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MATHEMATICAL SCIENCE

Division of Power Series: Recursive and Non-Recursive Formulas

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Abstract: In this paper we propose a new formula to divide power series. We develop two versions of the formula: a recursive and a non-recursive one, the latter aiming to reduce the computational cost for high-order series truncation. To use the non-recursive formula we define certain fundamental sets of summation indexes. Additional non-trivial information about effects of repetition of the indexes are needed and contabilized within a coefficient γ in the formula, we explain how to calculate the coefficient γ for each summation index by constructing appropriate mappings between the fundamental sets of indexes previous defined.

Key words: Division of series, non-recursive formula, power series, recursive formula.

INTRODUCTION

Operating with power series is important in applied mathematics, physics and engineering since many problems of real life lead to complicated calculations with smooth functions, which can be approximated by truncated power series. There are many works on power series and operations with them. The Cauchy product (discrete convolution given in (4)) of two power series, allows calculation of the n-th coefficient of the product without needs to calculate the previous coefficients $0 \leq k \leq n-1$. For division of power series, in general, the classical algorithms calculate iteratively the n-th coefficient of the quotient, *i.e.*, they calculate all the previous coefficient $k < n$ of the quotient series. For instance, the traditional algebraic method to divide power series is iterated "polynomial long division", which contain cumbersome calculations especially for high-order truncated series, such iterative method has elevated computational cost (it requires more than $n(n-1)$ multiplications and $n(n-1)/2$ additions). Other classical methods to calculate the quotient P/Q of power series are based in the previous calculation of the power series that represents the reciprocal $1/Q$ and then perform the product $P \cdot 1/Q$ by the Cauchy formula. Kaluza 1928, Lamperti 1958 have important works on the coefficients of reciprocal power series. From a computational point of view, Sieveking 1972 used the reciprocal $1/Q$ to develop a fast iterative algorithm for division of power series, which needs less than $7(n-1)$ nonscalar multiplications to obtain the first n coefficients of the quotient's power series. Kung 1974, showed that root-finding iterations can be used in the field of power series and obtained a class of algorithms for computing reciprocals of power series with no more than $4(n-1)$ nonscalar multiplications. He also showed that Sieveking's algorithm for computing reciprocals is the same that Newton iteration. Brent & Kung 1978 used the Newton's method for the computation of the first

n coefficients of the quotient series in computational time $O(n \ log \ n)$. In this paper we propose a new formula (in closed-form)to divide power series. We develop two versions of the formula: a recursive and a non-recursive one, the latter aiming to reduce the computational cost for high-order series truncation; all (high-order) necessary multi-indexes sets and coefficients can be pre-calculated, stored, and used to divide power series. With our non-recursive formula, the calculation of the n-th coefficient of the quotient power series is independent of the calculation of previous coefficients. This is an advantage over the methods before mentioned. We show two examples that illustrate the usefulness of our new formula (in its non-recursive version) to calculate one of the coefficients of the division power series without having to calculate the previous coefficients. For the particular case of the reciprocal $1/Q$ of a power series, the problem of calculate each coefficient a_k of the reciprocal through our non-recursive formula, reduces to evaluate a $k+1$ degree multivariate rational function in $k+1$ variables. Horner's Rule or similar strategies could be used in order to evaluate the polynomials efficiently, see Carnicer & Gasca 1990. Implementation of algorithms based on our formulas, detailed studies about computational complexity and establish comparisons with existing computational methods to divide power series will be part of future works.

Our formula to divide power series can be applied also to differential calculus of smooth functions. Since Analytic functions f, g can be represented by convergent power series via Taylor's expansion, and their high order derivatives are related with the coefficients of their power series, our non-recursive formula can be applied to represent the n-th derivative of the fraction f/g in terms of the derivatives f^k, g^k, $0 \leq k \leq n$. As far as we known, the only formula that exists for this type of calculation (but not in the context of power series) was provided by Quaintance & Gould 2016 (see Section 8.3, equation 8.33). This formula relates high order derivatives of the quotient f/g with the derivatives of the functions f, g and other non-linear expressions of these functions. He did not used power series, instead, he obtained his formula as application of the generalized chain rule and Hoppe's formula.

MOTIVATION

Of course, developing a formula for a quotient of power series have a theoretical value itself as a powerful tool in mathematical analysis. Nevertheless, we are specially motivated on potential applications to asymptotic methods which are widely used to solve partial differential equations and requires formal manipulations and operation with series through the process of obtaining the solutions. Maslov 1977, proposed an asymptotic method that can be applied to obtain three types of singular solutions (shocks, vortices and infinitely narrow solitons) for distinct problems modeled by hyperbolic conservation laws. This asymptotic method uses smooth power series expansions of the solutions in a neighborhood of the singularity to obtain the well known Hugoniot-Maslov chain to be satisfied by the coefficients of the asymptotic expansion; see, for instance, Maslov 1977, Dobrokhotov 1999, Bulatov et al. 1997, Dobrokhotov et al. 2004. Of course, the nonlinearity of the flux functions of the conservation laws leads to complicated calculations involving power series such as: iterated products of power series, n-power of a power series, quotient between power series, and other nonlinear operations between them. We refer Alvarez & Valiño-Alonso 2001, Rodríguez-Bermúdez & Valiño-Alonso 2007, Bernard et al. 2012, Rodríguez-Bermúdez & Valiño-Alonso 2018 for examples where iterated products and n-power of power series were important to perform the asymptotic method. When the flux function of the conservation law contains fractions; as in Buckley-Leverett equation modeling two-phase flow in porous media with applications to oil engineering, division of power series are needed to perform the asymptotic of the solutions. Motivated by that, we propose here both a recursive and a non-recursive versions of a formula to divide power series, which can be applied in those cases. See Rodríguez-Bermúdez et al. 2021 for an important application of our non-recursive formula when performing asymptotic methods to obtain shock-type solutions for the Buckley-Leverett equation.

We remark that in this work, we are not worried about convergence of the series, we focus only on formal division of series when it is possible.

In the third section, we propose a recursive formula for the quotient of power series, the validity of the formula was proved by induction method. In the fourth section, we propose an alternative non-recursive formula to calculate the quotient of the power series, such way of calculation has significant less computational cost than the original recursive version; however, it requires additional information about repetition of summation indexes; we explain how to take into account such information in the formula by defining a set of suitable mappings. In the fifth section we illustrate with two examples the usefulness of our formulas.

A PRACTICAL FORMULA TO DIVIDE POWER SERIES: RECURSIVE VERSION

Given two power series $\sum c_k\tau^k$ and $\sum b_k\tau^k$, we provide a formula for their quotient $\sum a_k\tau^k = \sum c_k\tau^k / \sum b_k\tau^k$.

Proposition 1. *Consider the power series $\sum c_k\tau^k$ and $\sum b_k\tau^k$, and assume that the first coefficient b_0 of the series $\sum b_k\tau^k$ is not zero. The coefficients a_k of the series corresponding to the quotient $\sum a_k\tau^k = \sum c_k\tau^k / \sum b_k\tau^k$ are given by*

$$a_k = \frac{1}{{b_0}^{k+1}} \sum_{m=0}^{k} c_m \alpha(k,m), \tag{1}$$

where for each m and all $k \geq m$ the polynomials $\alpha(k,m)$ are defined by the following recurrence formula

$$\alpha(k,m) = -\sum_{i=1}^{k-m} \alpha(k-i,m) b_0^{i-1} b_i \tag{2}$$

with $\alpha(m,m) = b_0^m$. We also define $\alpha(k,m) = 0,$ for $k < m$.

Proof. First we focus in a product of two convergent power series, given the series $\sum_{k=0}^{\infty} a_k\tau^k$ and $\sum_{k=0}^{\infty} b_k\tau^k$ their product is of course another convergent power series:

$$\sum_{k=0}^{\infty} c_k\tau^k = \left(\sum_{k=0}^{\infty} a_k\tau^k\right)\left(\sum_{k=0}^{\infty} b_k\tau^k\right). \tag{3}$$

If we perform the product term by term, we obtain by straight forward calculations (Cauchy product)

$$c_k = \sum_{i=0}^{k} a_i b_{k-i}, \forall k. \tag{4}$$

We start from the above expression for the product, in order to obtain a formula for the quotient $\dfrac{\sum c_k \tau^k}{\sum b_k \tau^k}$. We will illustrate the recursive calculation of the first four coefficients a_0, a_1, a_2, a_3 of the quotient. First we calculate the polynomial functions $\alpha(k,m)$, defined in (2) for $k = 0,1,2,3$ and $m \leq k$

$$\begin{aligned} &\alpha(0,0) = 1,\ \alpha(1,0) = -b_1,\ \alpha(1,1) = b_0,\ \alpha(2,0) = b_1^2 - b_0 b_2, \\ &\alpha(2,1) = -b_0 b_1,\ \alpha(2,2) = b_0^2, \\ &\alpha(3,0) = -b_1^3 + 2b_0 b_1 b_2 - b_0^2 b_3,\ \alpha(3,1) = b_0 b_1^2 - b_0^2 b_2, \\ &\alpha(3,2) = -b_0^2 b_1,\ \alpha(3,3) = b_0^3. \end{aligned} \tag{5}$$

Taking into account the expressions in (5), the product rule (3)-(4) allows the recursive calculation of the coefficients a_k as follows

$$\begin{aligned} c_0 &= a_0 b_0 \Rightarrow a_0 = \frac{c_0}{b_0} = \frac{1}{b_0}\left[c_0 \alpha(0,0)\right], \\ c_1 &= a_0 b_1 + a_1 b_0 \Rightarrow a_1 = \frac{c_1 - \frac{c_0}{b_0} b_1}{b_0} = \frac{c_1}{b_0} - \frac{c_0 b_1}{{b_0}^2} \\ &\qquad = \frac{c_1 b_0 - c_0 b_1}{{b_0}^2} = \frac{1}{b_0}\left[c_0 \alpha(1,0) + c_1 \alpha(1,1)\right], \\ c_2 &= a_0 b_2 + a_1 b_1 + a_2 b_0 \Rightarrow a_2 = \frac{c_2 - \frac{c_0}{b_0} b_2 - \frac{c_1 b_0 - c_0 b_1}{{b_0}^2} b_1}{b_0} \\ &\qquad = \frac{1}{b_0^3}\left[c_2 b_0^2 - c_0 b_0 b_2 - c_1 b_0 b_1 + c_0 b_1^2\right] \\ &\qquad = \frac{1}{b_0^3}\left[c_0 \alpha(2,0) + c_1 \alpha(2,1) + c_2 \alpha(2,2)\right], \\ c_3 &= a_0 b_3 + a_1 b_2 + a_2 b_1 + a_3 b_0 \Rightarrow a_3 = \frac{c_3 - a_0 b_3 - a_1 b_2 - a_2 b_1}{b_0}, \quad \text{thus} \\ a_3 &= \frac{c_3}{b_0} - \frac{\frac{c_0}{b_0} b_3}{b_0} - \frac{\left(\frac{c_1 b_0 - c_0 b_1}{{b_0}^2}\right) b_2}{b_0} - \frac{\frac{1}{b_0^3}\left(c_2 b_0^2 - c_0 b_0 b_2 - c_1 b_0 b_1 + c_0 b_1^2\right) b_1}{b_0} \\ &= \frac{1}{b_0^4}\left[c_3 b_0^3 - c_0 b_0^2 b_3 - c_1 b_0^2 b_2 + c_0 b_0 b_1 b_2 - c_2 b_0^2 b_1 + c_0 b_0 b_1 b_2 + c_1 b_0 b_1^2 - c_0 b_1^3\right] \\ &= \frac{1}{b_0^4}\left[c_0 \alpha(3,0) + c_1 \alpha(3,1) + c_2 \alpha(3,2) + c_3 \alpha(3,3)\right]. \end{aligned}$$

Thus, the relation $a_k = \dfrac{1}{{b_0}^{k+1}} \displaystyle\sum_{m=0}^{k} c_m \alpha(k,m)$ holds for $k = 0,1,2,3$. Of course, in order to prove this for general k we have to perform a formal mathematics induction, we assume that $a_k = \dfrac{1}{{b_0}^{k+1}} \displaystyle\sum_{m=0}^{k} c_m \alpha(k,m)$ for a given k, and we will prove that the same argument is true for the $k+1$ coefficient.

Indeed, (4) for $k+1$ gives

$$c_{k+1} = \sum_{i=0}^{k+1} a_i b_{k-i+1}, \forall k. \tag{6}$$

Since $b_0 \neq 0$, (6) can be solved for a_{k+1} to obtain

$$a_{k+1} = \frac{1}{b_0^{k+2}}\Big[c_{k+1}b_0^{k+1} - \sum_{j=0}^{k}\sum_{m=0}^{j} c_m \alpha(j,m) b_0^{k-j} b_{k-j+1}\Big]. \tag{7}$$

Using the polynomial functions $\alpha(j,m)$ defined in (2) and taking into account that $\alpha(j,m)=0$ for $m>j$, we interchange the summation symbols and obtain

$$a_{k+1} = \frac{1}{b_0^{k+2}}\left[c_{k+1}b_0^{k+1} + \sum_{m=0}^{k} c_m\Big[-\sum_{j=m}^{k}\alpha(j,m)b_0^{k-j}b_{k-j+1}\Big]\right]. \tag{8}$$

Now we use definition in (2) after setting $i=k-j$ to obtain

$$a_{k+1} = \frac{1}{b_0^{k+2}}\left[c_{k+1}\alpha(k+1,k+1) + \sum_{m=0}^{k} c_m \alpha(k+1,m)\right]. \tag{9}$$

Therefore,

$$a_{k+1} = \frac{1}{b_0^{k+2}}\left[\sum_{m=0}^{k+1} c_m \alpha(k+1,m)\right]. \tag{10}$$

completing the induction proof.

□

A NON-RECURSIVE FORMULA

In order to calculate a_k for a high order k, the recursive combination (1)-(2) in general produces high computational cost and we do not recommend it. An alternative non-recursive way to calculate the coefficients a_k of the quotient series can be performed by using (1) together with the following formula for $\alpha(k,m)$

$$\alpha(k,m) = \sum_{(S_0,\cdots,S_k)\in\Omega_{k,m}} (-1)^{k+S_0}\gamma(\vec{S})\prod_{i=0}^{k} b_i^{S_i}, \tag{11}$$

with $\gamma(\vec{S})$ being a natural number which reflects contribution of a repeated index $\vec{S}\in\Omega_{k,m}$ in the summation.

The fundamental sets $\Omega_{k,m}$ of indices, are defined below

$$\Omega_{k,m} = \Big\{(S_0,\cdots,S_k)\in\mathbb{N}^{k+1}, \quad \text{such that} \quad \sum_{n=0}^{k} S_n = k, \quad \sum_{n=1}^{k} nS_n = k-m\Big\}. \tag{12}$$

Remark. *Notice that* $\Omega_{k,m}=\emptyset$ *for* $k<m$.

Remark. *Notice that the polynomials* $\alpha(k,m)$ *defined in* (11) *represent a* k*- degree homogeneous polynomial in* $k+1$ *variables* $b_0, b_1, ..., b_k$.

Example 1. *For $k = 1, 2, 3, 4$ and $m = 0$ we have the sets of indices:*

$$\Omega_{4,0} = \{(0,4,0,0,0); (1,2,1,0,0); (2,1,0,1,0); (2,0,2,0,0); \tag{13}$$

$$(3,0,0,0,1)\}, \tag{14}$$

$$\Omega_{3,0} = \{(0,3,0,0); (1,1,1,0,); (2,0,0,1)\}, \tag{15}$$

$$\Omega_{2,0} = \{(0,2,0); (1,0,1)\}, \tag{16}$$

$$\Omega_{1,0} = \{(0,1)\}, \tag{17}$$

$$\Omega_{0,0} = \{0\}. \tag{18}$$

Developing the recurrence formula (2) *for $k = 1, k = 2, k = 3, k = 4$, etc... we rewrite the expressions for $\alpha(k,m)$ in the format of* (11)*, indeed*

$$\alpha(0,0) = 1, \quad \textit{here } \gamma(0) = 1 \textit{ in } \Omega_{0,0}. \tag{19}$$

$$\alpha(1,0) = -\alpha(0,0)b_1 = -b_1, \quad \textit{here } \gamma((0,1)) = 1 \textit{ in } \Omega_{1,0}. \tag{20}$$

$$\begin{aligned} \alpha(2,0) &= -\alpha(1,0)b_1 - \alpha(0,0)b_0b_2 = b_1^2 - b_0b_2, \\ &\textit{here } \gamma((0,2,0)) = 1, \gamma((1,0,1)) = 1 \textit{ in } \Omega_{2,0}. \end{aligned} \tag{21}$$

$$\begin{aligned} \alpha(3,0) &= -\alpha(2,0)b_1 - \alpha(1,0)b_0b_2 - \alpha(0,0)b_0^2b_3 \\ &= -b_1^3 + 2b_0b_1b_2 - b_0^2b_3, \\ &\textit{here } \gamma((0,3,0,0)) = 1, \gamma((2,0,0,1)) = 1, \\ &\gamma((1,1,1,0)) = 2 \textit{ in } \Omega_{3,0}. \end{aligned} \tag{22}$$

$$\begin{aligned} \alpha(4,0) &= -\alpha(3,0)b_1 - \alpha(2,0)b_0b_2 - \alpha(1,0)b_0^2b_3 \\ &- \alpha(0,0)b_0^3b_4 \\ &= b_1^4 - 3b_0b_1^2b_2 + 2b_0^2b_1b_3 + b_0^2b_2^2 - b_0^3b_4, \\ &\textit{here } \gamma((0,4,0,0,0)) = 1, \gamma((2,0,2,0,0)) = 1, \\ &\gamma((3,0,0,0,1)) = 1, \gamma((2,1,0,1,0)) = 2, \\ &\gamma((1,2,1,0,0)) = 3 \textit{ in } \Omega_{4,0}. \end{aligned} \tag{23}$$

Of course, in the generic case, the difficulty to calculate the proper value of the coefficient $\gamma(\vec{S})$ for each arbitrary index $\vec{S} \in \Omega_{k,m}$ represents the main disadvantage of this non-recursive formula given in (11) when compared to the recursive formula in (2). However, for special cases where all the coefficients $\gamma(\vec{S})$ (which appear in the calculations) are known "a priori" (11) can be used, providing a fast and efficient computational way of dividing power series.

Calculating the Coefficients $\gamma(\vec{S})$

Now we explain how to calculate $\gamma(\vec{S})$ in order to apply the non-recursive formula in (11). We proceed as follows.

For each j fixed ($m \le j < k$) and for each set of indexes $(S_0, \cdots, S_j) \in \Omega_{j,m}$ we have that $S_0 + S_1 + \cdots + S_j = j$ and $S_1 + 2S_2 + \cdots + jS_j = j - m$.

We define the following map $\psi_j^{k,m} : \Omega_{j,m} \to \Omega_{k,m}$, such that $(S_0, \cdots, S_j) \mapsto (\tilde{S}_0, \cdots, \tilde{S}_k)$ as follows

(i) In the case $j = 0$, we set $\tilde{S}_0 = S_0 + k - 1$; $\tilde{S}_i = 0$, for $i = 1, \cdots, k-1$; $\tilde{S}_k = 1$;

(ii) In the case $1 \le j < (k-1)/2$, we set $\tilde{S}_0 = S_0 + k - j - 1$; $\tilde{S}_i = S_i$, for $i = 1, \cdots, j$; $\tilde{S}_{j+1} = \cdots = \tilde{S}_{k-j-1} = 0$; $\tilde{S}_{k-j} = 1$; $\tilde{S}_{k-j+1} = \cdots = \tilde{S}_k = 0$.

(iii) In the case $j = (k-1)/2$, we set $\tilde{S}_0 = S_0 + (k-1)/2$; $\tilde{S}_i = S_i$, for $i = 1, \cdots, (k-1)/2$; $\tilde{S}_{[(k-1)/2]+1} = 1$; $\tilde{S}_{[(k-1)/2]+2} = \cdots = \tilde{S}_k = 0$.

(iv) In the case $(k-1)/2 < j \le k-1$, we set $\tilde{S}_0 = S_0 + k - j - 1$; $\tilde{S}_i = S_i$ for $i = 1, \cdots, j$ with $i \ne k - j$; $\tilde{S}_i = S_i + 1$ for $i = k - j$, $\tilde{S}_{j+1} = \cdots = \tilde{S}_k = 0$.

Indeed, notice that in all above cases

$$\tilde{S}_0 + \tilde{S}_1 + \cdots + \tilde{S}_k = (S_0 + k - j - 1) + S_1 + \cdots + S_j + 1 = k, \tag{24}$$

$$\begin{aligned}\tilde{S}_1 + 2\tilde{S}_2 + \cdots + k\tilde{S}_k &= S_1 + 2S_2 + \cdots + jS_j + (k-j) \\ &= k - m.\end{aligned} \tag{25}$$

Thus, we have that $(\tilde{S}_0, \cdots, \tilde{S}_k) \in \Omega_{k,m}$.

Definition 1. *Given a vector of indexes $\vec{S} \in \Omega_{k,m}$, the coefficient $\gamma(\vec{S})$ is defined by a natural number which quantify all the inverse-image-composition-paths of the form $[(\psi_m^{i,m})^{-1} \circ \cdots \circ (\psi_j^{k,m})^{-1}]$, $m < i \le j < k$, connecting the element $\vec{S}$ in $\Omega_{k,m}$ with the element $(m, 0, ..., 0)$ in $\Omega_{m,m}$.*

Example 2. *For $m = 0$, $k = 4$, we have*

$$\Omega_{0,0} = \{0\}, \tag{26}$$

$$\Omega_{1,0} = \{(0,1)\}, \tag{27}$$

$$\Omega_{2,0} = \{(0,2,0); (1,0,1)\}, \tag{28}$$

$$\Omega_{3,0} = \{(0,3,0,0); (1,1,1,0,); (2,0,0,1)\}, \tag{29}$$

$$\Omega_{4,0} = \{(0,4,0,0,0); (1,2,1,0,0); (2,1,0,1,0); (2,0,2,0,0); (3,0,0,0,1)\}. \tag{30}$$

For each set $\Omega_{i,0}$, $i = 1,2,3,4$, let us to apply the maps

$$\psi_j^{i,0} : \Omega_{j,0} \to \Omega_{i,0} \ \ \textit{for} \ \ 0 \le j < i$$

$$\begin{aligned}
\text{codomain } \Omega_{1,0} :& \psi_0^{1,0}(0) = (0,1), \\
\text{codomain } \Omega_{2,0} :& \psi_0^{2,0}(0) = (1,0,1); \psi_1^{2,0}((0,1)) = (0,2,0), \\
\text{codomain } \Omega_{3,0} :& \psi_0^{3,0}(0) = (2,0,0,1); \psi_1^{3,0}((0,1)) = (\mathbf{1,1,1,0}); \\
& \psi_2^{3,0}((0,2,0)) = (0,3,0,0); \psi_2^{3,0}((1,0,1)) = (\mathbf{1,1,1,0}), \\
\text{codomain } \Omega_{4,0} :& \psi_0^{4,0}(0) = (3,0,0,0,1); \psi_1^{4,0}((0,1)) = (\mathbf{2,1,0,1,0}); \\
& \psi_2^{4,0}((0,2,0)) = (\mathbf{1,2,1,0,0}); \psi_2^{4,0}((1,0,1)) = (2,0,2,0,0); \\
& \psi_3^{4,0}((0,3,0,0)) = (0,4,0,0,0); \psi_3^{4,0}((1,1,1,0)) = (\mathbf{1,2,1,0,0}); \\
& \psi_3^{4,0}((2,0,0,1)) = (\mathbf{2,1,0,1,0}).
\end{aligned}$$

Since there are two inverse image composition-paths connecting the element $(1,1,1,0)$ *(red color) in* $\Omega_{3,0}$ *with the element* 0 *in* $\Omega_{0,0}$, *we have* $\gamma((1,1,1,0)) = 2$. *Indeed, notice that*

$$[(\psi_0^{1,0})^{-1} \circ (\psi_1^{3,0})^{-1}](1,1,1,0) = 0,$$
$$[(\psi_0^{2,0})^{-1} \circ (\psi_2^{3,0})^{-1}](1,1,1,0) = 0.$$

As in the previous case, there are two inverse image composition-paths connecting the element $(2,1,0,1,0)$ *(red color) in* $\Omega_{4,0}$ *with the element* 0 *in* $\Omega_{0,0}$, *we have* $\gamma((2,1,0,1,0)) = 2$.

$$[(\psi_0^{1,0})^{-1} \circ (\psi_1^{4,0})^{-1}](2,1,0,1,0) = 0,$$
$$[(\psi_0^{3,0})^{-1} \circ (\psi_3^{4,0})^{-1}](2,1,0,1,0) = 0.$$

By the other hand, there are three inverse image composition-paths connecting the element $(1,2,1,0,0)$ *(blue color) in* $\Omega_{4,0}$ *with the element* 0 *in* $\Omega_{0,0}$, *we have* $\gamma((1,2,1,0,0)) = 3$. *Indeed, notice that*

$$[(\psi_0^{1,0})^{-1} \circ (\psi_1^{2,0})^{-1} \circ (\psi_2^{4,0})^{-1}](1,2,1,0,0) = 0,$$
$$[(\psi_0^{2,0})^{-1} \circ (\psi_2^{3,0})^{-1} \circ (\psi_3^{4,0})^{-1}](1,2,1,0,0) = 0,$$
$$[(\psi_0^{1,0})^{-1} \circ (\psi_1^{3,0})^{-1} \circ (\psi_3^{4,0})^{-1}](1,2,1,0,0) = 0.$$

The rest of the indexes (black color) have only one inverse image composition path so they have $\gamma(\vec{S}) = 1$.

APPLICATIONS

Example 3. *Consider the Taylor series at* $x = 0$ *of the functions* e^x *and* $\cos x$,

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n \tag{31}$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \tag{32}$$

By using the non-recursive formula for the division of power series we will calculate directly the 5*-th coefficient (*$k = 4$*) of the quotient series.*

Notice that in this case, $c_0 = 1$, $c_1 = 1$, $c_2 = 1/2$, $c_3 = 1/6$, $c_4 = 1/24$ *while* $b_0 = 1$, $b_1 = 0$, $b_2 = -1/2$, $b_3 = 0$, $b_4 = 1/24$. *We define the vector of coefficients* $\vec{b} = (1, 0, -1/2, 0, 1/24)$ *and* $\vec{c} = (1, 1, 1/2, 1/6, 1/24)$.

We evaluate the k-degree homogeneous polynomials $\alpha(k,m)$ *(in their non-recursive form* (11)*) for* $k = 4$, $m = 0,1,2,3,4$ *at the given vector* $\vec{b}$*; see the Appendix section to see all the fundamental sets* $\Omega_{k,m}$ *and the corresponding coefficient* γ *of each index. Thus, we have* $\alpha(4,0) = 5/24$, $\alpha(4,1) = 0$, $\alpha(4,2) = 1/2$, $\alpha(4,3) = 0$, $\alpha(4,4) = 1$. *Therefore, by using* (1) *and the given vector* $\vec{c}$, *we obtain* $a_4 = 1/2$, i.e., *the* 5*-th coefficient of the quotient power series.*

Example 4 (Calculating the Reciprocal $1/Q$)**.** *As an important special case, we can use our formula to obtain the Reciprocal power series* $1/Q$ *of a given power series* $Q = \sum b_i x^i$. *For this case, we have*

$c_0 = 1$ and $c_i = 0$, for $1 \leq i$; therefore the coefficient a_k of the reciprocal power series is the rational function in the variables $b_0, b_1, ...b_k$

$$a_k = \frac{\alpha(k,0)}{b_0^{k+1}}. \tag{33}$$

For instance if $Q = \cos x$, since the coefficient $b_0 = 1$, the 5-th coefficient of the reciprocal power series $1/Q$ is just $a_4 = \alpha(4,0) = 5/24$; i.e., *this is the 5-th coefficient of the Taylor series for the $\sec x$ function.*

Remark. *In order to evaluate efficiently the multivariate polynomial $\alpha(4,0)$ at the given vector $\vec{b}$, one can use Horner's Rule or similar strategies, see for instance (Carnicer & Gasca 1990). The problem of finding the best strategy to reduce the computational cost of evaluating the polynomials $\alpha(k,m)$ will be the focus of future works. However, in order to illustrate the usefulness of our method, for this special example we rewrite the polynomial $\alpha(4,0)$ in (23), as follows:*

$$\alpha(4,0) = (b_1)^2((b_1)^2 - 3b_0b_2) + (b_0)^2(b_2^2 - b_0b_4 + 2b_1b_3) \tag{34}$$

Thus we only need of 8 nonscalar multiplications and 4 additions to evaluate the above polynomial. For a case in which $b_0 \neq 1$, the calculation of a_4 via (33) requires three additional operations (two multiplications and one division) totalizing 11 nonscalar multiplications/divisions and 4 additions. These are few operations when compared with traditional methods like polynomials "long division" which requires at least 21 multiplications and 10 additions to obtain iteratively the coefficients $a_i, i = 0, 1, 2, 3, 4$ of the quotient power series.

Remark. *If we want to calculate each of the first 5 coefficients a_0, a_1, a_2, a_3, a_4 of the reciprocal power series $1/Q$, it would be necessary to evaluate all the polynomials $\alpha(0,0)$, $\alpha(1,0)$, $\alpha(2,0)$, $\alpha(3,0)$ and $\alpha(4,0)$ detailed in (19)-(23) and then to use the formula (33). These calculations require 14 nonscalar multiplications, 4 divisions and 7 additions; these are less operations than polynomials "long division" method and contain a similar amount of nonscalar multiplications than Sieveking's algorithm (Sieveking 1972) or Kung's algorithms (Kung 1974) to obtain the first 5 coefficients of the reciprocal's power series .*

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APPENDIX

$$\Omega_{4,0} = \{(0,4,0,0,0);(1,2,1,0,0);(2,1,0,1,0);(2,0,2,0,0);(3,0,0,0,1)\}, \quad (35)$$

$$\Omega_{4,1} = \{(1,3,0,0,0);(2,1,1,0,0);(3,0,0,1,0)\}, \quad (36)$$

$$\Omega_{4,2} = \{(2,2,0,0,0);(3,0,1,0,0)\}, \quad (37)$$

$$\Omega_{4,3} = \{(3,1,0,0,0)\}, \quad (38)$$

$$\Omega_{4,4} = \{(4,0,0,0,0)\}. \quad (39)$$

The coefficients $\gamma(\vec{S})$ for the ordered fundamental sets (35)-(39) are stored in the following vectors

$\vec{\gamma}_{4,0} = (1,3,2,1,1)$, $\vec{\gamma}_{4,1} = (1,2,1)$, $\vec{\gamma}_{4,2} = (1,1)$, $\vec{\gamma}_{4,3} = 1$, $\vec{\gamma}_{4,4} = 1$