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MATHEMATICAL SCIENCES

Height estimates and half-space theorems for hypersurfaces in product spaces of the type $\mathbb{R} \times M^n$

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Abstract: We obtain height estimates and half-space theorems concerning a wide class of hypersurfaces immersed into a product space $\mathbb{R} \times M^n$, the so-called generalized linear Weingarten hypersurfaces, which extends that one having some constant higher order mean curvature.

Key words: Product spaces, generalized linear Weingarten hypersurfaces, height estimates, half-space theorems.

1 - INTRODUCTION

The last decades have seen a steadily growing interest in the study of a priori estimates for the height function of constant mean curvature compact graphs or, more generally, compact hypersurfaces with boundary having some constant higher order mean curvature. This problem has gained special attention, being considered by several authors probably motivated by the fact that these estimates turn out to be a very useful tool in order to investigate existence and uniqueness results for complete hypersurfaces with constant higher order mean curvature, as well as to obtain information on the topology at infinity of such hypersurfaces (see, for instance, Aledo et al. 2008, 2010, Alías & Dajczer 2007, Cheng & Rosenberg 2005, Espinar et al. 2009, Heinz 1969, Hoffman et al. 2006, Korevaar et al. 1992, 1989, Rosenberg 1993).

A height estimate of compact graphs with positive constant mean curvature in the Euclidean space $\mathbb{R}^{n+1}$ and boundary in a hyperplane, were first obtained by Heinz 1969. More specifically, denoting by H the mean curvature, Heinz proved that the height of such a graph can rise at most $1/H$. More than twenty years after that, Korevaar et al. 1992 obtained a sharp bound for compact graphs and for compact embedded hypersurfaces in the hyperbolic space $\mathbb{H}^{n+1}$ with nonzero constant mean curvature and boundary in a horosphere. More generally, given an arbitrary Riemannian manifold M^n, height estimates in the product space $\mathbb{R} \times M^n$ for constant mean curvature compact embedded hypersurfaces with boundary in a slice were exhibited by Hoffman et al. 2006 and Aledo et al. 2008, for $n = 2$, and by Alías & Dajczer 2007, for an arbitrary dimension n.

Regarding hypersurfaces having some constant higher order mean curvature, this was done firstly by Rosenberg 1993, who proved height estimates for compact embedded hypersurfaces with

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zero boundary values either in the Euclidean space or in the hyperbolic space, generalizing the previous estimates of Heinz and Korevaar. Later on, Cheng & Rosenberg 2005 were able to generalize these estimates for compact graphs with some constant higher order mean curvature in the product manifold $\mathbb{R} \times M^n$, with boundary in a slice. As application of their height estimates, they used the Alexandrov's reflection technique to prove that a noncompact properly embedded hypersurface having constant higher order mean curvature in $\mathbb{R}\times M^n$, where M^n is a compact manifold with sectional curvature bounded from below, has at least two ends or, equivalently, it cannot lie in a half-space. The same technique was used by Hoffman et al. 2006 in order to obtain some information on the topology at infinity of properly embedded surfaces of constant mean curvature in $\mathbb{R} \times M^2$. More recently, Rosenberg et al. 2013 showed that an entire minimal graph with nonnegative height function in a product space $\mathbb{R} \times M^n$, whose base M^n is a complete Riemannian manifold having non-negative Ricci curvature and with sectional curvature bounded from below, must be a slice.

Proceeding with the picture described above, in this paper our aim is to obtain height estimates and half-space theorems of a wide class of hypersurfaces immersed into a product space $\mathbb{R} \times M^n$, which extends that one having some constant higher order mean curvature. Precisely, we consider in $\mathbb{R} \times M^n$ *generalized linear Weingarten* hypersurfaces, by meaning that there exists a linear relation involving some of the corresponding higher order mean curvatures (for more details, see Section 3). We point out that our results offer improvements of those ones obtained in Alías & Dajczer 2007, Alías et al. 2016, Cheng & Rosenberg 2005 and Hoffman et al. 2006. Furthermore, we are able to prove half-space theorems for complete noncompact generalized linear Weingarten hypersurfaces in $\mathbb{R}\times M^n$, generalizing some results of Cheng & Rosenberg 2005 and Hoffman et al. 2006. Recently, the authors proved similar results for the case of hypersurfaces immersed into warped product manifolds (see de Lima & de Lima 2018). However, as we will see, the results presented there do not contemplate those obtained here.

This manuscript is organized in the following way: In Section 2 we introduce some basic facts and notations that will appear in the proofs of our results. In particular, we recall some geometric conditions which guarantee the ellipticity of the linearized operator of the higher order mean curvature (see Lemmas 2 and 3). In Section 3, we establish our first main results concerning height estimates of compact generalized linear Weingarten hypersurfaces in $\mathbb{R} \times M^n$ (see Theorems 1 and 2). In Section 4, as application of our height estimates, we prove half-space theorems related to noncompact generalized linear Weingarten hypersurface immersed in $\mathbb{R} \times M^n$, supposing that the fiber M^n is compact (see Theorems 3 and 4). Finally, when M^n is not necessarily compact, using a generalized version of the Omori-Yau maximum principle for trace type differential operators, we prove other half-space theorem, which is of independent interest by itself (see Theorem 5).

2 - PRELIMINARIES

In this section we will introduce some basic facts and notations that will appear along the paper. In this sense, along this work we will always consider M^n a (connected) n-dimensional Riemannian manifold and $I \subset \mathbb{R}$ an open interval in $\mathbb{R}$. Let us denote by $\overline{M}^{n+1} = I \times M^n$ the product manifold endowed with the Riemannian metric

$$\langle\, ,\, \rangle = \pi_I^*(dt^2) + \pi_M^*(\langle\, ,\, \rangle_M),$$

where π_I and π_M denote the canonical projections from $I \times M^n$ onto each factor, $\langle\, ,\, \rangle_M$ is the Riemannian metric on the fiber M^n and I is endowed with the metric dt^2. Observe that ∂_t is a unitary vector field globally defined on $\overline{M}^{n+1} = I \times M^n$, which determines on $\overline{M}^{n+1}$ a codimension one foliation by totally geodesic slices $\{t\} \times M$.

Throughout this paper, we will study (connected) two-sided hypersurfaces $\psi : \Sigma^n \to \overline{M}^{n+1}$ immersed into the product Riemannian manifold $\overline{M}^{n+1} = I \times M^n$, which means that there exists a unitary normal vector field N globally defined on Σ^n. As usual, we also denote by $\langle\, ,\, \rangle$ the metric of Σ^n induced via ψ. In this setting, we consider two particular functions naturally attached to the two-sided hypersurface Σ^n, namely, the (vertical) height function $h = \pi_{\mathbb{R}} \circ \psi$ and the angle function $\Theta = \langle N, \partial_t \rangle$.

Let us denote by $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ the shape operator (or Weingarten endomorphism) of Σ^n in $\overline{M}^{n+1} = I \times M^n$ with respect to N, which is given by $AX = -\overline{\nabla}_X N$, where $\overline{\nabla}$ stands for the Levi-Civita connection of $\overline{M}^{n+1}$. A fact well known is that the curvature tensor R of the hypersurface Σ^n can be described in terms the shape operator A and of the curvature tensor $\overline{R}$ of the ambient space $\overline{M}^{n+1} = I \times M^n$ by the Gauss equation given by

$$R(X, Y)Z = (\overline{R}(X, Y)Z)^\top + \langle AX, Z\rangle AY - \langle AY, Z\rangle AX \tag{2.1}$$

for every tangent vector fields $X, Y, Z \in \mathfrak{X}(\Sigma)$, where $(\ )^\top$ denotes the tangential component of a vector field in $\mathfrak{X}(\overline{M}^{n+1})$ along Σ^n.

Associated with the shape operator A there are n algebraic invariants, which are the elementary symmetric functions S_r of its principal curvatures $\varkappa_1, \ldots, \varkappa_n$, given by

$$S_r = S_r(\varkappa_1, \ldots, \varkappa_n) = \sum_{i_1 < \ldots < i_r} \varkappa_{i_1} \cdots \varkappa_{i_r}, \qquad 1 \le r \le n.$$

As it is well known, the r-mean curvature H_r of the hypersurface Σ^n is defined by

$$\binom{n}{r} H_r = S_r(\varkappa_1, \ldots, \varkappa_n).$$

In particular, when $r = 1$,

$$H_1 = \frac{1}{n} \sum_i \varkappa_i = \frac{1}{n} \mathrm{tr}(A) = H$$

is just the mean curvature of Σ^n. When $r = 2$, H_2 defines a geometric quantity which is related to the (intrinsic) scalar curvature S of the hypersurface. For instance, when the ambient space has constant sectional curvature c, it follows from the Gauss equation that $S = (n-1)(c + H_2)$. In general, it also follows from Gauss equation of the hypersurface that when r is odd H_r is extrinsic (and its sign depends on the chosen orientation), while when r is even H_r is an intrinsic geometric quantity.

It is a classical fact that the higher order mean curvatures satisfy a very useful set of inequalities, usually alluded as Newton's inequalities. For future reference, we collect them here. A proof can be found in Hardy et al. 1989.

Lemma 1. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a two-sided hypersurface immersed into a product space $\overline{M}^{n+1} = I \times M^n$. For each $1 \le r \le n$, if $H_1, \ldots, H_r$ are nonnegative on Σ^n, then:*

(a) $H_r H_{r+2} \le H_{r+1}^2$;

(b) $H_1 \geq H_2^{1/2} \geq \ldots \geq H_r^{1/r}$,

and equality holds only at umbilical points.

For each $0 \leq r \leq n$, one defines the r-th *Newton transformation* $P_r : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ of the hypersurface Σ^n by setting $P_0 = I$ (the identity operator) and, for $1 \leq r \leq n$, via the recurrence relation

$$P_r = \binom{n}{r} H_r I - AP_{r-1},$$

Equivalently,

$$P_r = \sum_{j=0}^{r} \binom{n}{j} (-1)^{r-j} H_j A^{r-j},$$

so that the Cayley-Hamilton theorem gives $P_n = 0$. Observe also that when r is even, the definition of P_r does not depend on the chosen unitary normal vector field N, but when r is odd there is a change of sign in the definition of P_r. Moreover, it is easy to see that each P_r is a self-adjoint operator which commutes with shape operator A, that is, if a local orthonormal frame on Σ^n diagonalizes A, then it also diagonalizes each P_r. More specifically, if $\{E_1, \ldots, E_n\}$ is such a local orthonormal frame with $A(E_i) = \varkappa_i E_i$, then

$$P_r(E_i) = \mu_{i,r} E_i,$$

where

$$\mu_{i,r} = \sum_{i_1 < \cdots < i_r, i_j \neq i} \varkappa_{i_1} \cdots \varkappa_{i_r}.$$

It follows from here that for each $0 \leq r \leq n-1$, we have

$$\operatorname{tr}(P_r) = c_r H_r, \quad \text{with} \quad c_r = (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}.$$

Let ∇ stand for the Levi-Civita connection of the two-sided hypersurface Σ^n. Associated to each Newton transformation P_r, one has the second order linear differential operator $L_r : C^\infty(\Sigma) \to C^\infty(\Sigma)$ for $r = 0, 1, \ldots, n-1$, defined by

$$L_r u = \operatorname{tr}(P_r \circ \operatorname{hess}\ u),$$

where hess $u : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ denotes the self-adjoint linear operator metrically equivalent to the Hessian of u, Hess u, which are given by

$$\operatorname{hess}\ u(X) = \nabla_X \nabla u \ \text{ and } \ \operatorname{Hess}\ (X, Y) = \langle \operatorname{hess}\ u(X), Y \rangle,$$

respectively, for all $X, Y \in \mathfrak{X}(\Sigma)$. In particular, $L_0 = \Delta$, the Laplacian of Σ^n, which is always an elliptic operator in divergence form. More generally, it is well known that the operator L_r is elliptic if and only if P_r is positive definite.

For our applications, it will be useful to have some geometric conditions which guarantee the ellipticity of L_r when $r \geq 1$. For $r = 1$, the next lemma assures the ellipticity of L_1 (see Lemma 3.10 of Elbert 2002).

Lemma 2. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a two-sided hypersurface immersed into a product space $\overline{M}^{n+1} = I \times M^n$. If $H_2 > 0$ on Σ, then L_1 is elliptic or, equivalently, P_1 is positive definite (for an appropriate choice of the orientation N).*

When $r \geq 2$, the following lemma give us sufficient conditions to guarantee the ellipticity of L_r. The proof is given in Proposition 3.2 of Cheng & Rosenberg 2005 (see also Proposition 3.2 of Barbosa & Colares 1997).

Lemma 3. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a two-sided hypersurface (with or without boundary) immersed into a product space $\overline{M}^{n+1} = I \times M^n$ with $H_{r+1} > 0$ on Σ^n, for some $2 \leq r \leq n-1$. If there exists an interior point p of Σ^n such that all the principal curvatures at p are nonnegative, then for all $1 \leq k \leq r$ the operator L_k is elliptic and the $(k+1)$-mean curvature H_{k+1} is positive.*

Next, we close this section with the following formulas, which will be essential for the proofs of our main results (for more details see Proposition 6 and Lemma 27 of Alías et al. 2013).

Proposition 1. *Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a two-sided hypersurface immersed into a product space $\overline{M}^{n+1} = I \times M^n$. For every $r = 0, \ldots, n-1$:*

(a) The height function satisfies

$$\mathrm{Hess}\ h(X,Y) = \Theta\langle AX, Y\rangle \quad \text{and} \quad L_r h = c_r \Theta H_{r+1},$$

where $c_r := (n-r)\binom{n}{r} = (r+1)\binom{n}{r+1}$.

(b) The angle function satisfies

$$\begin{aligned} L_r\Theta &= -\frac{c_r}{r+1}\langle \nabla H_{r+1}, \nabla h\rangle - \frac{c_r\Theta}{r+1}\left(nH_1H_{r+1} - (n-r-1)H_{r+2}\right) \\ &- \Theta\sum_{i=1}^{n}\mu_{i,r}K_M(N^*, E_i^*)|N^* \wedge E_i^*|^2, \end{aligned}$$

where $\{E_1, \ldots, E_n\}$ is an orthonormal frame on Σ^n diagonalizing A, K_M denotes the sectional curvature of the fiber M^n, $\mu_{i,r}$ stands for the eigenvalues of P_k and, for every vector field $X \in \mathfrak{X}(\overline{M})$, X^* is the orthogonal projection on TM.

3 - HEIGHT ESTIMATES OF GENERALIZED LINEAR WEINGARTEN HYPERSURFARCES

This section is devoted to establish our results concerning to estimates of the height function h of a wide class of two-sided hypersurfaces immersed into a product Riemannian manifold $\overline{M}^{n+1} = \mathbb{R} \times M^n$, which extends that one having some constant higher order mean curvature. Specifically, let us consider $\psi : \Sigma^n \to \overline{M}^{n+1}$ a two-sided hypersurface immersed into a product space $\overline{M}^{n+1} = \mathbb{R} \times M^n$. We say that Σ^n is (r,s)-*linear Weingarten*, for some $0 \leq r \leq s \leq n-1$, if there exist nonnegative real numbers $b_r, \ldots, b_s$ (at least one of them nonzero) such that the following linear relation holds on Σ^n:

$$\sum_{k=r}^{s} b_k H_{k+1} = d \in \mathbb{R}. \tag{3.1}$$

Thus, naturally attached to a (r,s)-linear Weingarten two-sided hypersurface we have the constant d given by (3.1). We note that the (r,r)-linear Weingarten two-sided hypersurfaces are exactly the two-sided hypersurfaces having $d = H_{r+1}$ constant. On the other hand, if the ambient space has zero sectional curvature and taking into account that in this case $\overline{S} = H_2$, where $\overline{S}$ stands for the normalized scalar curvature of Σ^n, we observe that the $(0,1)$-linear Weingarten two-sided hypersurfaces are called simply linear Weingarten two-sided hypersurfaces. Throughout this paper, we will always denote by d the constant given by equation (3.1).

Now, we are in position to state and prove our first main result. More precisely, we will establish an estimate for the height function concerning (r,s)-linear Weingarten two-sided hypersurface in a product space $\mathbb{R} \times M^n$.

Theorem 1. *Let $\overline{M}^{n+1} = \mathbb{R} \times M^n$ be a product space whose the fiber M^n has nonnegative sectional curvature K_M. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a compact (r,s)-linear Weingarten two-sided hypersurface with $(s+1)$-mean curvature $H_{s+1} \neq 0$ on Σ^n, for some $0 \leq s \leq n-1$, and boundary $\partial\Sigma^n$ contained into the slice $\{t_0\} \times M^n$ for some $t_0 \in \mathbb{R}$. Suppose that the angle function Θ does not change sign on Σ^n. Then,*

(a) Either $\max h \neq t_0$ *and*
$$\Sigma^n \subset \left[t_0, t_0 + \frac{1}{\min |H_1|}\right] \times M^n,$$

(b) or $\min h \neq t_0$ *and*
$$\Sigma^n \subset \left[t_0 - \frac{1}{\min |H_1|}, t_0\right] \times M^n.$$

Proof. First of all it is clear from our hypothesis on the $(s+1)$-mean curvature that either $\max h \neq t_0$ or $\min h \neq t_0$. So, we begin by assuming that $\max h \neq t_0$ and let us choose an interior point p_0 of Σ^n such that the height function reaches its maximum and the orientation so that $\Theta \leq 0$. Then, Proposition 1 yields

$$0 \geq \operatorname{Hess}\ h(p_0)(v,v) = \Theta(p_0)\langle Av, v\rangle(p_0), \quad \forall v \in T_{p_0}\Sigma,$$

that is, at p_0 all the principal curvatures are nonnegative. Since we are assume that $H_{s+1} \neq 0$ on Σ^n, we must have $H_{s+1} > 0$ on Σ^n. In particular, we can apply Lemma 3 (or Lemma 2 if $s = 1$) to guarantee the ellipticity of the operator L_k for every $k = r, \ldots, s$ and H_{k+1} is positive on Σ^n for every $0 \leq k \leq s$. So, for instance, we have

$$L_s h = c_s \Theta H_{s+1} \leq 0$$

and, consequently, by the weak maximum principle we obtain that $h \geq t_0$ on Σ^n.

Now let us consider on Σ^n the smooth function $\varphi = ch + \Theta$, where $c \in \mathbb{R}$ is a positive constant to be chosen in an appropriate way. Then Proposition 1 gives

$$\begin{aligned} L_k\varphi &= -\frac{c_k}{k+1}\langle \nabla H_{k+1}, \nabla h\rangle - \Theta\binom{n}{k+1}\Big(nH_1H_{k+1} - (n-k-1)H_{k+2} \\ &\quad -(k+1)cH_{k+1}\Big) - \Theta\sum_{i=1}^{n}\mu_{i,k}K_M(N^*, E_i^*)|N^* \wedge E_i^*|^2, \end{aligned} \tag{3.2}$$

where $\{E_1,\dots,E_n\}$ is an orthonormal frame on Σ^n diagonalizing A with $P_kE_i=\mu_{i,k}E_i$, for every $i=1,\dots,n$ and $k=r,\dots,s$, and X^* denotes the orthogonal projection on TM.

Since H_{k+1} is positive for every $k=0,\dots,s$, from Lemma 1 we get

$$H_1H_{k+1}-H_{k+2}\geq H_1H_{k+1}-H_{k+1}^2H_k^{-1}=\frac{H_{k+1}}{H_k}\left(H_1H_k-H_{k+1}\right).$$

By using once more Lemma 1 it follows from here that

$$H_1H_{k+1}-H_{k+2}\geq\frac{H_{k+1}}{H_k}\left(H_1H_k-H_k^{(k+1)/k}\right)=H_{k+1}(H_1-H_k^{1/k})\geq 0.$$

Then, the previous inequality implies that

$$\begin{aligned} nH_1H_{k+1}-(n-k-1)H_{k+2}-(k+1)cH_{k+1} &= (k+1)H_{k+1}(H_1-c)\\ &+ (n-k-1)(H_1H_{k+1}-H_{k+2})\\ &\geq (k+1)H_{k+1}(H_1-c)\geq 0, \end{aligned} \tag{3.3}$$

provided that $c:=\min H_1$.

On the other hand, since the operator L_k is elliptic for every $k=r,\dots,s$ or, equivalently, P_k is positive definite, we get that its eigenvalues $\mu_{i,k}$ are all positive on Σ^n. Then, by our assumption on the sectional curvature K_M of the fiber M^n we have

$$\sum_{i=1}^{n}\mu_{i,k}K_M(N^*,E_i^*)|N^*\wedge E_i^*|^2\geq 0,$$

Hence, by using (3.3) and taking into account that $\Theta\leq 0$, we infer from (3.2) and the previous inequality that

$$L_k\varphi\geq-\frac{c_k}{k+1}\langle\nabla H_{k+1},\nabla h\rangle. \tag{3.4}$$

Proceeding, we introduce the following second order linear differential operator $L:C^\infty(\Sigma)\to C^\infty(\Sigma)$ defined by

$$\begin{aligned} L &= \sum_{k=r}^{s}(k+1)c_k^{-1}b_kL_k\\ &= \operatorname{tr}(P\circ\operatorname{hess}), \end{aligned}$$

where the tensor $P:\mathfrak{X}(\Sigma)\to\mathfrak{X}(\Sigma)$ is given by

$$P=\sum_{k=r}^{s}(k+1)c_k^{-1}b_kP_k.$$

Since $(k+1)c_k^{-1}b_k>0$ for every $k=r,\dots,s$ and each operator L_k is elliptic (equivalently, each P_k is positive definite) we see that the operator P is positive definite and, consequently, the operator L is elliptic too. So, equation (3.4) and the fact that Σ^n is (r,s)-linear Weingarten imply that

$$L\varphi\geq 0.$$

By using once more the weak maximum principle for the elliptic operator L we get

$$\varphi \leq \max_{\partial\Sigma} \varphi \leq ct_0,$$

that is,

$$c(h - t_0) \leq 1.$$

Therefore, we conclude that

$$h \leq t_0 + \frac{1}{\min H_1}.$$

This proves (a).

In the case $\min h \neq t_0$, we choose an interior point q_0 of Σ^n satisfying $\min h = h(q_0)$ and the orientation so that $\Theta \geq 0$. Then,

$$0 \leq \text{Hess } h(q_0)(v, v) = \Theta(q_0)\langle Av, v\rangle(q_0), \quad \forall v \in T_{q_0}\Sigma,$$

that is, at q_0 all the principal curvatures must be nonnegative. So, reasoning as in the previous case we see that each operator L_k is elliptic for every $k = r, \ldots, s$, H_{k+1} is positive on Σ^n for every $0 \leq k \leq s$ and $h \leq t_0$ on Σ^n.

Besides, keeping the notation of case (a), it follows that $\varphi = ch + \Theta$ satisfies, by equations (3.2), (3.3) and our assumption on K_M,

$$L_k\varphi \leq -\frac{c_k}{k+1}\langle \nabla H_{k+1}, \nabla h\rangle,$$

which implies that $L\varphi \leq 0$. Therefore, by weak maximum principle we conclude that

$$\varphi \geq \min_{\partial\Sigma} \varphi \geq ct_0,$$

that is,

$$h \geq t_0 - \frac{1}{\min H_1}.$$

This finishes the proof of the theorem. □

Remark 1. We observe that the estimate given in Theorem 1 is sharp in the sense that it is reached by the hemisphere $\Sigma_+ = \{x \in \mathbb{S}^n \,;\, x_1 \geq 0\}$ of the standard sphere $\mathbb{S}^n$ in $\mathbb{R}^{n+1}$. Indeed, it follows easily that Σ_+ is a totally umbilical hypersurface (in fact, it is a vertical graph) with $H_1 = 1$, boundary $\{0\} \times \mathbb{S}^{n-1} \subset \{0\} \times \mathbb{R}^n$ and has the maximum height 1.

We observe that Theorem 1 above does not follow from Theorem 1 of de Lima & de Lima 2018, because there the authors just assume that $H_{s+1} > 0$ and, with this constraint, it is not possible to obtain item (b) above.

Let us also point out that when $s = r$, that is, the hypersurface has constant $(r+1)$-mean curvature H_{r+1}, Theorem 1 improves the estimate obtained in Theorem 4.1(i) of Cheng & Rosenberg 2005. Indeed, it is easy to see that the inequality

$$\frac{1}{\min H_1} \leq \frac{1}{H_{r+1}^{1/(r+1)}}$$

holds for every $r = 0, \ldots, n-1$. Moreover, this result is also an extension of Theorem 3.5 of Alías & Dajczer 2007 (case $\alpha = 0$) and Proposition 1 of Hoffman et al. 2006 (case $\tau = 0$).

It is still worth pointing out that the same argument done in the proof of Theorem 4.2 of Cheng & Rosenberg 2005 by using the Alexandrov reflection technique, enable us to get the following consequence of Theorem 1 concerning compact embedded (r,s)-linear Weingarten hypersurfaces:

Corollary 1. *Let $\overline{M}^{n+1} = \mathbb{R} \times M^n$ be a product space whose the fiber M^n has nonnegative sectional curvature K_M. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a compact embedded (r,s)-linear Weingarten two-sided hypersurface with $(s+1)$-mean curvature $H_{s+1} \neq 0$ on Σ^n, for some $0 \leq s \leq n-1$. Suppose that the angle function Θ does not change sign on Σ^n. Then Σ^n is symmetric about some slice $\{t_0\} \times M^n$, $t_0 \in \mathbb{R}$, and the extrinsic vertical diameter of Σ^n is no more than $\frac{2}{\min |H_1|}$.*

Proceeding, we are able to relax the assumption on the sectional curvature K_M of the fiber M^n letting it be bounded from below by a negative constant. For this, we will assume that the mean curvature satisfies a certain condition, which holds automatically when the sectional curvature of the fiber is nonnegative. In what follows, we will denote by $c = \min H_1$. So, we get the following result.

Theorem 2. *Let $\overline{M}^{n+1} = \mathbb{R} \times M^n$ be a product space whose the fiber M^n has sectional curvature satisfying $K_M \geq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a compact (r,s)-linear Weingarten two-sided hypersurface with $(s+1)$-mean curvature $H_{s+1} \neq 0$ on Σ^n, for some $0 \leq s \leq n-1$, and boundary $\partial\Sigma^n$ contained into the slice $\{t_0\} \times M^n$ for some $t_0 \in \mathbb{R}$. Suppose that the angle function Θ does not change sign on Σ and $c(r+1) \min H_{k+1} > \alpha(s+1) \max H_k$ for every $k = r, \dots, s$. Then,*

(a) *Either* $\max h \neq t_0$ *and*

$$\Sigma^n \subset \left[t_0, t_0 + \frac{(r+1)d}{(r+1)dc - (s+1)\alpha\beta}\right] \times M^n,$$

where d is given by (3.1) *and* $\beta = \sum_{k=r}^{s} b_k \max H_k$.

(b) *or* $\min h \neq t_0$ *and*

$$\Sigma^n \subset \left[t_0 - \frac{(r+1)d}{(r+1)dc - (s+1)\alpha\beta}, t_0\right] \times M^n,$$

where d is given by (3.1) *and* $\beta = \sum_{k=r}^{s} b_k \max H_k$.

Remark 2. We note that in the case of hypersurfaces having constant $(r+1)$-mean curvature H_{r+1}, our assumption on c in Theorem 2 becomes $cH_{r+1} > \alpha \max H_r$. In particular, it is weaker than assumption (7.77) of Theorem 7.19 of Alías et al. 2016. Moreover, the constant $\frac{(r+1)d}{(r+1)dc-(s+1)\alpha\beta}$ is just given by $\frac{H_{r+1}}{cH_{r+1}-\alpha \max H_r}$. Furthermore, by Lemma 1 we have $c \geq H_{r+1}^{1/(r+1)}$, which implies that

$$\frac{H_{r+1}}{cH_{r+1} - \alpha \max H_r} \leq \frac{H_{r+1}}{H_{r+1}^{(r+2)/(r+1)} - \alpha \max H_r}.$$

In this setting, our estimate improves that one given in Theorem 7.19 of Alías et al. 2016 for the compact case.

On the other hand, we also observe that, since K_M can be negative in Theorem 2, it does not follow from Theorem 1 of de Lima & de Lima 2018.

Proof of Theorem 2. In what follows, we keep the notations established in Theorem 1. Let us suppose $\max h \neq t_0$ first. Then, as in the previous result, taking the angle function Θ nonpositive, it is easy to see that the operator L_k is elliptic for every $k = r, \ldots, s$, the $(k+1)$-mean curvature H_{k+1} is positive for every $0 \leq k \leq s$ and $h \geq t_0$. Besides, by equations (3.4) and (3.3) we get that the function φ defined in Theorem 1 satisfies

$$L_k\varphi \geq -\frac{c_k}{k+1}\langle \nabla H_{k+1}, \nabla h\rangle - \Theta \sum_{i=1}^{n} \mu_{i,k} K_M(N^*, E_i^*)|N^* \wedge E_i^*|^2. \tag{3.5}$$

Since the eigenvalues $\mu_{i,k}$ are all positive on Σ^n and using our assumption on K_M we have

$$\mu_{i,k} K_M(N^*, E_i^*)|N^* \wedge E_i^*|^2 \geq -\mu_{i,k}\alpha |N^* \wedge E_i^*|^2, \tag{3.6}$$

for every $i = 1, \ldots, n$ and $k = r, \ldots, s$. With a straightforward computation, we can show that

$$|N^* \wedge E_i^*|^2 = |N^*|^2 |E_i^*|^2 - \langle N^*, E_i^*\rangle^2 = |\nabla h|^2 - \langle E_i, \nabla h\rangle^2 \leq 1,$$

which jointly with (3.6) imply that

$$\sum_{i=1}^{n} \mu_{i,k} K_M(N^*, E_i^*)|N^* \wedge E_i^*|^2 \geq -\alpha \mathrm{tr}\ \ P_k = -\alpha c_k H_k \geq -\alpha c_k \max H_k.$$

From here and (3.5) we infer that

$$L\varphi \geq \sum_{k=r}^{s} (k+1)\alpha\Theta b_k \max H_k \geq (s+1)\alpha\beta\Theta, \tag{3.7}$$

where $\beta = \sum_{k=r}^{s} b_k \max H_k$. On the other hand, by using Proposition 1 we get that

$$Lh = \sum_{k=r}^{s} (k+1) c_k^{-1} b_k L_k h = \sum_{k=r}^{s} (k+1)\Theta b_k H_{k+1} \leq (r+1) d\Theta. \tag{3.8}$$

So, let us consider on Σ^n the smooth function given by

$$\tilde{\varphi} = \varphi - \frac{(s+1)\alpha\beta}{(r+1)d} h = \frac{(r+1)dc - (s+1)\alpha\beta}{(r+1)d} h + \Theta.$$

Then, equations (3.7) and (3.8) yield

$$L\tilde{\varphi} = L\varphi - \frac{(s+1)\alpha\beta}{(r+1)d} Lh \geq (s+1)\alpha\beta\Theta - (s+1)\alpha\beta\Theta = 0.$$

Hence, we can apply once more the weak maximum principle to conclude that

$$\tilde{\varphi} \leq \max_{\partial\Sigma} \tilde{\varphi} \leq \frac{(r+1)dc - (s+1)\alpha\beta}{(r+1)d} t_0,$$

that is,

$$\frac{(r+1)dc - (s+1)\alpha\beta}{(r+1)d}(h - t_0) \leq 1. \tag{3.9}$$

Now, the assumption on c gives

$$\begin{aligned}(r+1)dc-(s+1)\alpha\beta &= (r+1)c\sum_{k=r}^{s} b_k H_{k+1}-(s+1)\alpha\sum_{k=r}^{s} b_k \max H_k\\ &= \sum_{k=r}^{s} b_k\left((r+1)cH_{k+1}-(s+1)\alpha\max H_k\right)>0.\end{aligned}$$

Therefore, from equation (3.9) we arrive to

$$h\leq t_0+\frac{(r+1)d}{(r+1)dc-(s+1)\alpha\beta}$$

as desired.

Finally, the case $\min h\neq t_0$ follows as above and this finishes the proof of the theorem. □

As consequence of Theorem 2, the analogue of Corollary 1 also holds in this situation:

Corollary 2. *Let $\overline{M}^{n+1}=\mathbb{R}\times M^n$ be a product space whose the fiber M^n has sectional curvature satisfying $K_M\geq-\alpha$, for some positive constant $\alpha\in\mathbb{R}$. Let $\psi:\Sigma^n\to\overline{M}^{n+1}$ be a compact embedded (r,s)-linear Weingarten two-sided hypersurface with $(s+1)$-mean curvature $H_{s+1}\neq0$ on Σ^n, for some $0\leq s\leq n-1$. Suppose that the angle function Θ does not change sign on Σ and $c(r+1)\min H_{k+1}>\alpha(s+1)\max H_k$ for every $k=r,\dots,s$. Then Σ^n is symmetric about some slice $\{t_0\}\times M^n$, $t_0\in\mathbb{R}$, and the extrinsic vertical diameter of Σ^n is no more than $\frac{2(r+1)d}{(r+1)dc-(s+1)\alpha\beta}$.*

4 - HALF-SPACE THEOREMS

The aim of this section is to give nonexistence results, in the form of half-space theorems, concerning complete two-sided hypersurfaces in the product Riemannian manifold $\mathbb{R}\times M^n$. We point out that our results do not assume that some higher order mean curvature of the hypersurface is constant. In this setting, when the fiber M^n is compact, our results generalize those one obtained by Cheng & Rosenberg 2005 and Hoffman et al. 2006 for the case in which the mean curvature or some higher order mean curvature is constant (see Theorems 3 and 4 below). Moreover, in the case in which M^n is not necessarily compact, by using a generalized version of the Omori-Yau maximum principle for trace type differential operators, we prove other interesting half-space theorem (see Theorem 5 below).

According to Hoffman et al. 2006, we say that a two-sided hypersurface in a product space $\mathbb{R}\times M^n$ *lies in an upper or lower half-space* if it is, respectively, contained in a region of $\mathbb{R}\times M^n$ of the form

$$[a,+\infty)\times M^n \qquad \text{or} \qquad (-\infty,a]\times M^n,$$

for some real number $a\in\mathbb{R}$.

As an application of Theorem 1 we get the following result:

Theorem 3. *Let $\overline{M}^{n+1}=\mathbb{R}\times M^n$ be a product space whose fiber M^n is compact and has nonnegative sectional curvature. Let $\psi:\Sigma^n\to\overline{M}^{n+1}$ be a noncompact (r,s)-linear Weingarten two-sided properly immersed hypersurface with $(s+1)$-mean curvature bounded away from zero, for some $0\leq s\leq n-1$, and such that its angle function does not change sign. Then, Σ^n cannot lie in a half-space.*

Proof. Let us assume by contradiction that Σ^n lies in an upper half-space, that is, $\Sigma^n \subset [a, +\infty) \times M^n$, for some $a \in \mathbb{R}$. For any number $t_0 > a$, we denote by Σ_{t_0} the hypersurface

$$\Sigma_{t_0} = \{(t, p) \in \Sigma^n \ ; \ t \leq t_0\}.$$

Then, Σ_{t_0} is a compact (r, s)-linear Weingarten two-sided hypersurface with boundary contained into the slice $\{t_0\} \times M$ and $\min h \neq t_0$, because M^n is compact and the immersion is proper. Hence, by Theorem 1 we must have $H_{s+1} > 0$ on Σ_{t_0} and $\Sigma_{t_0} \subset [t_0 - \frac{1}{c(t_0)}, t_0] \times M^n$, where $c(t_0) = \min_{\Sigma_{t_0}} H_1 > 0$, that is,

$$t_0 - a \leq \frac{1}{c(t_0)}.$$

Because H_{s+1} is bounded away from zero we get $\inf H_{s+1} > 0$, which implies $\inf H_1 > 0$. Thus

$$t_0 - a \leq \frac{1}{c(t_0)} \leq \frac{1}{\inf H_1}.$$

Then choosing t_0 large enough we reached a contradiction.

Finally, if Σ^n is contained into a lower half-space, we may apply the same argument above to arrive at a contradiction. □

Similarly, we can reason as in Theorem 3 to obtain as consequence of Theorem 2 the following result, where we keep the notation $c = \min H_1$.

Theorem 4. *Let $\overline{M}^{n+1} = \mathbb{R} \times M^n$ be a product space whose fiber M^n is compact with sectional curvature satisfying $K_M \geq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a noncompact (r, s)-linear Weingarten two-sided properly immersed hypersurface with bounded away from zero $(s + 1)$-mean curvature, for some $0 \leq s \leq n - 1$, and such that its angle function does not change sign. Suppose that $c(r + 1) \min H_{k+1} > \alpha(s + 1) \max H_k$ for every $k = r \dots, s$. Then, Σ^n cannot lie in a half-space.*

In order to treat the case in which the fiber is not compact, we will make use of a generalized version of the Omori-Yau maximum principle for trace type differential operators proved in Alías et al. 2016. Let Σ^n be a Riemannian manifold and let $\mathcal{L} = \mathrm{tr}(\mathcal{P} \circ \mathrm{hess})$ be a semi-elliptic operator, where $\mathcal{P} : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is a positive semi-definite symmetric tensor. Following the terminology introduced by Pigola et al. 2005, we say that the Omori-Yau maximum principle holds on Σ^n for the operator $\mathcal{L}$ if, for any function $u \in C^2(\Sigma)$ with $u^* = \sup u < +\infty$, there exists a sequence of points $(p_j) \subset \Sigma^n$ satisfying

$$u(p_j) > u^* - \frac{1}{j}, \quad |\nabla u(p_j)| < \frac{1}{j} \quad \text{and} \quad \mathcal{L}u(p_j) < \frac{1}{j}$$

for every $j \in \mathbb{N}$. Equivalently, for any smooth function $u \in C^2(\Sigma)$ with $u_* = \inf u > -\infty$ there exists a sequence of points $(p_j) \subset \Sigma^n$ satisfying

$$u(p_j) < u_* + \frac{1}{j}, \quad |\nabla u(p_j)| < \frac{1}{j} \quad \text{and} \quad \mathcal{L}u(p_j) > -\frac{1}{j}$$

for every $j \in \mathbb{N}$.

We quote a suitable version of the Omori-Yau maximum principle for trace type differential operators on a complete noncompact Riemannian manifold (see Theorem 6.13 of Alías et al. 2016).

Lemma 4. *Let Σ^n be a complete noncompact Riemannian manifold; let $o \in \Sigma^n$ be a reference point and denote by r_o the Riemannian distance function from o. Assume that the sectional curvature of Σ^n satisfies*

$$K_\Sigma \geq -G^2(r_o), \tag{4.1}$$

with $G \in C^1([0,+\infty))$ satisfying

$$G(0) > 0, \quad G'(t) \geq 0 \quad \text{and} \quad \frac{1}{G(t)} \notin L^1(+\infty). \tag{4.2}$$

Let $\mathcal{P}$ be a positive semi-definite symmetric tensor on Σ^n. If $\sup tr(\mathcal{P}) < +\infty$, then the Omori-Yau maximum principle holds on Σ^n for the semi-elliptic operator $\mathcal{L} = tr(\mathcal{P} \circ hess)$.

In particular, Lemma 4 remains true if we replace condition (4.1) by the stronger condition of Σ^n having sectional curvature bounded from below by a constant.

Remark 3. As it is well known, especially significant examples of functions G satisfying the condition (4.2) in Lemma 4 are given by (see, for instance Alías et al. 2016 and Pigola et al. 2005)

$$G(t) = t\prod_{j=1}^{N} \log^j(t), \quad t \gg 1,$$

where $\log^j$ stands for the j-th iterated logarithm.

Now, we are in ready to state and prove our last half-space theorem.

Theorem 5. *Let $\overline{M}^{n+1} = \mathbb{R} \times M^n$ be a product space whose fiber M^n has sectional curvature satisfying $K_M \geq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact (r,s)-linear Weingarten two-sided hypersurface with positive $(s+1)$-mean curvature, for some $1 \leq r \leq s \leq n-1$. Suppose that $\sup |H_r| < +\infty$ and, if $s \geq 2$, there exists an elliptic point in Σ^n. Assume further that the shape operator satisfies $|A| \leq G(r_o)$, where $G \in C^1([0,+\infty))$ satisfies (4.2) and r_o is the distance function from a reference point of Σ^n. The following holds:*

(a) either $\sup \Theta > 0$ or Σ^n cannot lie in an upper half-space;

(b) either $\inf \Theta < 0$ or Σ^n cannot lie in a lower half-space.

Proof. We begin stating that the sectional curvature K_Σ of Σ satisfies the assumption (4.1) of Lemma 4. Indeed, denoting by $\overline{K}$ the sectional curvature of the ambient space, it follows from Gauss equation (2.1) that if $\{X, Y\}$ is an orthonormal basis for an arbitrary plane tangent to Σ^n, then

$$\begin{aligned} K_\Sigma(X,Y) &= \overline{K}(X,Y) + \langle AX, X\rangle\langle AY, Y\rangle - \langle AX, Y\rangle^2 \\ &\geq \overline{K}(X,Y) - |AX||AY| - |AX|^2 \\ &\geq \overline{K}(X,Y) - 2|A|^2, \end{aligned} \tag{4.3}$$

where the last inequality follows from the fact that

$$|AX|^2 \leq \text{tr}(A^2)|X|^2 = |A|^2$$

for every unitary vector X tangent to Σ^n. Taking into account that

$$\overline{K}(X,Y) = K_M(X^*,Y^*)|X^* \wedge Y^*|^2$$

we obtain from our hypothesis on K_M that $\overline{K}(X,Y) \geq -\alpha$, because $|X^* \wedge Y^*|^2 \leq |X \wedge Y|^2 \leq 1$. Hence, since the shape operator satisfies $|A| \leq G(r_o)$, equation (4.3) yields

$$K_\Sigma \geq -\alpha - 2G^2(r_o),$$

which concludes the claim.

We prove part (a) first. To do this, we assume that $\Theta \leq 0$ and argue by contradiction, that is, we suppose that Σ^n lies in an upper half-space. Equivalently, the height function of Σ^n satisfies $h_* = \inf h > -\infty$.

We set the second order linear differential operator $\mathcal{L} : C^\infty(\Sigma) \to C^\infty(\Sigma)$ by

$$\mathcal{L} = \sum_{k=r}^{s} c_k^{-1} b_k L_k = \mathrm{tr}\ \ (\mathcal{P} \circ \mathrm{hess}),$$

where the tensor $\mathcal{P} : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is given by

$$\mathcal{P} = \sum_{k=r}^{s} c_k^{-1} b_k P_k.$$

Since Σ^n has an elliptic point (that is, all the principal curvatures are positive in such a point), the operator L_k is elliptic for every $k = r, \ldots, s$ or, equivalently, P_k is positive definite. Then, $\mathcal{P}$ is a positive linear combination of the P_k's, so that it is positive definite. Thus, $\mathcal{L}$ is a trace type elliptic operator. Besides, by using the identity $\mathrm{tr}(P_k) = c_k H_k$ we obtain from Lemma 1 that

$$\mathrm{tr}(\mathcal{P}) = \sum_{k=r}^{s} b_k H_k \leq \sum_{k=r}^{s} b_k H_r^{k/r},$$

which implies that $\sup \mathrm{tr}(\mathcal{P}) < +\infty$. Hence, we are ready to apply Lemma 4 to guarantee that the Omori-Yau maximum principle holds on Σ^n for the operator $\mathcal{L}$. Then, there exists a sequence of points $(p_j) \subset \Sigma^n$ having the following properties:

$$\lim h(p_j) = h_*, \quad |\nabla h(p_j)| < \frac{1}{j} \quad \textit{and} \quad \mathcal{L}h(p_j) > -\frac{1}{j}.$$

In particular, by Proposition 1 we get

$$-\frac{1}{j} < \mathcal{L}h(p_j) = \sum_{k=r}^{s} \Theta(p_j) b_k H_{k+1}(p_j) = d\Theta(p_j).$$

Since $|\nabla h|^2 = 1 - \Theta^2$, we see that $\Theta(p_j) \to -1$. So, taking limits we conclude that $d \leq 0$, which gives a contradiction.

In case (b), we reason again by contradiction, that is, by assuming that $\Theta \geq 0$ and Σ^n is contained into a lower half-space so that the height function satisfies $h^* = \sup h < +\infty$. Then, reasoning as in part (a), it is not difficult to see once more that $d \leq 0$, characterizing a contradiction. This concludes the proof of the theorem. □

Let us observe that the proof of Theorem 5 remains true with the stronger assumption that K_Σ is bounded from below by a constant, which implies the validity of the Omori-Yau's maximum principle. For instance, reasoning as in the proof of Theorem 5 we see that K_Σ is bounded from below since $\sup|A|^2 < +\infty$. On the other hand, the hypothesis on H_r in Theorem 5, $\sup|H_r| < \infty$, can be replaced by $\sup H_1 < \infty$, because of Lemma 1. In this case, taking into account the relation

$$|A|^2 = n^2H_1^2 - n(n-1)H_2,$$

it follows that the condition $\sup|A|^2 < +\infty$ is equivalent to $\sup H_1 < +\infty$. This proves the following result:

Corollary 3. *Let $\overline{M}^{n+1} = \mathbb{R}\times M^n$ be a product space whose fiber M^n has sectional curvature satisfying $K_M \geq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact (r,s)-linear Weingarten two-sided hypersurface with positive $(s+1)$-mean curvature, for some $1 \leq r \leq s \leq n-1$. Suppose that $\sup|H_1| < +\infty$ and, if $s \geq 2$, there exists an elliptic point in Σ^n. The following holds:*

(a) either $\sup\Theta > 0$ or Σ^n cannot lie in an upper half-space;

(b) either $\inf\Theta < 0$ or Σ^n cannot lie in a lower half-space.

In the case of hypersurfaces having constant mean curvature the assumption of existence of an elliptic point can be dropped as follows.

Corollary 4. *Let $\overline{M}^{n+1} = \mathbb{R}\times M^n$ be a product space whose fiber M^n has sectional curvature satisfying $K_M \geq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact two-sided hypersurface with positive constant mean curvature and such that $\inf H_2 > -\infty$. The following holds:*

(a) either $\sup\Theta > 0$ or Σ^n cannot lie in an upper half-space;

(b) either $\inf\Theta < 0$ or Σ^n cannot lie in a lower half-space.

In other words we have:

(a') There is no complete noncompact two-sided hypersurface having positive constant mean curvature, angle function nonpositive and contained into an upper half-space;

(b') There is no complete noncompact two-sided hypersurface having positive constant mean curvature, angle function nonnegative and contained into a lower half-space.

Proof. It is enough to prove part (a). We suppose that $\Theta \leq 0$ and let us reason by contradiction, that is, $\inf h = h_* > -\infty$. As in the proof of Theorem 5 and by remark above, we might see that the Omori-Yau maximum principle holds on Σ^n for the Laplacian. Then, there is a sequence of points $(p_j) \subset \Sigma^n$ satisfying

$$\lim h(p_j) = h_*, \quad |\nabla h| < \frac{1}{j} \quad \text{and} \quad \Delta h(p_j) > -\frac{1}{j}.$$

By applying Proposition 1 we find

$$-\frac{1}{j} < \Delta h(p_j) = n\Theta(p_j)H_1$$

Since the angle function is nonpositive, taking limits here we conclude that $H_1 \leq 0$, which gives a contradiction. □

More generally, for hypersurfaces having some constant higher order mean curvature we get the following result:

Corollary 5. *Let $\overline{M}^{n+1} = \mathbb{R} \times M^n$ be a product space whose fiber M^n has sectional curvature satisfying $K_M \geq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact two-sided hypersurface with positive constant $(r+1)$-mean curvature, for some $1 \leq r \leq n-1$. Suppose that $\sup H_1 < +\infty$ and, if $r \geq 2$, there exists an elliptic point in Σ^n. The following holds:*

(a) either $\sup \Theta > 0$ or Σ^n cannot lie in an upper half-space;

(b) either $\inf \Theta < 0$ or Σ^n cannot lie in a lower half-space.

In other words we have:

(a') there is no complete noncompact two-sided hypersurface having $H_{r+1} > 0$, an elliptic point, with $\sup H_1 < +\infty$, angle function nonpositive and contained into an upper half-space;

(b') there is no complete noncompact two-sided hypersurface having $H_{r+1} > 0$, an elliptic point, with $\sup H_1 < +\infty$, angle function nonnegative and contained into a lower half-space.

Finally we collect (a) and (b) in the previous corollaries in order to obtain the following result.

Corollary 6. *Let $\overline{M}^{n+1} = \mathbb{R} \times M^n$ be a product space whose fiber M^n has sectional curvature satisfying $K_M \geq -\alpha$, for some positive constant $\alpha \in \mathbb{R}$. Let $\psi : \Sigma^n \to \overline{M}^{n+1}$ be a complete noncompact two-sided hypersurface with positive constant $(r+1)$-mean curvature, for some $0 \leq r \leq n-1$. Suppose that $\sup H_1 < +\infty$ and, if $r \geq 2$, there exists an elliptic point in Σ^n. In addition, if $r = 0$ assume that $\inf H_2 \geq -\infty$. Then, either Θ does not vanishes identically or Σ^n cannot lie in a half-space. In other words, there is no complete noncompact two-sided hypersurface having $H_{r+1} > 0$, an elliptic point, with $\sup H_1 < +\infty$, angle function vanishes identically and contained into a half-space.*

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