.

INTRODUCTION

Bibliographical review and preliminaries

The concept of log-symmetry appears when a random variable presents the same distribution as its reciprocal, or in terms of ordinary symmetry regarding the distribution of the logged random variable; see Jones (2008). The class of distributions having this property is called log-symmetric and it has been used to describe the behavior of strictly positive data. The log-symmetric class encompasses bimodal distributions as special cases, and distributions that possess either lighter or heavier tails than the log-normal distribution, which is a particular case of this class; see e.g., Vanegas & Paula (2016b). Some examples of log-symmetric distributions are: log-normal, log-Student-*t*, log-logistic, log-Laplace, log-Cauchy, log-power-exponential, log-slash, harmonic law, Birnbaum-Saunders, and Birnbaum-Saunders-*t*; see e.g., Crow & Shimizu (1988), Birnbaum & Saunders (1969), Rieck & Nedelman (1991), Johnson et al. (1994, 1995), Díaz-García & Leiva (2005), Marshall & Olkin (2007), Jones (2008), and Vanegas & Paula (2016b).

The class of log-symmetric distributions has been primarily used in the regression context. Vanegas & Paula (2016a) proposed log-symmetric regression models which allow both the median and skewness (or the relative dispersion) be described using an arbitrary number of non-parametric additive components. Vanegas & Paula (2016b) studied some interesting properties of the log-symmetric class of distributions. Vanegas & Paula (2016c) proposed an extension to allow the presence of non-informative left or right-censored data in log-symmetric regression models. Medeiros & Ferrari (2017) discussed the issue of testing hypothesis in symmetric and log-symmetric regression models. In special, the authors considered the Wald, likelihood ratio, score and gradient tests for this purpose. Finally, Ventura et al. (2019) analyzed movie business data using log-symmetric regression models.

High frequency financial data on transactions have been modeled primarily by autoregressive conditional duration (ACD) models, which were proposed by Engle & Russell (1998). These models are commonly used to capture the clustering structure and they can be seen as the counterpart of GARCH models for modeling trade duration (TD) data; see Liu & Heyde (2008). We strongly recommend reading the works by Pacurar (2008) and Bhogal & Variyam (2019), which are literature reviews on ACD models. Some characteristics concerning TD data are: (C1) the irregular nature with respect to the way they are collected; (C2) the diurnal intra-day pattern; (C3) the large number of observations; (C4) the probability density function (PDF) with asymmetric shape; and (C5) the hazard rate (HR) with inverse bathtub shape (unimodal); see e.g., Leiva et al. (2014). Extensions of the original ACD model proposed by Engle & Russell (1998) are basically based on the following aspects: (A1) the distributional assumption in order to yield an asymmetric PDF and an unimodal HR; (A2) the linear form for the conditional mean or median dynamics; (A3) and the time series properties; see, for example, Bauwens & Giot (2000), Grammig & Maurer (2000), Meitz & Terasvirta (2006), Chiang (2007), Pacurar (2008), Bhatti (2010), Leiva et al. (2014), Diana (2015), Dionne et al. (2015), Zheng et al. (2016), Saulo et al. (2019), and Mishra & Ramanathan (2017).

Recently, Saulo & Leao (2017) proposed a family of ACD models based on the class of log-symmetric models. The log-symmetric ACD models encompass all the log-symmetric distributions cited at the beginning of this introduction as special cases, that is, they encompass highly competitive performance models in the literature. For example, the log-normal-ACD, Birnbaum-Saunders-ACD and Birnbaum-Saunders-*t*-ACD models; see Xu (2013) and Leiva et al. (2014). The log-symmetric ACD models are written in terms of a conditional median duration rather than a conditional mean duration - the typical parameter used in the literature is the mean. The use of the median is more interesting because it is a measure of central tendency better than the mean, besides the median-based approach provides additional protection against outliers; see Saulo et al. (2019). On the other hand, the log-symmetric family provides a wide range of asymmetric distributions with HR with inverse bathtub shape. Therefore, characteristics (C4) and (C5) and aspects (A1) and (A2) are addressed by the log-symmetric ACD models.

The flexibility provided by the log-symmetric family makes its corresponding ACD models an important area to be explored in the literature. In this context, this paper deals with the problem of Bayesian inference for log-symmetric-ACD models. The estimation methodology is based on the Hamiltonian Monte Carlo (HMC) method, which generates chains both with little dependence and high probability of acceptance (Neal 2011). The main advantage of log-symmetric ACD models is the robustness property of the median, namely, it is not affected by extremes or outliers. In terms of predictions, it implies that they will not be significantly affected by freak events; see Saulo et al. (2019).

Log-symmetric distributions

A continuous and positive random variable X follows a log-symmetric distribution if its PDF is given by

$$f_X(x;\theta,\phi,g(\cdot)) = \begin{cases} \frac{1}{\sqrt{\phi}x} g\left(a^2(x)\right), & x > 0, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $a(x) = \log\left([x/\theta]^{1/\sqrt{\phi}}\right)$, $\theta > 0$ is a scale parameter, ϕ is a power parameter and g is a density generator with $g(u) > 0$ for $u > 0$ and $\int_0^\infty u^{-1/2} g(u)\mathrm{d}u = 1$. Note that g may involve an extra parameter ξ or an extra parameter vector $\boldsymbol{\xi}$. In this case, the notation $X \sim \mathrm{LS}(\theta,\phi,g)$ is used. For example, in order to obtain a random variable X following a log-normal, log-Student-t (having ξ degrees of freedom), log-Laplace or log-slash distribution, we use, respectively, $g(u) \propto \exp\left(-\frac{1}{2}u\right)$, $g(u) \propto \left(1+\frac{u}{\xi}\right)^{-\frac{\xi+1}{2}}$, $g(u) \propto \exp\left(-\frac{1}{2}u^{\frac{1}{2}}\right)$, $g(u) \propto \mathrm{IGF}\left(\xi+\frac{1}{2},\frac{u}{2}\right)$, where $\mathrm{IGF}(a,x) = \frac{1}{x^a}\int_0^x \exp(-t)t^{a-1}\mathrm{d}t$ is the incomplete gamma function for $a > 0$ and $x \geq 0$. The quantile function of the log-symmetric distribution is given by

$$t_X(q;\theta,\phi,g(\cdot)) = \theta\exp\left[\sqrt{\phi}v_\xi(q)\right], \tag{2}$$

where $v_\xi(q)$ is the $q \times 100$th quantile of $V = \frac{(Y-\mu)}{\sqrt{\phi}} \sim \mathrm{S}(\mu=0,\phi=1,g(\cdot))$, with the notation S referring to symmetrical distributions.

Some statistical properties related to a random variable X following a log-symmetric distribution, namely $X \sim \mathrm{LS}(\theta,\phi,g)$, are: (P1) $X^\star = \left(\frac{X}{\theta}\right)^{1/\sqrt{\phi}} \sim \mathrm{LS}(\theta=1,\phi=1,g)$ is standard log-symmetric distributed; (P2) $cX \sim \mathrm{LS}(c\theta,\phi,g)$, with $c > 0$; (P3) $X^c \sim \mathrm{LS}(\theta^c, c^2\phi, g)$, with $c \neq 0$; (P4) θ is the the median of the distribution of X; and (P5) setting $Y = \log(X)$; (Vanegas & Paula 2015). With this, we obtain a symmetric random variable whose distribution belongs to the symmetric class with PDF given by

$$f_Y(y;\mu,\phi,g(\cdot)) = \frac{1}{\sqrt{\phi}} g\left(\frac{(y-\mu)^2}{\phi}\right), \quad y \in \mathbb{R}, \tag{3}$$

where $\mu = \log(\theta) \in \mathbb{R}$ is a location parameter, $\phi > 0$ is a dispersion parameter and g is as in (2); see Fang et al. (1990) and we write $Y \sim \mathrm{S}(\mu,\phi,g)$. The properties (P2) and (P3) say that the log-symmetric distribution is closed under scale and reciprocal transformations, respectively.

Organization of the paper

The rest of this paper proceeds as follows. The log-symmetric ACD models are formulated in Section LOG-SYMMETRIC ACD MODELS, with parameters estimated by the Bayesian approach using HMC algorithm. In Section NUMERICAL EVALUATION we present a numerical evaluation of the proposed model considering the (i) evaluation of this model via Monte Carlo (MC) simulations and; (ii) application of a real-world high-frequency financial data. Some concluding remarks and possible future research are mentioned in Section CONCLUDING REMARKS.

LOG-SYMMETRIC ACD MODELS

Consider a sequence of successive times $T_1, \ldots, T_n$ at which market events, or trades, occur. Then, the duration or time elapsed between X_i and X_{i-1}, for $i = 1, \ldots, n$, is given by $X_i = T_i - T_{i-1}$. The family of log-symmetric ACD models introduced by Saulo & Leao (2017) considers a dynamic point process in terms of a conditional median duration

$$\theta_i = t_X(0.5; \theta_i, \phi_i, g),$$

where $t_X(\cdot)$ is the quantile function (QF) of the log-symmetric distribution presented in (2) and Ω_{i-1} is an information set including all information available until time T_{i-1}. The log-symmetric ACD model is defined by

$$X_i = \theta_i\, \varepsilon_i^{\sqrt{\phi_i}}, \quad i = 1, \ldots, n, \tag{4}$$

where $\{\varepsilon_i\}$ are independent identically distributed (IID) random variables following a log-symmetric distribution with median and power equal to one, denoted by $\varepsilon_i \overset{\text{IID}}{\sim} \mathrm{LS}(1, 1, g)$, and θ_i and ϕ_i are the median and skewness of the X_i distribution, respectively, as $X_i \overset{\text{IND}}{\sim} \mathrm{LS}(\theta_i, \phi_i, g)$, then the X_is are independent (IND) not identically distributed. The linear form of (4) is given by

$$\underbrace{\log(X_i)}_{Y_i} = \underbrace{\log(\theta_i)}_{\mu_i} + \sqrt{\phi_i}\,\underbrace{\log(\varepsilon_i)}_{\epsilon_i}, \quad i = 1, \ldots, n, \tag{5}$$

where $\epsilon_i \overset{\text{IND}}{\sim} \mathrm{S}(0, 1, g(\cdot))$, namely $\{\epsilon_i\}$, are IID random variables following a standard symmetric distribution with PDF given by (3). Then, we write $Y_i \overset{\text{IND}}{\sim} \mathrm{S}(\mu_i, \phi_i, g)$.

The component θ_i in (4) is defined in terms of autoregressive (AR) and moving average (MA) processes, of order p and q respectively, as

$$\log(\theta_i) = \varpi + \sum_{j=1}^{p} \alpha_j \log(\theta_{i-j}) + \sum_{j=1}^{q} \beta_j \left(\frac{X_{i-j}}{\theta_{i-j}}\right), \tag{6}$$

where $\varpi > 0$, $\alpha_j \geq 0$ and $\beta_j \geq 0$. Then, the notation LSACD(p, q) is used. The order of lags for LSACD(p, q) models, in general, are set as $p = 1$ and $q = 1$, because a higher order does not improve the model fit; see Bhatti (2010). Thereby, in the following, any LSACD$(p = 1, q = 1)$ model is simply denoted as LSACD. Moreover, for simplicity's sake, it is assumed that $\phi_i = \phi$, for $i = 1, \ldots, n$.

Bayesian inference

The Bayesian inference is based on the Bayes theorem

$$\pi(\boldsymbol{\theta}, \phi_i \mid X) \propto f_{X|\theta,\phi}(x_i|\theta_i, \phi_i; g(\cdot))\pi(\boldsymbol{\theta}, \phi_i),$$

where $\pi(\boldsymbol{\theta}, \phi_i \mid X)$ is the posterior distribution and $\pi(\boldsymbol{\theta}, \phi_i)$ is the prior distribution. We will use the following estimator

$$\hat{\boldsymbol{\theta}} = \mathbb{E}_{\boldsymbol{\theta}|y}[\boldsymbol{\theta}], \tag{7}$$

which minimizes the expected squared error of the estimate. The Equation (7) is analytically intractable. Therefore, we adopt HMC sampling strategies for obtaining samples from the joint posterior distributions and adopt the sample mean of the HMC simulation as an estimator of $\mathbb{E}_{\boldsymbol{\theta}|y}[\boldsymbol{\theta}]$. In the next subsection, we present the HMC methodology.

The HMC is a method which combines alternately Gibbs updates and Metropolis ones and avoids the random walk behavior. The main advantage of using HMC as opposed to other methods is to generate chains both with little dependence and high probability of acceptance (Neal 2011), especially when compared to the Metropolis-Hastings algorithm. The HMC method is implemented in the `R` package **`rstan`**; see www.R-project.org and R-Team (2019). Consider a random vector $\boldsymbol{\theta} \in \mathbb{R}^k$ as position variables (parameters) and $\boldsymbol{r} \in \mathbb{R}^k$ an independent auxiliary random vector, with $\boldsymbol{r} \sim N_k(0, M)$. The joint PDF of $(\boldsymbol{\theta}, \boldsymbol{r})$ is given by

$$\pi(\boldsymbol{\theta}, \boldsymbol{r}) \propto \exp(-H(\boldsymbol{\theta}, \boldsymbol{r})), \tag{8}$$

where $H(\boldsymbol{\theta}, \boldsymbol{r}) = U(\boldsymbol{\theta}) + K(\boldsymbol{r})$ is a Hamiltonian function, $U(\boldsymbol{\theta}) = -\log[\pi(\boldsymbol{\theta} \mid y)\pi(\boldsymbol{\theta})]$ is called the *potential energy* and $K(\boldsymbol{r}) = \boldsymbol{r}M^{-1}\boldsymbol{r}$ is called the *kinetic energy*.

This method considers a candidate to $(\boldsymbol{\theta}, \boldsymbol{r})$ which is generated in two stages before being subjected to a Metropolis acceptance step. These stages are presented in Algorithm 1

Algorithm 1 Steps before to acceptance Metropolis

Step 1. Generate a random number of a normal distribution with mean $\mathbf{0}$ and covariance matrix $\boldsymbol{M}$ independently of θ.

Step 2. The joint system $(\boldsymbol{\theta}, \mathbf{r})$ made up of the current parameter values $\boldsymbol{\theta}$, and new momentum $\mathbf{r}$ through Hamiltonian dynamics operates. The system is evolved via Hamilton's equations:

$$\begin{aligned} \frac{\partial \boldsymbol{\theta}}{\partial t} &= \frac{H(\boldsymbol{\theta}, \boldsymbol{r})}{\partial \boldsymbol{r}} = \frac{\partial K(\boldsymbol{r})}{\partial \boldsymbol{r}}, \\ \frac{\partial \boldsymbol{r}}{\partial t} &= \frac{H(\boldsymbol{\theta}, \boldsymbol{r})}{\partial \boldsymbol{\theta}} = -\frac{\partial U(\boldsymbol{\theta})}{\partial \boldsymbol{\theta}}. \end{aligned}$$

Hamilton's equations must be approximated by discretizing time, using some small step-size, ε. The leapfrog method has been used to solve the Hamilton's equations, which works as follows:

$$\begin{aligned} r_i(t + \varepsilon/2) &= r_i(t) - (\varepsilon/2)\frac{\partial U(\boldsymbol{\theta}(t))}{\partial \theta_i}, \\ \theta_i(t + \varepsilon) &= \theta_i(t + \varepsilon/2) - \varepsilon\frac{\partial K(\boldsymbol{r}(t + \varepsilon/2))}{\partial r_i}, \\ r_i(t + \varepsilon) &= r_i(t + \varepsilon/2) - (\varepsilon/2)\frac{\partial U(\boldsymbol{\theta}(t + \varepsilon))}{\partial \theta_i}. \end{aligned}$$

It applies L leapfrog steps, a total of L_ε time is simulated. The resulting state at the end of the simulation (L repetitions of the three steps above) is denoted by $(\boldsymbol{\theta}^*, \boldsymbol{r}^*)$. Applying the Metropolis algorithm, the state (θ^*, ω^*) is then accepted as the next state of the Markov chain with probability

$$\mathbb{P}(\boldsymbol{\theta}, \boldsymbol{\omega}; \boldsymbol{\theta}^*, \boldsymbol{\omega}^*) = \min\{1, \exp\{H(\boldsymbol{\theta}, \boldsymbol{\omega}) - H(\boldsymbol{\theta}^*, \boldsymbol{\omega}^*)\}\}.$$

NUMERICAL EVALUATION

In this section, we carry out a simulation study to evaluate the performance of the Bayesian estimators of some log-symmetric ACD models. Then, we illustrate the proposed methodology by applying it to a real-world high-frequency financial data set. This data set refers to price durations of BASF-SE stock on 19th April 2016 downloaded from the Dukascopy site (www.dukascopy.com). We consider the ACD models based on the following log-symmetric distributions: log-normal (LNACD), log-Laplace (LLACD), log-Student-t (LtACD) and log-slash (LSACD). To estimate the parameters of the model, we used the HMC algorithm using `rstan` in the `R` software. In Stan's programming language the convergence of a Markov chain to a stationary distribution by no-U-turn sampler (NUTS) which is even more efficient at exploring the posterior; see Hoffman & Gelman (2014). The interested reader in Stan programming language and HMC algorithm is referred to Hoffman & Gelman (2014) and Carpenter et al. (2017).

A simulation study

The scenario considers: sample size $n \in \{500, 2000, 5000\}$, vector of true parameters $(\omega, \alpha, \beta, \phi)^\top = \{0.10,\ 0.90,\ 0.10, 0.50\}$ for LNACD and LLACD models; $(\omega, \alpha, \beta, \phi, \xi)^\top = \{0.10,\ 0.90,\ 0.10, 4.00\}$ for LtACD and LSACD models, and the following independent prior distributions: $\omega \sim \mathrm{N}(0, 100)\mathbb{I}(\omega > 0)$; $\alpha \sim \mathrm{N}(0, 100)\mathbb{I}(0 \leq \alpha \leq 1)$; $\beta \sim \mathrm{N}(0, 100)\mathbb{I}(0 \leq \beta \leq 1)$; $\mu \sim \mathrm{N}(0, 100)$; $\xi \sim \mathrm{N}(0, 100)\mathbb{I}(\xi > 2)$ if $\varepsilon_t \sim$ log-t_ξ; and $\xi \sim \mathrm{N}(0, 100)\mathbb{I}(\xi > 1)$ if $\varepsilon_t \sim$ log-slash$_\xi$.

For each value of the parameter and sample size, we report the empirical values for the mean, median, standard deviation (SD) and the percentage of data sets where the true parameter value was contained inside the Bayesian 90% and 95% credible intervals: this is the MC estimate of frequentist coverage for an interval estimator. The estimates computed by the Bayesian approach are presented by Tables I and II. As the sample size increases, the Bayesian estimators become more efficient. Therefore, in general, all of these results show the good performance of the Bayesian estimators of the corresponding parameters.

Analysis of high-frequency financial transaction data

We now illustrate the log-symmetric ACD models by analyzing the BASF-SE data set. A data adjustment is necessary due to the fact that these data often exhibit certain diurnal patterns; see Tsay (2002). We used the `R` package `ACDm` (see Belfrage 2015) to perform the diurnal adjustment of the BASF-SE data. We simulated two chains with 1500 iterations and discarded the first 500 as burn-in. Table III presents some descriptive statistics for the BASF-SE data set, including central tendency statistics, standard deviation (SD), coefficient of variation (CV), skewness (CS) and kurtosis (CK). From Table III, we observe the right skewed nature and high kurtosis level of the data distribution. The skewness is ratified by the histogram showed by Figure 1(a).

A tool to characterize the shape of a hazard rate is the scaled total time on test (TTT) function; see Aarset (1987). The hazard rate of a random variable X is defined by $h(x) = f(x)/[1 - F(x)]$, where f and F are the PDF and cumulative distribution function (CDF) of X. The scaled TTT function is defined by $W(u) = H^{-1}(u)/H^{-1}(1)$, for $0 \leq u \leq 1$, where $H^{-1}(u) = \int_0^{F^{-1}(u)}[1 - F(y)]\mathrm{d}y$, with F^{-1} being the inverse function of the CDF of X. An approximation for W is obtained by plotting the points $[k/n, W_n(k/n)]$, with

$$W_n(k/n) = \frac{\sum_{i=1}^{k} x_{(i)} + [n - k]x_k}{\sum_{i=1}^{n} x_{(i)}}, \quad k = 1, \ldots, n,$$

and $x_{(i)}$ being the ith observed order statistic. Figure 1(b) suggests that the hazard rate for the BASF-SE data set is inverse-bathtub-shaped, as expected; see Leiva et al. (2014).

Table I. Summary statistics from simulated LNACD and LLACD data for the indicated estimators and *n*.

Statistic	LNACD			LLACD		
	n			n		
	500	2000	5000	500	2000	5000
	$\widehat{\omega}$			$\widehat{\omega}$		
True value	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
Mean	0.1607	0.1176	0.0973	0.1928	0.1071	0.0998
Median	0.1467	0.1104	0.0963	0.1848	0.0956	0.0992
SD	0.1196	0.0571	0.0275	0.0892	0.0614	0.0165
CR90%	0.8838	0.8866	0.9022	0.7606	0.8930	0.9004
CR95%	0.9194	0.9318	0.9486	0.8430	0.9404	0.9474
	$\widehat{\alpha}$			$\widehat{\alpha}$		
True value	0.9000	0.9000	0.9000	0.9000	0.9000	0.9000
Mean	0.8195	0.8831	0.9004	0.8346	0.9135	0.8988
Median	0.8268	0.8872	0.9011	0.8386	0.9197	0.8986
SD	0.0608	0.0310	0.0143	0.0437	0.0322	0.0073
CR90%	0.6774	0.8552	0.9022	0.6056	0.9050	0.8932
CR95%	0.7760	0.9072	0.9502	0.7220	0.9400	0.9506
	$\widehat{\beta}$			$\widehat{\beta}$		
True value	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
Mean	0.1686	0.1146	0.1026	0.1344	0.0684	0.1052
Median	0.1682	0.1138	0.1025	0.1340	0.0667	0.1052
SD	0.0289	0.0157	0.0083	0.0203	0.0126	0.0038
CR90%	0.2262	0.7710	0.8830	0.4852	0.6102	0.8932
CR95%	0.3326	0.8552	0.9346	0.6202	0.7276	0.9455
	$\widehat{\phi}$			$\widehat{\phi}$		
True value	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
Mean	0.5212	0.5022	0.4945	0.4391	0.4479	0.4985
Median	0.5209	0.5021	0.4944	0.4383	0.4477	0.4987
SD	0.0161	0.0078	0.0049	0.0195	0.0129	0.0086
CR90%	0.6374	0.8878	0.8956	0.3196	0.8684	0.8974
CR95%	0.7506	0.9392	0.9492	0.4300	0.9264	0.9448

Table II. **Summary statistics from simulated L*t*ACD and LSACD data for the indicated estimators and *n*.**

Statistic	L*t*ACD			LSACD		
	n			*n*		
	500	2000	5000	500	2000	5000
	$\widehat{\omega}$			$\widehat{\omega}$		
True value	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
Mean	0.0557	0.0985	0.1035	0.2890	0.1224	0.0960
Median	0.0530	0.0985	0.1034	0.2639	0.1221	0.0926
SD	0.0355	0.0021	0.0046	0.1769	0.0425	0.0289
CR90%	0.6292	0.7988	0.8116	0.7702	0.8104	0.9066
CR95%	0.7622	0.8804	0.9540	0.8506	0.8826	0.9536
	$\widehat{\alpha}$			$\widehat{\alpha}$		
True value	0.9000	0.9000	0.9000	0.9000	0.9000	0.9000
Mean	0.9162	0.9001	0.8999	0.7501	0.8857	0.8991
Median	0.9180	0.9001	0.8998	0.7636	0.8864	0.9007
SD	0.0180	0.0007	0.0022	0.0872	0.0218	0.0148
CR90%	0.7542	0.8998	0.8982	0.5314	0.7608	0.9080
CR95%	0.8588	0.9470	0.9476	0.6543	0.8438	0.9528
	$\widehat{\beta}$			$\widehat{\beta}$		
True value	0.1000	0.1000	0.1000	0.1000	0.1000	0.1000
Mean	0.1020	0.0993	0.0991	0.1975	0.1061	0.1042
Median	0.1019	0.0992	0.0991	0.1974	0.1059	0.1038
SD	0.0121	0.0013	0.0016	0.0320	0.0119	0.0077
CR90%	0.8396	0.8520	0.8926	0.7920	0.7970	0.8814
CR95%	0.9094	0.9192	0.9430	0.8065	0.8750	0.9348
	$\widehat{\phi}$			$\widehat{\phi}$		
True value	0.5000	0.5000	0.5000	0.5000	0.5000	0.5000
Mean	0.4912	0.5070	0.5137	0.5987	0.4986	0.4995
Median	0.4909	0.5069	0.5138	0.6006	0.4943	0.4987
SD	0.0296	0.0140	0.0087	0.0269	0.0303	0.0123
CR90%	0.5236	0.8556	0.8854	0.3420	0.8960	0.9036
CR95%	0.6430	0.9210	0.9416	0.5260	0.9476	0.9496
	$\widehat{\xi}$			$\widehat{\xi}$		
True value	4.0000	4.0000	4.0000	4.0000	4.0000	4.0000
Mean	3.8302	3.5650	3.9848	5.5243	4.1830	3.9987
Median	3.7381	3.5486	3.9756	5.6100	4.9140	3.9790
SD	0.7377	0.2866	0.2217	0.8520	0.4132	1.4082
CR90%	0.5270	0.9092	0.9024	0.5809	0.9096	0.9023
CR95%	0.6512	0.9584	0.9536	0.6460	0.9160	0.9429

Table III. **Summary statistics for the BASF-SE data.**

n	Minimum	Median	Mean	Maximum	SD	CV	CS	CK
2194	0.061	0.682	1.067	9.776	1.167	109.35%	2.521	8.902

Figure 1(c) shows the usual and adjusted box-plots, where the latter is useful in cases when the data follow a skew distribution, since a significant number of observations can be classified as atypical when they are not; see Hubert & Vanderveeken (2008). From Figure 1(c), we note that potential outliers considered by the usual box-plot are not influential when we observe the adjusted box-plot.

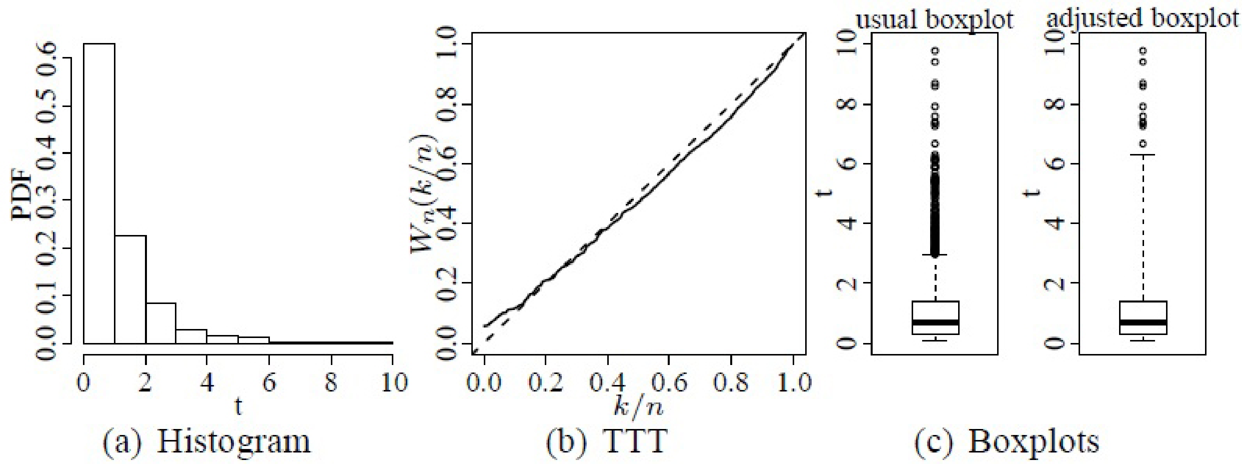

Figure 1. Histogram (left), TTT plot (center) and boxplots (right) for the BASF-SE data.

Table IV reports the Bayesian estimates of the LNACD, L*t*ACD, LLACD and LSACD model parameters (with estimated standard errors in parentheses). Furthermore, we report the Expected Bayesian Information Criterion (EBIC), Deviance Information Criterion (DIC), Watanabe-Akaike Information Criterion (WAIC) and Bayesian Information Criterion (BIC); see Akaike (1992), Watanabe (2010), Schwarz et al. (1978) and Spiegelhalter et al. (2002).

Table IV. Estimates (with SE in parentheses) for fit to the BASF-SE data.

	LNACD	L*t*ACD	LLACD	LSACD
ϖ	−0.3228 (0.0831)	−0.3211 (0.0853)	−0.2122 (0.0507)	−0.2203 (0.0821)
α	0.4748 (0.1673)	0.4809 (0.1681)	0.6646 (0.1145)	0.5819 (0.1632)
β	0.0507 (0.0107)	0.0516 (0.0117)	0.0564 (0.0117)	0.0512 (0.0113)
ϕ	1.0694 (0.0164)	1.0597 (0.0187)	0.8720 (0.0183)	0.9641 (0.0163)
ξ		6.7067 (1.0554)		7.8025 (4.9732)
AIC	6543.0070	6546.5520	6787.1950	6604.3071
BIC	6562.6380	6571.0910	6806.8260	6643.2816
WAIC	6539.0000	6541.1000	6789.8000	6671.2800
DIC	6524.3930	6590.7080	6815.9870	6695.7233

From Table IV, note that the LNACD model provides the better adjustment compared to the other models based on the values of EBIC, DIC, WAIC and BIC. Table V reports the estimate of effective sample size and *R* statistic of Gelman et al. (1992), we observed that the generated chains were efficient.

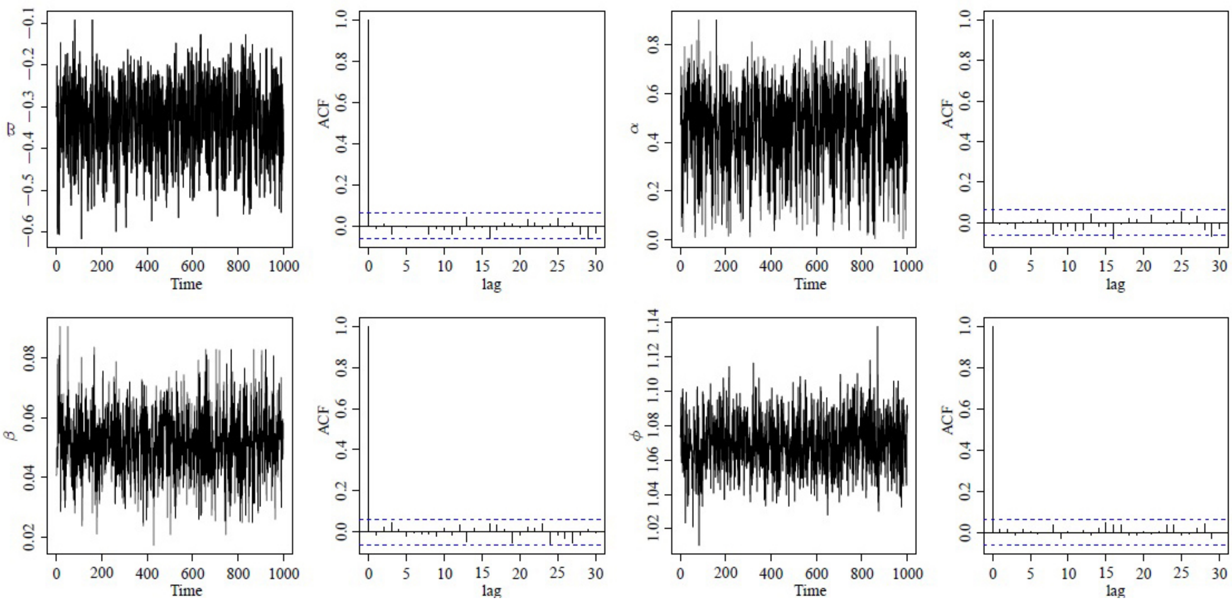

Figure 2. **Trace plots and ACF for the LNACD model.**

Table V. **Estimate of effective sample size and** R **statistic.**

	LNACD		L*t*ACD		LLACD		LSACD	
	n_eff	$\widehat{R}$	n_eff	$\widehat{R}$	n_eff	$\widehat{R}$	n_eff	$\widehat{R}$
ϖ	380	1.00	563	1.01	531	1.01	571	1.01
α	313	1.00	624	1.01	579	1.01	631	1.01
β	384	1.01	748	1.00	515	1.00	683	1.00
ϕ	358	1.00	1065	1.00	1087	1.00	895	1.00
ξ			2698	1.00	947	1.00	607	1.01

Figure 2 shows the sample autocorrelation function (ACF) and trace plots for the parameters from LNACD model. From this figure, we can notice the absence of autocorrelation and that the chains converge to their stationary distributions also observed by the statistic R of the Table V.

CONCLUDING REMARKS

We have discussed Bayesian inference for log-symmetric autoregressive conditional duration models, which are based on the conditional median duration. We have employed Hamiltonian Monte Carlo sampling strategies to obtain the estimates of the model parameters. A Monte Carlo simulation study was carried out to evaluate the behavior of the Bayesian estimates of the corresponding parameters. We have applied the proposed models to a real-world data set of financial transactions from the German DAX stock exchange.

As part of future research, it would be of interest to propose an outlier detection procedure to detect and estimate outlier effects for these models; see Chiang & Wang (2012). Also, influence diagnostic tools can be extended to log-symmetric autoregressive conditional duration models; see Leiva et al. (2014). Work on these issues is currently in progress and we hope to report some findings in a future paper.

Acknowledgments

The research was partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico (CNPq) and Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES) grants from the Brazilian government and by Fundação de Amparo à Pesquisa do Estado do Amazonas (FAPEAM) grants from the government of the state of Amazonas, Brazil.

References

AARSET M. 1987. How to identify a bathtub hazard rate. IEEE Transactions on Reliability 36: 106-108.

AKAIKE H. 1992. Information theory and an extension of the maximum likelihood principle. In: Breakthroughs in statistics, Springer, p. 610-624.

BAUWENS L & GIOT P. 2000. The logarithmic ACD model: An application to the bid-ask quote process of three NYSE stocks. Ann Econ Stat 60: 117-149.

BELFRAGE M. 2015. R package ACDm: Tools for autoregressive conditional duration model. https://cran.r-project.org/web/packages/ACDm.

BHATTI C. 2010. The Birnbaum-Saunders autoregressive conditional duration model. Math Comput Simul 80: 2062-2078.

BHOGAL SK & VARIYAM RT. 2019. Conditional duration models for high-frequency data: A review on recent developments. J Econ Surv 33(1): 252-273.

BIRNBAUM ZW & SAUNDERS SC. 1969. A new family of life distributions. J Appl Probab 6: 319-327.

CARPENTER B, GELMAN A, HOFFMAN MD, LEE D, GOODRICH B, BETANCOURT M, BRUBAKER M, GUO J, LI P & RIDDELL A. 2017. Stan: A probabilistic programming language. J Stat Softw 76(1).

CHIANG MH. 2007. A smooth transition autoregressive conditional duration model. Stud Nonlinear Dyn Econom 11: 108-144.

CHIANG MH & WANG LM. 2012. Additive outlier detection and estimation for the logarithmic autoregressive conditional duration model. Commun Stat: Simul Comput 41: 287-301.

CROW EL & SHIMIZU K. 1988. Lognormal Distributions: Theory and Applications. New York, US: Dekker.

DIANA T. 2015. Measuring the impact of traffic flow management on interarrival duration: An application of autoregressive conditional duration. J Air Transp Manag 42: 219-225.

DÍAZ-GARCÍA J & LEIVA V. 2005. A new family of life distributions based on elliptically contoured distributions. J Stat Plan Inference 128: 445-457.

DIONNE G, PACURAR M & ZHOU X. 2015. Liquidity-adjusted intraday value at risk modeling and risk management: An application to data from Deutsche Börse. J Bank Finance 59: 202-219.

ENGLE R & RUSSELL J. 1998. Autoregressive conditional duration: A new method for irregularly spaced transaction data. Econometrica 66: 1127-1162.

FANG KT, KOTZ S & NG KW. 1990. Symmetric Multivariate and Related Distributions. London, UK: Chapman and Hall.

GELMAN A, RUBIN DB ET AL. 1992. Inference from iterative simulation using multiple sequences. Stat Sci 7(4): 457-472.

GRAMMIG J & MAURER K. 2000. Non-monotonic hazard functions and the autoregressive conditional duration model. J Econom 3: 16-38.

HOFFMAN MD & GELMAN A. 2014. The No-U-Turn sampler: adaptively setting path lengths in Hamiltonian Monte Carlo. J Mach Learn Res 15(1): 1593-1623.

HUBERT M & VANDERVEEKEN S. 2008. Outlier detection for skewed data. J Chemom 22: 235-246.

JOHNSON N, KOTZ S & BALAKRISHNAN N. 1994. Continuous Univariate Distributions. Vol. 1. New York, US: Wiley.

JOHNSON N, KOTZ S & BALAKRISHNAN N. 1995. Continuous Univariate Distributions. Vol. 2. New York, US: Wiley.

JONES MC. 2008. On reciprocal symmetry. J Stat Plan Inference 138: 3039-3043.

LEIVA V, SAULO H, LEÃO J & MARCHANT C. 2014. A family of autoregressive conditional duration models applied to financial data. CSDA 79: 175-191.

LIU S & HEYDE CC. 2008. On estimation in conditional heteroskedastic time series models under non-normal distributions. Stat Pap 49: 455-469.

MARSHALL A & OLKIN I. 2007. Life Distributions. New York, US: Springer.

MEDEIROS FMC & FERRARI SLP. 2017. Small-sample testing inference in symmetric and log-symmetric linear regression models. Stat Neerl 71(3): 200-224.

MEITZ M & TERASVIRTA T. 2006. Evaluating models of autoregressive conditional duration. J Bus Econ Stat 24: 104-124.

MISHRA A & RAMANATHAN TV. 2017. Nonstationary autoregressive conditional duration models. Stud Nonlinear Dyn Econom 21(4): 1-22.

NEAL RM. 2011. MCMC using Hamiltonian dynamics. In: Handbook of Markov Chain Monte Carlo. Vol. 2. p. 113-162. Boca Raton: Chapman and Hall-CRC Press.

PACURAR M. 2008. Autoregressive conditional durations models in finance: A survey of the theoretical and empirical literature. J Econ Surv 22: 711-751.

R-TEAM. 2019. R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing. Austria: Vienna. URL www.R-project.org.

RIECK J & NEDELMAN J. 1991. A log-linear model for the Birnbaum-Saunders distribution. Technometrics 3: 51-60.

SAULO H & LEAO. 2017. On log-symmetric duration models applied to high frequency financial data. Econ Bull 37: 1089-1097.

SAULO H, LEAO J, LEIVA V & AYKROYD RG. 2019. Birnbaum-Saunders autoregressive conditional duration models applied to high-frequency financial data. Stat Pap 60: 1605-1629.

SCHWARZ G ET AL. 1978. Estimating the dimension of a model. The annals of statistics 6(2): 461-464.

SPIEGELHALTER DJ, BEST NG, CARLIN BP & VAN DER LINDE A. 2002. Bayesian measures of model complexity and fit. J R Stat Soc: Series B (Stat Methodol) 64(4): 583-639.

TSAY RS. 2002. Analysis of Financial Time Series. New Jersey, US: Wiley.

VANEGAS L & PAULA G. 2016a. An extension of log-symmetric regression models: R codes and applications. J Stat Comput Simul 86: 1709-1735.

VANEGAS L & PAULA G. 2016b. Log-symmetric distributions: statistical properties and parameter estimation. Braz J Probab Stat 30: 196-220.

VANEGAS L & PAULA G. 2016c. Log-symmetric regression models under the presence of non-informative left-or right-censored observations. Test, p. 1-24.

VANEGAS LH & PAULA GA. 2015. A semiparametric approach for joint modeling of median and skewness. Test 24(1): 110-135.

VENTURA M, SAULO H, LEIVA V & MONSUETO SE. 2019. Log-symmetric regression models: information criteria and application to movie business and industry data. Appl Stoch Models Bus Ind 35: 963-977.

WATANABE S. 2010. Asymptotic equivalence of Bayes cross validation and widely applicable information criterion in singular learning theory. J Mach Learn Res 11: 3571-3594.

XU Y. 2013. The lognormal autoregressive conditional duration (LNACD) model and a comparison with an alternative ACD models. Available at SSRN: https://ssrncom/abstract=2382159 doi:10.2139/ssrn.2382159.

ZHENG Y, LI Y & LI G. 2016. On Fréchet autoregressive conditional duration models. J Stat Plan Inference 175: 51-66.

How to cite
LEÃO J, PAIXÃO R, SAULO H & LEAO T. 2021. Bayesian inference for the log-symmetric autoregressive conditional duration model. An Acad Bras Cienc 93: e20190301. DOI 10.1590/0001-3765202120190301.

Manuscript received on March 19, 2019;
accepted for publication on July 18, 2019

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