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On non-Kupka points of codimension one foliations on $\mathbb{P}^3$

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ABSTRACT

We study the singular set of a codimension one holomorphic foliation on $\mathbb{P}^3$. We find a local normal form for these foliations near a codimension two component of the singular set that is not of Kupka type. We also determine the number of non-Kupka points immersed in a codimension two component of the singular set of a codimension one foliation on $\mathbb{P}^3$.

Key words: holomorphic foliations, Kupka sets, non-Kupka points.

1 - INTRODUCTION

A *regular codimension one holomorphic foliation* on a complex manifold M, can be defined by a triple $\{(\mathfrak{U}, f_\alpha, \psi_{\alpha\beta})\}$ where

1. $\mathfrak{U} = \{U_\alpha\}$ is an open cover of M.
2. $f_\alpha : U_\alpha \to \mathbb{C}$ is a holomorphic submersion for each α.
3. A family of biholomorphisms $\{\psi_{\alpha\beta} : f_\beta(U_{\alpha\beta}) \to f_\alpha(U_{\alpha\beta})\}$ such that

$$\psi_{\alpha\beta} = \psi_{\beta\alpha}^{-1}, \quad f_\beta|_{U_\alpha\cap U_\beta} = \psi_{\beta\alpha} \circ f_\alpha|_{U_\alpha\cap U_\beta} \quad \text{and} \quad \psi_{\alpha\gamma} = \psi_{\alpha\beta} \circ \psi_{\beta\gamma}.$$

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Dedicated to José Seade in his 60 birthday.

Since $df_\alpha(x) = \psi'_{\alpha\beta}(f_\beta(x)) \cdot df_\beta(x)$, the set $F = \bigcup_\alpha Ker(df_\alpha) \subset TM$ is a subbundle. Also $[\psi'_{\alpha\beta}(f_\beta)] \in \check{H}^1(\mathfrak{U}, \mathcal{O}^*)$ define a line bundle $N = TM/F$. The family of 1–forms $\{df_\alpha\}$ glue to a global section $\omega \in H^0(M, \Omega^1(N))$. We have

$$0 \to F \to TM \xrightarrow{\{df_\alpha\}} N \to 0, \quad 0 \to \mathcal{F} \to \Theta \xrightarrow{\{df_\alpha\}} \mathcal{N} \to 0, \quad [\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$$

where $\mathcal{F} = \mathcal{O}(F)$, $\Theta = \mathcal{O}(TM)$ and $\mathcal{N} = \mathcal{O}(N)$. We also obtain

$$\wedge^n TM^* = det(F^*) \otimes N^*, \qquad \Omega^n_M := K_M = det(\mathcal{F}^*) \otimes \mathcal{N}^*, \quad n = dim(M).$$

Definition 1.1. Let M be a compact complex manifold of dimension n. A *singular codimension one holomorphic foliation on* M, may be defined by one of the following ways:

1. A pair $\mathcal{F} = (S, \mathcal{F})$, where $S \subset M$ is an analytic subset of codim$(S) \geq 2$, and $\mathcal{F}$ is a regular codimension one holomorphic foliation on $M \setminus S$.
2. A class of global sections $[\omega] \in \mathbb{P}H^0(M, \Omega^1(L))$, where $L \in Pic(M)$ such that
   (a) the singular set $S_\omega = \{p \in M | \omega_p = 0\}$ has codim$(S_\omega) \geq 2$.
   (b) $\omega \wedge d\omega = 0$ in $H^0(M, \Omega^3(L^{\otimes 2}))$.

   In this case, we denote by $\mathcal{F}_\omega = (S_\omega, \mathcal{F}_\omega)$ the foliation represented by ω.
3. An exact sequence of sheaves
   $$0 \to \mathcal{F} \to \Theta \to \mathcal{N} \to 0, \quad [\mathcal{F}, \mathcal{F}] \subset \mathcal{F}$$
   where $\mathcal{F}$ is a reflexive sheaf of rank $rk(\mathcal{F}) = n - 1$ with torsion free quotient $\mathcal{N} \simeq \mathcal{J}_S \otimes L$, where $\mathcal{J}_S$ is an ideal sheaf for some closed scheme S.

These three definitions are equivalents.

Remark 1.2. Let $\omega \in H^0(M, \Omega^1(L))$ be a section.

1. The section ω may be defined by a family of 1-forms
   $$\omega_\alpha \in \Omega^1(U_\alpha), \quad \omega_\alpha = \lambda_{\alpha\beta}\omega_\beta \text{ in } U_{\alpha\beta} = U_\alpha \cap U_\beta, \quad L = [\lambda_{\alpha\beta}] \in \check{H}^1(\mathfrak{U}, \mathcal{O}^*).$$
2. The section ω is a morphism of sheaves $\Theta \xrightarrow{\omega} L$. The kernel of ω is *the tangent sheaf* $\mathcal{F}$. The image of ω is a twisted ideal sheaf $\mathcal{N} = \mathcal{J}_{S_\omega} \otimes L$. It is called the *normal sheaf of* $\mathcal{F}$.
3. As in the non-singular case, the following equality of line bundles holds
   $$K_M = \Omega^n_M = det(\mathcal{F}^*) \otimes \mathcal{N}^* = K_\mathcal{F} \otimes L^{-1}, \quad det(N) \simeq L$$
   where $K_M, K_\mathcal{F} = det(\mathcal{F}^*)$ are the canonical sheaf of M and $\mathcal{F}$ respectively.

We denote by

$$\mathcal{F}(M, L) \;=\; \{[\omega] \in \mathbb{P}H^0(M, \Omega^1(L)) \,|\, \text{codim}(S_\omega) \geq 2, \quad \omega \wedge d\omega = 0\,\}$$

$$\mathcal{F}(n, d) \;=\; \{[\omega] \in \mathbb{P}H^0(\mathbb{P}^n, \Omega^1(d+2)) \,|\, \text{codim}(S_\omega) \geq 2, \quad \omega \wedge d\omega = 0\}.$$

The number $d \geq 0$ is called the *degree of the foliation* represented by ω.

1.1 - STATEMENT OF THE RESULTS

In the sequel, M is a compact complex manifold with $dim(M) \geq 3$. We will use any of the above definitions for foliation. The singular set will be denoted by S. Observe that S decomposes as

$$S = \bigcup_{k=2}^{n} S_k \quad \text{where} \quad \text{codim}(S_k) = k.$$

For a foliation $\mathcal{F}$ on M represented by $\omega \in \mathcal{F}(M, L)$, the Kupka set (Kupka 1964, De Medeiros 1977) is defined by

$$K(\omega) = \{p \in M \mid \omega(p) = 0,\, d\omega(p) \neq 0\}.$$

We recall that for points near $K(\omega)$ the foliation $\mathcal{F}$ is biholomorphic to a product of a dimension one foliation in a transversal section by a regular foliation of codimension two (Kupka 1964) and in particular we have $K(\omega) \subset S_2$.

In this note, we focus our attention on the set of non-Kupka points $NK(\omega)$ of ω. The first remark is

$$NK(\omega) = \{p \in M | \omega(p) = 0,\, d\omega(p) = 0\} \supset S_3 \cup \cdots \cup S_n.$$

We analyze three cases, one in each section, the last two being the core of the work.

1. $S_2 = K(\omega)$, then $NK(\omega) = S_3 \cup \cdots \cup S_n$.
2. There is an irreducible component $Z \subset S_2$ such that $Z \cap K(\omega) = \emptyset$.
3. For a foliation $\omega \in \mathcal{F}(3, d)$. Let $Z \subset S_2$ be a connected component such that $Z \setminus Z \cap K(\omega)$ is a finite set of points.

The first case has been considered in Brunella (2009), Calvo-Andrade (1999, 2016), Calvo-Andrade and Soares (1994), Cerveau and Lins Neto (1994). Let $\omega \in \mathcal{F}(n, d)$ be a foliation with $K(\omega) = S_2$ and connected, then ω has a meromorphic first integral. In the generic case, the leaves define a *Lefschetz* or *a Branched Lefschetz Pencil*. The non-Kupka points are isolated singularities $NK(\omega) = S_n$. In this note, we present a new and short proof of this fact when the transversal type of $K(\omega)$ is radial.

In the second section, we study the case of a non-Kupka irreducible component of S_2. These phenomenon arise naturally in the intersection of irreducible components of $\mathcal{F}(M, L)$. The following result is a local normal form for ω near the singular set and is a consequence of a result of Loray (2006).

Theorem 1. *Let $\omega \in \Omega^1(\mathbb{C}^n, 0)$, $n \geq 3$, be a germ of integrable 1–form such that codim$(S_\omega) = 2$, $0 \in S_\omega$ is a smooth point and $d\omega = 0$ on S_ω. If $j^1_0\omega \neq 0$, then or either*

1. *there exists a coordinate system $(x_1, \ldots, x_n) \in \mathbb{C}^n$ such that*

$$j^1_0(\omega) = x_1 dx_2 + x_2 dx_1$$

and $\mathcal{F}_\omega$ is biholomorphic to the product of a dimension one foliation in a transversal section by a regular foliation of codimension two, or

2. *there exists a coordinate system* $(x_1, \dots, x_n) \in \mathbb{C}^n$ *such that*

$$\omega = x_1 dx_1 + g_1(x_2)(1 + x_1 g_2(x_2))dx_2,$$

such that $g_1, g_2 \in \mathcal{O}_{\mathbb{C},0}$ *with* $g_1(0) = g_2(0) = 0$*, or*

3. ω *has a non-constant holomorphic first integral in a neighborhood of* $0 \in \mathbb{C}^n$.

The alternatives are not exclusives. The following example was suggest by the referee and show that the case (3) of Theorem 1 cannot be avoid.

Example 1.3. Let ω be a germ of a 1-form at $0 \in \mathbb{C}^3$ defined by

$$\omega = xdx + (1 + xf)df$$

where $f(x, y, z) = y^2 z$. We have

$$\omega = xdx + 2yz(1 + xy^2 z)dy + y^2(1 + xy^2 z)dz.$$

The singular set of ω is $\{x = y = 0\}$ and $\{x = z = y^2 = 0\}$, therefore the singular set has an embedding point $\{x = z = y^2 = 0\}$ and $d\omega$ vanish along $\{x = y = 0\}$. We will show that ω has a holomorphic first integral F in a neighborhood of $0 \in \mathbb{C}^3$. In fact, let $t = f(x, y, z) = y^2 z$ and set $\varphi : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ defined by

$$\varphi(x, y, z) = (x, t).$$

Let $\eta = xdx + (1 + xt)dt$ be 1-form at $0 \in \mathbb{C}^2$, note that $\omega = \varphi^*(\eta)$ and moreover $\eta(0, 0) \neq 0$, this implies that η is non-singular at $0 \in \mathbb{C}^2$ and by Frobenius theorem η has a holomorphic first integral $H(x, t)$ on $(\mathbb{C}^2, 0)$. Defining $H_1(x, y, z) := H(x, f(x, y, z)) = H(x, y^2 z)$, we get H_1 is a holomorphic first integral for ω in a neighborhood of $0 \in \mathbb{C}^3$.

We apply Theorem 1 to a codimension one holomorphic foliation of the projective space with empty Kupka set.

About the third case, consider a foliation $\omega \in \mathcal{F}(3, d)$. Let Z be a connected component of S_2. We count the number $|Z \cap NK(\omega)|$ of non-Kupka points of ω in $Z \subset S_2$.

Theorem 2. *Let* $\omega \in \mathcal{F}(3, d)$ *be a foliation and* $Z \subset S_2$ *a connected component of* S_2*. Suppose that* Z *is a local complete intersection and* $Z \setminus Z \cap K(\omega)$ *is a finite set of points, then* $d\omega|_Z$ *is a global section of* $K_Z^{-1} \otimes K_{\mathcal{F}}|_Z$ *and the associated divisor* $D_\omega = \sum_{p \in Z} ord_p(d\omega) \cdot p$ *has degree*

$$\deg(D_\omega) = \deg(K_{\mathcal{F}}) - \deg(K_Z).$$

Note that the section $d\omega|_Z$ vanishes exactly in the non-Kupka points of ω in Z then the above theorem determine the number $|Z \cap NK(\omega)|$ (counted with multiplicity) of non-Kupka points of ω in Z.

2 - THE SINGULAR SET

Let $\omega \in \mathcal{F}(M, L)$ be a codimension one holomorphic foliation then singular set of ω may be written as

$$S = \bigcup_{j=2}^{n} S_j \qquad \text{where} \qquad \text{codim}(S_j) = j.$$

The fact that $K(\omega) \subset S_2$ implies that $S_3 \cup \ldots \cup S_n \subset NK(\omega)$. To continue we focus in the components of singular set of ω of dimension at least three.

2.1 - SINGULAR SET OF CODIMENSION AT LEAST THREE

We recall the following result due to B. Malgrange.

Theorem 2.1. (Malgrange 1976) *Let ω be a germ at $0 \in \mathbb{C}^n$, $n \geq 3$ of an integrable 1–form singular at 0, if codim$(S_\omega) \geq 3$, then there exist $f \in \mathcal{O}_{\mathbb{C}^n,0}$ and $g \in \mathcal{O}^*_{\mathbb{C}^n,0}$ such that*

$$\omega = gdf \quad \text{on a neighborhood of } 0 \in \mathbb{C}^n.$$

We have the following proposition.

Proposition 2.2. *Let $\omega \in \mathcal{F}(M, L)$ be a foliation and let $p \in S_n$ an isolated singularity, then any germ of vector field tangent to the foliation vanishes at p.*

Proof. Let $\omega = gdf, \quad g \in \mathcal{O}^*_p,\ f \in \mathcal{O}_p$ be a 1-form representing the foliation at p. Let $\mathbf{X} \in \Theta_p$ be a vector field tangent to the foliation, i.e., $\omega(\mathbf{X}) = 0$. If $\mathbf{X}(p) \neq 0$ there exists a coordinate system with $z(p) = 0$ and $\mathbf{X} = \partial/\partial z_n$, then

$$0 = \omega(\mathbf{X}) = g \cdot \left(\sum_{i=1}^{n} (\partial f/\partial z_i) dz_i (\partial/\partial z_n) \right) = g \cdot (\partial f/\partial z_n), \text{ therefore } \partial f/\partial z_n \equiv 0,$$

and $f = f(z_1, \ldots, z_{n-1})$, but this function does not have an isolated singularity. □

Now, we begin our study of the irreducible components of codimension two of the singular set of ω. Note that, given a section $\omega \in H^0(M, \Omega^1(L))$, along the singular set, the equation $\omega_\alpha = \lambda_{\alpha\beta}\omega_\beta$ implies $d\omega_\alpha|_S = (\lambda_{\alpha\beta} d\omega_\beta)|_S$. Then

$$\{d\omega_\alpha\} \in H^0(S, (\Omega^2_M \otimes L)|_S). \tag{2.1}$$

2.2 - THE KUPKA SET

These singularities has bee extensively studied and the main properties have been established in (Kupka 1964, De Medeiros 1977).

Definition 2.3. For $\omega \in \mathcal{F}(M, L)$. The Kupka set is

$$K(\omega) = \{ p \in M \,|\, \omega(p) = 0, \quad d\omega(p) \neq 0 \}.$$

The following properties of Kupka sets, are well known (De Medeiros 1977).

1. $K(\omega)$ is smooth of codimension two.
2. $K(\omega)$ has *local product structure* and the tangent sheaf $\mathcal{F}$ is locally free near $K(\omega)$.
3. $K(\omega)$ is subcanonically embedded and

$$\wedge^2 N_{K(\omega)} = L|_{K(\omega)}, \quad K_{K(\omega)} = (K_M \otimes L)|_{K(\omega)} = K_{\mathcal{F}}|_{K(\omega)}.$$

Let $\omega \in \mathcal{F}(n,d)$ be a foliation with $S_2 = K(\omega)$. By Calvo-Andrade and Soares (1994), there exists a pair (V,σ), where V is a rank two holomorphic vector bundle and $\sigma \in H^0(\mathbb{P}^n, V)$, such that

$$0 \longrightarrow \mathcal{O} \xrightarrow{\sigma} V \longrightarrow \mathcal{J}_K(d+2) \to 0 \quad \text{with} \quad \{\sigma = 0\} = K$$

and the total Chern class

$$c(V) = 1 + (d+2)\cdot \mathbf{h} + \deg(K(\omega))\mathbf{h}^2 \in H^*(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}[\mathbf{h}]/\mathbf{h}^{n+1}.$$

In 2009, Marco Brunella proved that following result, which in a certain sense say that the local transversal type of the singular set of foliation determines its behavior globally. Here we present a new proof of this fact. The techniques used in the proof could be of independent interest.

Proposition 2.4. *Let $\omega \in \mathcal{F}(n,d)$ be a foliation with $S_2 = K(\omega)$, (connected if $n = 3$) and of radial transversal type. Then $K(\omega)$ is a complete intersection and ω has a meromorphic first integral.*

To prove Proposition 2.4, we requires the following lemma. This result may be well known but for lack of a suitable reference we include the proof in an appendix.

Lemma 2.5. *Let F be a rank two holomorphic vector bundle over $\mathbb{P}^2$ with $c_1(F) = 0$ and $c_2(F) = 0$. Then $F \simeq \mathcal{O} \oplus \mathcal{O}$, is holomorphically trivial.*

Now, we prove Proposition 2.4.

Proof of Proposition 2.4. Let (V,σ) be the vector bundle with a section defining the Kupka set as scheme. The radial transversal type implies (Calvo-Andrade and Soares 1994)

$$c(V) = 1 + (d+2)\cdot \mathbf{h} + \frac{(d+2)^2}{4}\cdot \mathbf{h}^2 = \left(1 + \frac{(d+2)\cdot \mathbf{h}}{2}\right)^2 \in H^*(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}[\mathbf{h}]/\mathbf{h}^{n+1}.$$

The vector bundle $E = V(-\frac{d+2}{2})$, has $c_1(E) = 0$ and $c_2(E) = 0$. Let $\xi : \mathbb{P}^2 \hookrightarrow \mathbb{P}^n$ be a linear embedding. By the preceding lemma we have

$$\xi^* E \simeq \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}$$

and by the Horrocks' criterion (Okonek et al. 1980),

$$E \simeq \mathcal{O}_{\mathbb{P}^n} \oplus \mathcal{O}_{\mathbb{P}^n}$$

is trivial and hence V splits as $\mathcal{O}_{\mathbb{P}^n}(\frac{d+2}{2}) \oplus \mathcal{O}_{\mathbb{P}^n}(\frac{d+2}{2})$ and K is a complete intersection. The existence of the meromorphic first integral follows from Theorem A of (Cerveau and Lins Neto 1994). □

If ω is such that $K(\omega) = S_2$ and connected, the set of non-Kupka points of ω is

$$NK(\omega) = S_3 \cup \cdots \cup S_n.$$

A generic rational map, that means, a *Lefschetz or a Branched Lefschetz Pencil* $\varphi : \mathbb{P}^n \dashrightarrow \mathbb{P}^1$, has only isolated singularities away its base locus. The singular set of the foliation defined by the fibers of φ is $S_n \cup S_2$. The Kupka set corresponds away from its base locus and $S_n = NK(\omega)$ are the singularities as a map. S_n is empty if and only if the degree of the foliation is 0. The number $\ell(S_n)$ of isolated singularities counted with multiplicities can be calculated by (Cukierman et al. 2006). If ω_p is a germ of form that defines the foliation at $p \in S_n$, we have

$$\ell(S_n) = \sum_{p \in S_n} \mu(\omega_p, p), \quad \mu(\omega, p) = dim_{\mathbb{C}} \frac{\mathcal{O}_p}{(\omega_1, \ldots, \omega_n)}, \quad \omega_p = \sum_{i=1}^{n} \omega_i dz_i.$$

We have that $c_n(\mathcal{F}) = \ell(S_n)$.

3 - FOLIATIONS WITH A NON-KUPKA COMPONENT

It is well known that $K(\omega) \subset \{p \in M |\, j^1_p \omega \neq 0\}$, but the converse is not true. Our first result describes the singular points with this property.

3.1 - A NORMAL FORM

Now, we analyze the situation when there is an irreducible non-Kupka component of S_2.

Proof of Theorem 1. By hypotheses, $d\omega(p) = 0$ for any $p \in S_\omega$. Since

$$\omega = \omega_1 + \cdots, \qquad d\omega = d\omega_1 + \cdots = 0,$$

we get $d\omega_1(p) = 0$ for any $p \in S_\omega$. Now, as $\omega_1 \neq 0$ and $\text{codim}(S_\omega) = 2$, we have $1 \leq \text{codim}(S_{\omega_1}) \leq 2$. We distinguish two cases.

1. $\text{codim}(S_{\omega_1}) = 2$: there is a coordinate system $(x_1, \ldots, x_n) \in \mathbb{C}^n$ such that
$$\omega_1 = x_1 dx_2 + x_2 dx_1.$$
2. $\text{codim}(S_{\omega_1}) = 1$: there is a coordinate system $(x, \zeta) \in \mathbb{C} \times \mathbb{C}^{n-1}$ such that $x(p) = 0$ and $\omega_1 = xdx$.

The first case is known, the foliation $\mathcal{F}_\omega$ is equivalent in a neighborhood of $0 \in \mathbb{C}^n$ to a product of a dimension one foliation in a transversal section by a regular foliation of codimension two (Cerveau and Mattei 1982).

In the second case, Loray's preparation theorem (Loray 2006), shows that there exists a coordinate system $(x, \zeta) \in \mathbb{C} \times \mathbb{C}^{n-1}$, a germ $f \in \mathcal{O}_{\mathbb{C}^{n-1},0}$ with $f(0) = 0$, and germs $g, h \in \mathcal{O}_{\mathbb{C},0}$ such that the foliation is defined by the 1–form

$$\omega = xdx + [g(f(\zeta)) + xh(f(\zeta))]df(\zeta). \tag{3.1}$$

Since $S_{\omega_1} = \{x = 0\}$ and $0 \in S_\omega$ is a smooth point, we can assume that $S_{\omega,p} = \{x = \zeta_1 = 0\}$, where $S_{\omega,p}$ is the germ of S_ω at $p = 0$. Therefore,

$$S_{\omega,p} = \{x = \zeta_1 = 0\} = \{x = g(f(\zeta)) = 0\} \cup \left\{ x = \frac{\partial f}{\partial \zeta_1} = \cdots = \frac{\partial f}{\partial \zeta_{n-1}} = 0 \right\}.$$

Hence, either $g(0) = 0$ and $\zeta_1 | f$, or $g(0) \neq 0$ and $\zeta_1 | \frac{\partial f}{\partial \zeta_j}$ for all $j = 1, \ldots, n-1$. In any case, we have $\zeta_1 | f$ and then $f(\zeta) = \zeta_1^k \psi(\zeta)$, where ψ is a germ of holomorphic function in the variable ζ; $k \in \mathbb{N}$ and ζ_1 does not divide ψ. We have two possibilities:

1^{st} **case.–** $\psi(0) \neq 0$. In this case, we consider the biholomorphism

$$G(x, \zeta) = (x, \zeta_1 \psi^{1/k}(\zeta), \zeta_2, \dots, \zeta_n) = (x, y, \zeta_2, \dots, \zeta_n)$$

where $\psi^{1/k}$ is a branch of the k^{th} root of ψ, we get $f \circ G^{-1}(x, y, \zeta_2, \dots, \zeta_n) = y^k$ and

$$G_*(\omega) = xdx + (g(y^k) + xh(y^k))ky^{k-1}dy = xdx + (g_1(y) + xh_1(y))dy,$$

where $g_1(y) = ky^{k-1}g(y^k)$, $h_1(y) = ky^{k-1}h(y^k)$. Therefore, $\tilde{\omega} := G_*(\omega)$ is equivalent to ω and moreover $\tilde{\omega}$ is given by

$$\tilde{\omega} = xdx + (g_1(y) + xh_1(y))dy \quad \text{with} \quad S_{\tilde{\omega}} = \{x = g_1(y) = 0\}. \tag{3.2}$$

Since $d\tilde{\omega} = h_1(y)dx \wedge dy$ is zero identically on $\{x = g_1(y) = 0\}$, we get $g_1|h_1$, so that $h_1(y) = (g_1(y))^m H(y)$, for some $m \in \mathbb{N}$ and such that $H(y)$ does not divided $g_1(y)$. Using the above expression for h_1 in (3.2), we have

$$\tilde{\omega} = xdx + g_1(y)(1 + x(g_1(y))^{m-1}H(y))dy = xdx + g_1(y)(1 + xg_2(y))dy,$$

where $g_2(y) = (g_1(y))^{m-1}H(y)$. Consider $\varphi : (\mathbb{C}, 0) \times (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^2, 0)$ defined by $\varphi(x, \zeta) = (x, y)$, then

$$\omega = \varphi^*(xdx + g_1(y)(1 + xg_2(y))dy). \tag{3.3}$$

2^{nd} **case.–** $\psi(0) = 0$. We have $S_{\omega,p} = \{x = \zeta_1 = 0\}$ and

$$\omega = xdx + (g(\zeta_1^k\psi) + xh(\zeta_1^k\psi))d(\zeta_1^k\psi), \tag{3.4}$$

therefore

$$\omega = xdx + (g(\zeta_1^k\psi) + xh(\zeta_1^k\psi))\zeta_1^{k-1}(k\psi d\zeta_1 + \zeta_1 d\psi). \tag{3.5}$$

Note that $g(0) \neq 0$, otherwise $\{x = \zeta_1\psi(\zeta) = 0\}$ would be contained in $S_{\omega,p}$, but it is contradiction because $S_{\omega,p} = \{x = \zeta_1 = 0\} \subsetneq \{x = \zeta_1\psi(\zeta) = 0\}$. Furthermore $k \geq 2$, because otherwise $\zeta_1|\psi$.

Let $\varphi : (\mathbb{C}, 0) \times (\mathbb{C}^{n-1}, 0) \to (\mathbb{C}^2, 0)$ be defined by

$$\varphi(x, \zeta) = (x, \zeta_1^k\psi(\zeta)) = (x, t),$$

then from (3.4), we get that

$$\omega = \varphi^*(\eta),$$

where $\eta = xdx + (g(t) + xh(t))dt$. Since $\eta(0,0) = g(0)dt \neq 0$, we deduce that η has a non-constant holomorphic first integral $F \in \mathcal{O}_{\mathbb{C}^2,0}$ such that $dF(0,0) \neq 0$. Therefore, $F_1(x, \zeta) = F(x, \zeta_1^k\psi(\zeta))$ is a non-constant holomorphic first integral for ω in a neighborhood of $0 \in \mathbb{C}^n$. □

3.2 - APPLICATIONS TO FOLIATIONS ON $\mathbb{P}^n$

In order to give some applications of Theorem 1, we need the Baum-Bott index associated to singularities of foliations of codimension one.

Let M be a complex manifold and let $\mathcal{G}_\omega = (S, \mathcal{G})$ be a codimension one holomorphic foliation represented by $\omega \in H^0(M, \Omega^1(L))$. We have the exact sequence

$$0 \to \mathcal{G} \to \Theta_M \xrightarrow{\omega} \mathcal{N}_{\mathcal{G}} \to 0, \quad \mathcal{N}_{\mathcal{G}} \simeq \mathcal{J}_S \otimes L.$$

Set $M^0 = M \setminus S$ and take $p_0 \in M^0$. Then in a neighborhood U_α of p_0 the foliation $\mathcal{G}$ is induced by a holomorphic 1–form ω_α and there exists a differentiable 1–form θ_α such that

$$d\omega_\alpha = \theta_\alpha \wedge \omega_\alpha$$

Let Z be an irreducible component of S_2. Take a generic point $p \in Z$, that is, p is a point where Z is smooth and disjoint from the other singular components. Pick B_p a ball centered at p sufficiently small, so that $S(B_p)$ is a sub-ball of B_p of codimension 2. Then the De Rham class can be integrated over an oriented 3-sphere $L_p \subset B_p^*$ positively linked with $S(B_p)$:

$$\mathrm{BB}(\mathcal{G}, Z) = \frac{1}{(2\pi i)^2} \int_{L_p} \theta \wedge d\theta.$$

This complex number is the *Baum-Bott residue of $\mathcal{G}$ along Z*. We have a particular case of the general Baum-Bott residues Theorem (Baum and Bott 1972), reproved by Brunella and Perrone (2011).

Theorem 3.1. *Let $\mathcal{G}$ be a codimension one holomorphic foliation on a complex manifold M. Then*

$$c_1(L)^2 = c_1^2(\mathcal{N}_\mathcal{G}) = \sum_{Z \subset S_2} BB(\mathcal{G}, Z)[Z],$$

where $\mathcal{N}_\mathcal{G} = \mathcal{J}_S \otimes L$ is the normal sheaf of $\mathcal{G}$ on M and the sum is done over all irreducible components of S_2.

In particular, if $\mathcal{G}$ is a codimension one foliation on $\mathbb{P}^n$ of degree d, then the normal sheaf $\mathcal{N}_\mathcal{G} = \mathcal{J}_S(d+2)$ and the Baum-Bott Theorem looks as follows

$$\sum_{Z \subset S_2} \mathrm{BB}(\mathcal{G}, Z) \deg[Z] = (d+2)^2.$$

Remark 3.2. If there exist a coordinates system $(U, (x, y, z_3, \ldots, z_n))$ around $p \in Z \subset S_2$ such that $x(p) = y(p) = 0$ and $S(\mathcal{G}) \cap U = Z \cap U = \{x = y = 0\}$. Moreover, if we assume that

$$\omega|_U = P(x,y)dy - Q(x,y)dx$$

is a holomorphic 1-form representing $\mathcal{G}|_U$. Then we can consider the $\mathcal{C}^\infty$ (1,0)-form θ on $U \setminus Z$ given by

$$\theta = \frac{(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y})}{|P|^2 + |Q|^2}(\bar{P}dx + \bar{Q}dy).$$

Since $d\omega = \theta \wedge \omega$, we get

$$\mathrm{BB}(\mathcal{G}, Z) = \frac{1}{(2\pi i)^2} \int_{L_p} \theta \wedge d\theta = \mathrm{Res}_0 \left\{ \frac{\mathrm{Tr}\big(D\mathbf{X}\big)\, dx \wedge dy}{PQ} \right\}, \tag{3.6}$$

where Res_0 denotes the Grothendieck residue, $D\mathbf{X}$ is the Jacobian of the holomorphic map $\mathbf{X} = (P, Q)$. It follows from Griffiths and Harris (1978) that if $D\mathbf{X}(p)$ is non-singular, then

$$\mathrm{BB}(\mathcal{G}, Z) = \frac{\mathrm{Tr}(D\mathbf{X}(p))^2}{\det(D\mathbf{X}(p))}.$$

In the situation explained above, the tangent sheaf $\mathcal{G}(U)$ is locally free and generated by the holomorphic vector fields

$$\mathcal{G}(U) = \left\langle \mathbf{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}, \frac{\partial}{\partial z_3}, \dots, \frac{\partial}{\partial z_n} \right\rangle$$

and the vector field $\mathbf{X}$ carries the information of the Baum–Bott residues.

The next result, in an application of Theorem 1

Theorem 3.3. *Let $\omega \in \mathcal{F}(M, L)$ be a foliation and $Z \subset S_2 \setminus K(\omega)$. Suppose that Z is smooth and $j^1_p\omega \neq 0$ for all $p \in Z$, then $BB(\mathcal{F}_\omega, Z) = 0$.*

Proof. We work in a small neighborhood U of $p \in Z \subset M$. According to Theorem 1 there exist a coordinate system $(x, y, z_3, \dots, z_n)$ at p such that $Z \cap U = \{x = y = 0\}$ and one has three cases. In the first case, $\mathcal{F}_\omega$ is the product of a dimension one foliation in a section transversal to Z by a regular foliation of codimension two and $j^1_p(\omega) = xdy + ydx$. In this case, it follows from (3.6) that $\mathrm{BB}(\mathcal{F}_\omega, Z) = 0$. In the second case

$$\omega = xdx + g_1(y)(1 + xg_2(y))dy,$$

where $g_1, g_2 \in \mathcal{O}_{\mathbb{C},0}$ and it follows from Lemma 3.9 of Cerveau and Lins Neto (2013) that

$$\mathrm{BB}(\mathcal{F}_\omega, Z) = \mathrm{Res}_{t=0}\left[\frac{(g_1(t)g_2(t))^2dt}{g_1(t)}\right] = \mathrm{Res}_{t=0}\left[g_1(t)(g_2(t))^2\right].$$

Since $g_1(y)(g_2(y))^2$ is holomorphic at $y = 0$, we get $\mathrm{BB}(\mathcal{F}_\omega, Z) = 0$. In the third case $\mathcal{F}_\omega$ has a holomorhic first integral in neighborhood of p and is known that $\mathrm{BB}(\mathcal{F}_\omega, Z) = 0$. □

The Baum-Bott formula implies the following result.

Corollary 3.4. *Let $\omega \in \mathcal{F}(n, d)$, $n \geq 3$, be a foliation with $K(\omega) = \emptyset$. Then there exists a smooth point $p \in S_2$ such that $j^1_p\omega = 0$.*

Proof. If for all smooth point $p \in S_2$ one has $j^1_p\omega \neq 0$, the above theorem shows that $\mathrm{BB}(\mathcal{F}_\omega, Z) = 0$ for all irreducible components $Z \subset S_2$. By Baum–Bott's theorem, we get

$$0 < (d+2)^2 = \sum_{Z \subset S_2} \mathrm{BB}(\mathcal{F}_\omega, Z)\deg[Z] = 0$$

which is a contradiction. Therefore there exists a smooth point $p \in S_2$ such that $j^1_p\omega = 0$. □

In particular, if $\omega \in \mathcal{F}(n, d)$, $n \geq 3$, is a foliation with $j^1_p\omega \neq 0$ for any $p \in \mathbb{P}^n$, then its Kupka set is not empty.

4 - THE NUMBER OF NON-KUPKA POINTS

Through this section, we consider codimension one foliations on $\mathbb{P}^3$, but some results remain valid to codimension one foliations on others manifolds of dimension three.

4.1 - SIMPLE SINGULARITIES

Let ω be a germ of 1–form at $0 \in \mathbb{C}^3$. We define the *rotational* of ω as the unique vector field **X** such that

$$rot(\omega) = \mathbf{X} \Longleftrightarrow d\omega = \imath_{\mathbf{X}} dx \wedge dy \wedge dz,$$

moreover ω is integrable if and only if $\omega(rot(\omega)) = 0$.

Let ω be a germ of an integrable 1–form at $0 \in \mathbb{C}^3$. We say that 0 is a *simple singularity* of ω if $\omega(0) = 0$ and either $d\omega(0) \neq 0$ or $d\omega$ has an isolated singularity at 0. In the second case, these kind of singularities, are classified as follows

1. *Logarithmic*. The second jet $j_0^2(\omega) \neq 0$ and the linear part of $\mathbf{X} = rot(\omega)$ at 0 has non zero eigenvalues.
2. *Degenerated*. The rotational has a zero eigenvalue, the other two are non zero and necessarily satisfies the relation $\lambda_1 + \lambda_2 = 0$.
3. *Nilpotent*. The rotational vector field **X**, is nilpotent as a derivation.

The structure near simple singularity is known (Calvo-Andrade et al. 2004). If $p \in S$ is a simple singularity and $d\omega(p) = 0$, then p is a singular point of S.

Theorem 4.1. *Let* $\omega \in \Omega^1(\mathbb{C}^3, 0)$*,* $n \geq 3$*, be a germ of integrable 1-form such that* ω *has a simple singularity at* 0 *then the tangent sheaf* $\mathcal{F} = Ker(\omega)$ *is locally free at* 0 *and it is generated by* $\langle rot(\omega), \mathbf{S}\rangle$*, where* **S** *has non zero linear part.*

Proof. Let ω be a germ at $0 \in \mathbb{C}^3$ of an integrable 1–form and 0 a simple non-Kupka singularity. Then $0 \in \mathbb{C}^3$ is an isolated singularity of $\mathbf{X} = rot(\omega)$. Consider the Koszul complex of the vector field **X** at 0

$$\mathbb{K}(\mathbf{X})_0 : 0 \to \Omega^3_{\mathbb{C}^3,0} \xrightarrow{\imath_{\mathbf{X}}} \Omega^2_{\mathbb{C}^3,0} \xrightarrow{\imath_{\mathbf{X}}} \Omega^1_{\mathbb{C}^3,0} \xrightarrow{\imath_{\mathbf{X}}} \mathcal{O}_{\mathbb{C}^3,0} \to 0$$

Since $\omega(\mathbf{X}) = 0$, then $\omega \in H^1(\mathbb{K}(\mathbf{X})_0)$ that vanishes because **X** has an isolated singularity at 0. Therefore, there exists $\theta \in \Omega^2_{\mathbb{C}^3,0}$ such that $\imath_{\mathbf{X}}\theta = \omega$. The map $\Theta_{\mathbb{C}^3,0} \ni \mathbf{Z} \mapsto \imath_{\mathbf{Z}} dx \wedge dy \wedge dz \in \Omega^2_{\mathbb{C}^3,0}$ is an isomorphism, hence

$$\omega = \imath_{\mathbf{X}}\theta, \quad \text{and} \quad \theta = \imath_{\mathbf{S}} dx \wedge dy \wedge dz, \quad \text{implies} \quad \omega = \imath_{\mathbf{X}}\theta = \imath_{\mathbf{X}}\imath_{\mathbf{S}} dx \wedge dy \wedge dz$$

and then, the vector fields $\{\mathbf{X}, \mathbf{S}\}$ generate the sheaf $\mathcal{F}$ in a neighborhood of 0. □

Let $\omega \in \mathcal{F}(3, d)$ be a foliation and $Z \subset S_2$ be a connected component of S_2. Assume that Z is a local complete intersection and has only simple singularities. We will calculate the number $|NK(\omega) \cap Z|$ of non-Kupka points in Z.

Proof of Theorem 2. Let $\mathcal{J}$ be the ideal sheaf of Z. Since Z is a local complete intersection, consider the exact sequence

$$0 \to \mathcal{J}/\mathcal{J}^2 \to \Omega^1 \otimes \mathcal{O}_Z \to \Omega^1_Z \to 0$$

Taking $\wedge^2$ and twisting by $L = K_{\mathbb{P}^3}^{-1} \otimes K_Z = K_Z(4)$ we get

$$0 \to \wedge^2 \mathcal{J}/\mathcal{J}^2 \otimes L \to \Omega^2_{\mathbb{P}^3}|_Z \otimes L \to \cdots$$

Since $Z \subset S$, the singular set, we have seen before that

$$d\omega|_Z \in H^0(Z, \wedge^2(\mathcal{J}/\mathcal{J}^2) \otimes L)$$

Now, from the equalities of sheaves

$$K_Z^{-1} \otimes K_{\mathbb{P}^3} \simeq \wedge^2(\mathcal{J}/\mathcal{J}^2), \quad \text{and} \quad L \simeq K_{\mathbb{P}^3}^{-1} \otimes K_{\mathcal{F}}$$

we have

$$H^0(Z, \wedge^2(\mathcal{J}/\mathcal{J}^2) \otimes L) = H^0(Z, K_Z^{-1} \otimes K_{\mathcal{F}}|_Z),$$

the non-Kupka points of ω in Z satisfies $d\omega|_Z = 0$, denoting

$$D_\omega = \sum_{p \in Z} ord_p(d\omega)$$

the associated divisor to $d\omega|_Z$, one has

$$\deg(D_\omega) = \deg(K_{\mathcal{F}}) - \deg(K_Z),$$

as claimed. □

Remark 4.2. The method of the proof works also in projective manifolds, and does not depends on the integrability condition.

4.2 - EXAMPLES

We apply Theorem 2 for some codimension one holomorphic foliations on $\mathbb{P}^3$ and determine the number of non-Kupka points.

Example 4.3 (Degree two logarithmic foliations)**.** Recall that the canonical bundle of a degree two foliation of $\mathbb{P}^3$ is trivial. There are two irreducible components of logarithmic foliations in the space of foliations of $\mathbb{P}^3$ of degree two: $\mathcal{L}(1,1,2)$ and $\mathcal{L}(1,1,1,1)$. We analyze generic foliations on each component.
Component $\mathcal{L}(1,1,2)$: let ω be a generic element of $\mathcal{L}(1,1,2)$ and consider its singular scheme $S = S_2 \cup S_3$. By Theorem 3 of Cukierman et al. (2006), we have $\ell(S_3) = 2$. On the other hand, S_2 has three irreducible components, two quadratics and a line, the arithmetic genus is $p_a(S_2) = 2$. Note that Theorem 2, implies that the number $|NK(\omega) \cap S_2|$, of non-Kupka points in S_2 is

$$|NK(\omega) \cap S_2| = deg(D_\omega) = \deg(K_{\mathcal{F}}) - \deg(K_{S_2}) = -\chi(S_2) = 2.$$

The non-Kupka points of the foliation $\mathcal{F}_\omega$ are $|NK(\omega)| = \ell(S_3) + |NK \cap S_2| = 4$.
Component $\mathcal{L}(1,1,1,1)$: let ω be a generic element of $\mathcal{L}(1,1,1,1)$ then the tangent sheaf is $\mathcal{O} \oplus \mathcal{O}$ and the singular scheme $S = S_2$ (Giraldo and Pan-Collantes 2010), moreover consists of 6 lines given the edges of a tetrahedron, obtained by intersecting any two of the four invariant hyperplanes H_i. The arithmetic genus is $p_a(S_2) = 3$, by Theorem 2, $|NK(\omega)| = |NK(\omega) \cap S_2| = 4$, corresponding to the vertices of the tetrahedron where there are simple singularities of logarithmic type.

Example 4.4 (The exceptional component $\mathcal{E}(3)$)**.** The leaves of a generic foliation $\omega \in \mathcal{E}(3) \subset \mathcal{F}(3,2)$, are the orbits of an action of **Aff**$(\mathbb{C}) \times \mathbb{P}^3 \to \mathbb{P}^3$ and its tangent sheaf is $\mathcal{O} \oplus \mathcal{O}$ (see Calvo-Andrade et al. 2004, Giraldo and Pan-Collantes 2010). Its singular locus $S = S_2$ has $deg(S) = 6$ and three irreducible components: a line L, a conic C tangent to L at a point p, and a twisted cubic Γ with L as an inflection line at p. Then $NK(\omega) = L \cap C \cap \Gamma = \{p\} \subset S$.

The arithmetic genus is $p_a(S) = 3$ and the canonical bundle of the foliation again is trivial, by Theorem 2, the number of non-Kupka points $|NK(\omega)| = 4$. Therefore the non-Kupka divisor $NK(\omega) \cap S = 4p$. If ω represents the foliation at p, then $\mu(d\omega, p) = \mu(rot(\omega), p) = 4$.

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5 - APPENDIX

We prove Lemma 2.5.

Proof. First, we see that $h^0(F) \geq 1$. By Riemann–Roch–Hirzebruch, we have

$$\chi(F) = h^0(F) - h^1(F) + h^2(F) = [ch(F) \cdot Td(\mathbb{P}^2)]_2 = 2,$$

then

$$h^0(F) + h^2(F) = [ch(F) \cdot Td(\mathbb{P}^2)]_2 + h^1(F) \geq [ch(F) \cdot Td(\mathbb{P}^2)]_2 = 2$$

By Serre duality (Griffiths and Harris 1978, Okonek et al. 1980), we get $h^2(F) = h^0(F(-3))$. Moreover $h^0(F) \geq h^0(F(-k))$ for all $k > 0$, hence $h^0(F) \geq 1$. Let $\tau \in H^0(F)$ be a non zero section, consider the exact sequence

$$0 \longrightarrow \mathcal{O} \xrightarrow{\cdot\tau} F \longrightarrow \mathcal{Q} \longrightarrow 0 \quad \text{with} \quad \mathcal{Q} = F/\mathcal{O}. \tag{5.1}$$

The sheaf $\mathcal{Q}$ is torsion free, therefore $\mathcal{Q} \simeq \mathcal{J}_\Sigma$ for some $\Sigma \subset \mathbb{P}^2$. The sequence (5.1), is a free resolution of the sheaf $\mathcal{Q}$ with vector bundles with zero Chern classes. From the definition of Chern classes for coherent sheaves (Baum and Bott 1972), we get $c(\mathcal{Q}) = 1$, in particular $\deg(\Sigma) = c_2(\mathcal{Q}) = 0$, we conclude that $\Sigma = \emptyset$ and $\mathcal{Q} \simeq \mathcal{O}$. Then F is an extension of holomorphic line bundles, hence it splits (Okonek et al. 1980, p. 15). □