# On The Existence of Levi Foliations

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**ABSTRACT**

Let $L \subset \mathbb{C}^2$ be a real 3 dimensional analytic variety. For each regular point $p \in L$ there exists a unique complex line $l_p$ on the space tangent to $L$ at $p$. When the field of complex line

$$p \mapsto l_p$$

is completely integrable, we say that $L$ is Levi variety. More generally; let $L \subset M$ be a real subvariety in an holomorphic complex variety $M$. If there exists a real 2 dimensional integrable distribution on $L$ which is invariant by the holomorphic structure $J$ induced by $M$, we say that $L$ is a Levi variety. We shall prove:

**Theorem.** *Let $\mathcal{L}$ be a Levi foliation and let $\mathcal{F}$ be the induced holomorphic foliation. Then, $\mathcal{F}$ admits a Liouvillian first integral.*

In other words, if $\mathcal{L}$ is a 3 dimensional analytic foliation such that the induced complex distribution defines an holomorphic foliation $\mathcal{F}$; that is, if $\mathcal{L}$ is a Levi foliation; then $\mathcal{F}$ admits a Liouvillian first integral—a function which can be constructed by the composition of rational functions, exponentiation, integration, and algebraic functions (Singer 1992). For example, if $f$ is an holomorphic function and if $\theta$ is real a 1-form on $\mathbb{R}^2$; then the pull-back of $\theta$ by $f$ defines a Levi foliation $\mathcal{L} : f^*\theta = 0$ which is tangent to the holomorphic foliation $\mathcal{F} : df = 0$.

This problem was proposed by D. Cerveau in a meeting (see Fernandez 1997).

**Key words:** Levi foliations, holomorphic foliations, singularities, Levi varieties.

**ANNOUNCEMENT**

Let $\mathcal{L}$ be a Levi foliation and let $\mathcal{F}$ be the holomorphic foliation tangent to $\mathcal{L}$. Note that if $h$ in an holomorphic function such that $\mathcal{F}$ is $h$-invariant ($h^*\mathcal{F} = \mathcal{F}$); then $\mathcal{L}$ is also $h$-invariant ($h^*\mathcal{L} = \mathcal{L}$). We shall mainly use that property in order to prove

Theorem. *Let $\mathcal{L}$ be a Levi foliation and let $\mathcal{F}$ be the induced holomorphic foliation. Then $\mathcal{F}$ admits a Liouvillian first integral.*

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We proceed as follows:

We first show that if $\mathcal{L}$ is a Levi foliation, there exists analytic real functions $g_1$, $g_2$ such that: if $G = g_1 + ig_2$, then the Levi foliation is defined by

$$\mathcal{L} : \overline{G}\omega + G\overline{w} = 0,$$

where $\omega$ is an holomorphic 1-form so that $\omega = 0$ defines the holomorphic foliation $\mathcal{F}$ tangent to the Levi foliation $\mathcal{L}$. We then verify that; if $\mathcal{F}^*$ is the holomorphic foliation obtained from $\mathcal{F}$ after a finite number of blow-ups, there exists a Levi foliation $\mathcal{L}^*$ tangent to $\mathcal{F}^*$. Therefore, by Seidenberg Theorem (Seindenberg 1968), we analyse the foliation $\mathcal{F}^*$ for which all singularities are reduced.

Let $D$ denote the divisor obtained on the process of reducing the singularity and let $D_j$ denote the irreducible curves with normal crossings such that $D = \cup D_j$. We consider the induced Levi foliation on sections transversal to the holomorphic foliation through each component $D_j$ of the divisor. We show that the holomorphic diffeomorfisms for which the Levi foliation is invariant must satisfy an equation on one variable of the type

$$h'(z) = t\frac{F(h)}{F}; \quad t \in \mathbb{R} \tag{*}$$

We can then find an holomorphic coordinate system $y$ on the section such that

$$F(y) = \frac{y^{k+1}}{1 - \lambda y^k}.$$

We refer to such coordinate system as a normalizable coordinate system. We verify that it is unique up to homographies.

If either $\lambda \neq 0$ or $k = 0$, then $t = 1$ for all solutions $h$ of the differential equation (*). Furthermore, if $k = 0$, then the group of solutions of the differential equation is a linear group. On both cases we have an abelian group for the group of solutions of (*). We can already conclude:

THEOREM A. *Let $p$ be a singularity of the foliation*

$$\mathcal{F} : \omega = \lambda x dy + y dx + \{\textit{higher order terms}\} = 0 \quad \lambda \in \mathbb{R}^* - \mathbb{Q}\,.$$

*Suppose there exists a Levi foliation $\mathcal{L}$ tangent to $\mathcal{F}$. Then the singularity is analytically equivalent to a linear singularity.*

PROOF. For if there exists a Levi foliation, the holonomy associated to the singularity must satisfy an equation as (*). If so, the order of $F$ at 0 cannot be but 1; that is, k=0. The holonomy is linearizable; as a result, so is the singularity (Mattei & Moussu 1980).

We still have to consider the case $\lambda = 0$. There are solutions for which $t \neq 1$, $(h'(0))^k = \frac{1}{t} \in \mathbb{R}$. These solutions are necessarily linearizable, but not those for which $t = 1$. The latter, though, also determine an abelian group. We shall then describe the abelian group of solutions of (*) for $t = 1, k > 0$.

We can take an holomorphic coordinate system $(x, y)$ such that the group of solutions of the differential equation is in normalizable coordinate system on each transversal section $x = cte$.

For an holomorphic vector field $X$, let $\exp X$ denote its exponential application, that is, its flow for $t = 1$:

$$\exp(\xi(z)\frac{\partial}{\partial z})(z) = z + f_1(z) + \frac{1}{2}f_2(z) + \frac{1}{3!}f_3(z) + \dots$$

satisfying

$$\begin{cases} f_1 = \xi, \\ f_n = \xi f'_{n-1}. \end{cases}$$

If $h$ is a diffeomorfism which satisfies

$$h'(z) = \frac{h^{k+1}}{1 - \lambda h^k}\frac{1 - \lambda y^k}{y^{k+1}}$$

then the $k$-th interate of h; $h^k$, is tangent to the identity. There exists $\mu$ such that $h^k$ is the exponential of the vector field:

$$Y = 2\pi i\mu\frac{y^{k+1}}{1 - \lambda y^k}\frac{\partial}{\partial y};$$

that is

$$h^k(w) = \exp(2\pi i\mu\frac{y^{k+1}}{1 - \lambda y^k}\frac{\partial}{\partial y})(w).$$

Consequently

$$h(w) = \exp(2\pi i\frac{\mu}{k}\frac{y^{k+1}}{1 - \lambda y^k}\frac{\partial}{\partial y})(\epsilon w);\ \epsilon^k = 1.$$

If

$$X = x\frac{\partial}{\partial x} + yf(x, y)\frac{\partial}{\partial y}$$

is the vector field which defines the holomorphic foliation; then the holonomy application is defined by

$$\exp 2\pi iX.$$

We have found two linear independent vector fields—$X$, $Y$ that define $h$. Therefore; they commute:

$$[X, Y] = 0\ .$$

We can describe $X$ to be so as to satisfy the commutability condition. We then show the local result:

THEOREM B. *Let $p$ be a singularity of the foliation*

$$\mathcal{F} : \omega = \lambda xdy + ydx + \{\textit{higher order terms}\} = 0, \lambda \in \mathbb{C}.$$

*Suppose there exists a Levi foliation $\mathcal{L}$ tangent to $\mathcal{F}$. Then the singularity is normalizable in the sense of Martinet and Ramis (1982), Martinet and Ramis (1983). In particular, $\omega$ admits an analytic integrating factor.*

PROOF. If $\lambda \in \mathbb{C} - \mathbb{R}$, the singularity is linerizable by Poincare's Theorem. If $\lambda \in \mathbb{R} - \mathbb{Q}$, we have proved (Theorem A) that is also a linerizable singularity. Thus, we have to prove the result for $\lambda \in \mathbb{Q}$; since the singularity is a reduced one, $\lambda \in \mathbb{Q}_+$. Let

$$-2\pi i Y(x_0, y) = -2\pi i \mu(x_0)^k \frac{y^{k+1}}{1 - \lambda\mu(x_0)^k y^k}\frac{\partial}{\partial y}$$

be the vector field whose exponential application determines the holonomy application on $x_0$. If there are two invariant curves through the singularity, then the vector field that defines the holomorphic distribution can be written as $x\frac{\partial}{\partial x} + yf(x, y)\frac{\partial}{\partial y}$. By solving the commutability condition $[X, Y] = 0$ :

$$\begin{aligned} 0 &= \left[x\frac{\partial}{\partial x} + yf(x, y)\frac{\partial}{\partial y}, \mu(x)^k\frac{y^{k+1}}{1 - \lambda\mu(x)^k y^k}\frac{\partial}{\partial y}\right] \\ &= \left(y\frac{1}{(1 - \lambda\mu(x)^k y^k)^2} d(\mu(x)^k y^k).(x, yf) - y\frac{\partial f}{\partial y}\mu(x)^k \frac{y^{k+1}}{1 - \lambda\mu(x)^k y^k}\right)\frac{\partial}{\partial y}\ . \end{aligned}$$

Let $f(x, y) = f(x, 0) + g(x, y)$, then $f$ must be as to satisfy

$$\begin{cases} f(x, 0) = \frac{\mu'(x)x}{\mu(x)}\ , \\ \frac{\partial}{\partial y}\log g = k\frac{1}{y(1-\lambda\mu^k y^k)} = k\frac{\partial}{\partial y}\log\left(\frac{y}{(1-\lambda\mu^k y^k)^{\frac{1}{k}}}\right); \end{cases}$$

which leads us to

$$f(x, y) = \frac{\mu'(x)x}{\mu(x)} + \delta(x)\frac{y^k}{1 - \lambda\mu(x)^k y^k}.$$

The foliation on the punctured neighborhood is defined by the following 1-form

$$\omega = xdy + y\left(\frac{\mu'(x)x}{\mu(x)} - \delta(x)\frac{y^k}{1 - \lambda\mu^k y^k}\right)dx$$

or still by

$$\begin{aligned} \frac{\mu}{x}\omega &= \mu dy + y(1 - \frac{\delta}{\mu' x}\frac{y^k\mu^k}{1 - \lambda\mu^k y^k})d\mu \\ &= \frac{\mu^{k+1}y^{k+1}}{1 - \lambda x^k y^k}\left(\frac{1 - \lambda\mu^k y^k}{\mu^k y^k}\frac{d(\mu^k y^k)}{(\mu^k y^k)^2} + \frac{\delta}{x}dx\right)\ . \end{aligned}$$

Necessarily $\delta$ has an holomorphic extension through 0 and $\mu^k$ has either an holomorphic or a meromorphic extension through 0. If it were meromorphic, the singularity would not be a reduced one, contradicting our hypotheses. The extension is then an holomorphic one. We have then a normal form for either cases:

If $\mu^k \in \mathcal{O}^*$, we have a saddle-node; if $\mu^k \in \mathcal{O} - \mathcal{O}^*$ and let $p$ be the order of the zero of $f$ at 0, we have a ressonant singularity.

If there is only one invariant curve through the singularity; the singularity is a saddle-node and the invariant curve is $y = 0$. Therefore the vector field that defines the holomorphic distribution can be written as $X = (x + h(y))\frac{\partial}{\partial x} + yf(x, y)\frac{\partial}{\partial y}$, $f(0) = 0$. The holonomy is defined by the exponential application of the vector field $\frac{x}{x+h(y)}X = x\frac{\partial}{\partial x} + \frac{yf(x,y)}{x+h(y)}\frac{\partial}{\partial y}$. The commutability condition $[\frac{x}{x+h(y)}X, Y] = 0$ implies that

$$\frac{x}{x + h(y)}[X, Y] = \left(d\frac{x}{x + h(y)}.Y\right)X\ .$$

By solving the equation just above, we obtain that $\frac{1}{f}$ must be an holomorphic function which contradicts $f(0) = 0$. □

Following, we prove results that will allow us to relate the first integrals obtained on the neighborhood of each component $D_j$.

THEOREM C. *Let $p$ be a singularity of the foliation $\mathcal{F} : \omega = 0$ and*

$$\omega = f dF \text{ is an holomorphic 1-form}$$

*where $F$ is a Liouvillian function and $f$ is an holomorphic integrating factor of $\omega$. There exists a Levi foliation defined by*

$$\mathcal{L} : \overline{f}(f dF) + f(\overline{f dF}).$$

*Furthermore, if $p$ is not a linearizable ressonant singularity, then any other Levi foliation must be of the type:*

$$\mathcal{L}_\lambda : \lambda\overline{f}(f dF) + \overline{\lambda} f(\overline{f dF}).$$

Note that $\Re(\lambda F)$ is a first integral of the Levi foliation $\mathcal{L}_\lambda$. We can then show:

COROLLARY. *Let $p$ be a singularity of the holomorphic foliation $\mathcal{F} : \omega = 0$. Let $F_j$ be Liouvillian functions and let $f_j$ be holomorphic functions such that*

$$\omega = f_j dF_j .$$

*Suppose there exists a Levi foliation $\mathcal{L}$ tangent to $\mathcal{F}$ and suppose that $\Re(F_1), \Re(F_2)$ are first integrals of $\mathcal{L}$. Then:*

$$\frac{dF_j}{F_j} = \frac{dF_i}{F_i}\ .$$

PROOF. Follows from $dF_i = \frac{f_j}{f_i} dF_j$ and $d(F_i + \overline{F_i}) \wedge d(F_j + \overline{F}_j) = 0$.

We are then able to show:

THEOREM D. *Let $\mathcal{F}$ be an holomorphic foliation and $\mathcal{L}$ be a Levi foliation tangent to $\mathcal{F}$. Suppose all singularities lie on an irredutible curve $S$; which is $\mathcal{F}$-invariant. Then $\mathcal{F}$ admits a Liouvillian first integral $I$ defined on a neighborhood of $S$. Furthermore, $d(I + \overline{I})$ defines a Levi foliation tangent to $\mathcal{F}$.*

PROOF. To show the existence of a Liouvillian first integral of $\mathcal{F}$ it is enough to show the existence of a Liouvillian first integral of the reduced foliation $\mathcal{F}^*$. Let $D = \cup D_j$ be the divisor obtained on the process of reducing the singularities. Let us fix a transversal section of $\mathcal{F}^*$ through $D_j$. Since there exists a Levi foliation tangent to $\mathcal{F}^*$, there exists a normal coordinate system on the section so that the holonomy applications determined by the singularities on $D_j$ satisfy (*).

For each $D_j$, we then find an holomorphic vector field $Z_j$ that defines the foliation $\mathcal{F}^*$ in a neighborhood of the divisor. Let $Y$ be the holomorphic vector on each transversal section which defines the holonomies. To find $Z_j$, all we have to do is solve the equation

$$[Z_j, Y] = 0.$$

The vector field $Z_j$ allows us to describe a Liouvillian first integral of the holomorphic foliation on a neighborhood of each irreduceble component $D_j$ of the divisor $D = \cup D_j$ obtained on the resolution of the singularity. Let $F_j$ be a Liouvillian first integral of the holomorphic foliation $\mathcal{F}^*$ on a neighborhood of the $D_j$ such that $\Re(F_j)$ is a first integral of $\mathcal{L}^*$. By Theorem b, for each

$$p \in D_i \cap D_j$$

we have

$$\frac{dF_1}{F_1} = \frac{dF_2}{F_2}.$$

Therefore

$$\omega^* = \{\frac{dF_i}{F_i}\}.$$

is a well defined closed 1-form. Thus

$$I = \exp \int \omega^*$$

is a Liouvillian first integral of the holomorphic foliation $\mathcal{F}^*$ and there is a Levi foliation $d(I + \overline{I}) = 0$; *The Theorem* is thereby proved.

**RESUMO**

Seja $L \subset \mathbb{C}^2$ uma variedade real de dimensão 3. Para todo ponto regular $p \in L$ existe uma única reta complexa $l_p$ no espaço tangente à $L$ em $p$. Quando o campo de linhas complexas

$$p \mapsto l_p$$

é completamente integrável, dizemos que $L$ é uma variedade de Levi. Mais geralmente, seja $L \subset M$ uma subvariedade real em uma variedade analítica complexa. Se existe uma distribuição real integrável de

dimensão 2 em $L$ que é invariante pela estrutura holomorfa $J$ induzida pela variedade complexa $M$, dizemos que $L$ é uma variedade de Levi. Vamos provar:

**Teorema.** *Seja $\mathcal{L}$ uma folheação de Levi e seja $\mathcal{F}$ a folheação holomorfa induzida. Então $\mathcal{F}$ tem integral primeira Liouvilliana.*

Em outras palavras, se $\mathcal{L}$ é uma folheação real de dimensão 3 tal que a folheação holomorfa induzida define uma folheação holomorfa $\mathcal{F}$; isto é, se $\mathcal{L}$ é uma folheação de Levi; então $\mathcal{F}$ admite uma integral primeira Liouvilliana – uma função que pode ser construida por composição de funções rationais, exponenciações, integrações e funções racionais (Singer 1992). Por exemplo, se $f$ é uma função holomorfa e se $\theta$ é uma 1-forma real em $\mathbb{R}^2$; então o pull-back de $\theta$ por $f$ define uma folheação de Levi: $\mathcal{L} : f^*\theta = 0$ a qual é tangente a folheação holomorfa $\mathcal{F} : df = 0$.

Este problema foi proposto por D. Cerveau em uma reunião (Fernandez 1997).

**Palavras-chave:** folheações de Levi, folheações holomorfas, singularidades, variedades de Levi.

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