BCG - Boletim de Ciências Geodésicas - On-Line version, ISSN 1982-2170
http://dx.doi.org/10.1590/S1982-21702015000400050

Article

MATHEMATICAL ANALYSIS OF THE ALGORITHMS USED IN MODERNIZED METHODS OF BUILDING MEASUREMENTS WITH RTN GNSS TECHNOLOGY

Análise matemática dos algoritmos aplicados nos métodos modernizados de medição predial com tecnologia RTN GNSS

Robert Krzyżek[1]

AGH University of Science and Technology,Faculty of Mining Surveying and Environmental Engineering, Department of Geomatics. 30 Mickiewicza Al., 30-059 Krakow, Poland.E-mail: rkrzyzek@agh.edu.pl[1]

Abstract:

The use of RTN GNSS surveying technology (*Real Time Kinematic Global Navigation Satellite System*) for direct measurements of building structures is extremely difficult, and often impossible. Therefore, the real-time technology is supported by indirect methods of measurements. One such solution is the method of *line-line intersection*. Having determined the position (the coordinates) of the so-called base points on the extension of the wall faces of a building, and having calculated the coordinates of intersection of the lines of relationships between the base points, the coordinates of corners of a building structure are obtained, which are relatively unreliable. In order to increase the reliability of these coordinates, the author proposes to add an innovative solution called ***vector translation*** to the classical method of *line-line intersection*. This paper proves the thesis on the increased reliability of determining the coordinates of corners of buildings by conducting an appropriate mathematical analysis of these formulas, using the algorithm developed by the author. The performed analysis shows that this innovative solution results in a smaller difference between the adjusted coordinates relative to the theoretical ones, than the difference captured from the "raw" measurement results with respect to the theoretical values.

Keywords: RTK-RTN GNSS, Algorithm, Vector Translation, Vector Addition, Half-angle Method, Measurement of a Building

Resumo:

A aplicação da tecnologia GNSS RTN (*Real Time Kinematic Global Navigation Satellite System*) para medição predial direta é extremamente difícil e muitas vezes impossível. Em razão disso, a tecnologia de medição em tempo real é suportada por métodos indiretos de operação. Uma destas soluções é a utilização do método de *interseção de retas*. Depois de especificada a localização (coordenadas) dos pontos-base na extensão da face das paredes da edificação e do cálculo das coordenadas das interseções de retas da inter-relação dos pontos-base, são obtidas as coordenadas dos cantos da edificação, que são pouco confiáveis. Para aumentar a confiabilidade destas coordenadas, o autor propõe na aplicação do método clássico de *interseção de retas* a inovadora solução chamada ***translação de vetores***. Um algoritmo específico dos métodos de ***translação de vetores*** e a prova empírica de sua aplicação são apresentados em outros artigos. Neste artigo é comprovada a tese do aumento da confiabilidade da determinação das coordenadas dos cantos da edificação com a utilização do algoritmo autoral, realizando apropriada análise matemática dos objetos projetados. A análise efetuada mostra que, após a aplicação da solução inovadora, foi alcançada uma diferença menor entre as coordenadas comparadas em relação aos valores teóricos do que as diferenças entre as coordenadas obtidas com os resultados "brutos" da medição em relação aos valores teóricos.

Palavras-chave: RTK-RTN GNSS, Algoritmo, Translação de Vetores, Soma de Vetores, Método Meio-ângulo, Medição Predial

1. Introduction

In surveying, the awareness of the performed activities is essential, as well as certainty that the captured measurement results will meet the accuracy requirements regulated by legal standards. In Poland, one of the most important laws, which all the surveying practices are based on, is the regulation, (MIA, 2011). The secondary legislation precisely defines the measurement methods used in surveying, and also allows for other solutions, which are unspecified directly but meet the accuracy requirements. One of the measurement technologies used in surveying is RTN GNSS measurement, which has been successfully used in a variety of ways for many years (Naus and Nowak, 2014) or (Uznański, 2010). However, the use of this surveying mode for the direct measurement of a building is relatively difficult, and sometimes even impossible. The reason lies behind a necessity of an open horizon during real-time measurements, and the building, naturally, makes the implementation of this task difficult. Scientists are attempting to solve such problems by constantly searching for innovative solutions designed to improve the latest surveying technologies, so that they could be used in all kinds of field conditions, while ensuring the reliability of positioning. The studies carried out in difficult observational conditions for GNSS measurements have been presented, for example, in (Bakuła, 2013) and (Pelc-Mieczkowska, 2012).

The paper, (Bakuła, 2013), presents an innovative solution for fast and ultrafast static measurements to determine the coordinates of a control point. The coordinates of this point were determined by three GNSS receivers positioned in a line on a special base at a distance of every 0.5m. The proposed method allowed reliably determining of the coordinates, despite difficult observational conditions (along the edge of a forest or entirely in the forest, in the vicinity of building structures or power transmission lines). Although the measurements were not carried out in RTN GNSS, it still could be noticed that the very idea of creating innovative solutions for surveying structures with limited accessibility for GNSS technology is still in high demand.

The research results presented in (Pelc-Mieczkowska, 2012), demonstrate that during the RTN GNSS measurements in difficult field conditions (the obscured horizon), outliers may occur, with their values of even up to a few meters. The author argues that systematic errors are likely to be the reason, resulting in the incorrect determination of uncertainty. Such a situation may occur in spite of low values of the parameters defining measurement accuracy. Therefore, the paper, (Pelc-Mieczkowska, 2012), suggests a need to develop a method for increasing the reliability of RTN GNSS positioning in difficult observational conditions and performing measurement checks.

One of the elements of such checks may be independent determination of coordinates of the so-called control points by a double initialization of the GNSS receiver which is closely associated with the requirements set out in the (MIA, 2011) regulation. The issue of measuring control points has been presented in detail in (Krzyżek et al., 2014b)

An additional factor favorably affecting the accuracy, continuity and integrity of positioning, as well as the efficiency and precision of the captured results, both in the conditions with the obscured and clear horizon, is the application of a multi-constellation system (GPS, GLONASS), which has been demonstrated and proven in the research studies, (Angrisano et al., 2013) and (Pirti et al., 2013).

Basing on the existing research studies and on the ones described above, the author conducted his own studies to support the RTN GNSS technology with indirect methods of measurement in difficult observational conditions. One such method is *line-line intersection*. Having determined the position (the coordinates) of the so-called base points on the extension of the wall faces of a building, and having calculated the coordinates of intersection of the lines of relationships between the base points, the coordinates of corners of a building structure are obtained. Unfortunately, the studies of (Krzyżek, 2015), have shown that reliability of the determined coordinates of the building is limited. Thus, for the measurement method to be fully functional, some modernization of the final algorithm for determining the coordinates of the corners of a building should be performed. Such a procedure called ***vector translation*** has been proposed by the author in the research that shows a detailed process of implementing the innovative solution and presents the conducted empirical proof of the confirmation of its implementation. This article is a continuation of the thesis to increase the reliability of the determined coordinates of corners of a building in the RTN GNSS mode using the algorithm called ***vector translation*** in the indirect method of *line-line intersection.*

In order to present a complete quality control for the proposed solutions, the internal and external reliability should be analyzed according to the Delft theory. This method was developed by Teunissen P., and derives its name from the university, where it was created and developed. The internal reliability is characterized by its ability to find a bias in the results of observations, and is represented by the MDB (*Minimal Detectable Bias*), i.e the value of observation error which can be detected by means of a statistical test. In the case of the external reliability, this is the influence of undetected errors on the final results of the calculations. Results of the research carried out by (Blesch, 2006), however, lead to a conclusion that the MDB does not say anything about the minimum error that can be detected by studying the observations according to the said theory. (Uznański, 2008) presents the conclusions of his research in a similar way. The main objective of his study was to provide a practical interpretation of the results of the tests on the reliability of geodetic networks, determined with GPS receivers by the Delft method. The research experiment intentionally included disturbances in observations of geodetic networks, which were used to analyze the response of the mathematical and stochastic models in relation to the final calculations, i.e. the adjusted coordinates. The final conclusions of research (Uznański, 2008) assessed the Delft theory to be a good tool for calculations and analyses of observations. Nevertheless, this method cannot provide absolute assurance as to the accuracy of the obtained results.

In accordance with the Delft theory, for the RTN GNSS measurements carried out in real time, there is an assumption that observation adjustments are random variables with normal distribution and the standard deviation is known. In this situation, observations from the real-time measurements cannot be regarded as values which are completely deterministic. The main reason for the occurrence of random errors is the limitations of the performed measurements and the accuracy of their implementation. However, if the measurement in the RTN GNSS mode is performed on each point with a double and independent initialization of the receiver and with the simultaneous measurement of at least 30 observation epochs, as well as the control points are measured according to MIA (2011), then the final measurement results (X, Y coordinates) in the RTN GNSS mode may be regarded as deterministic values. Although this is only an assumption, it is commonly implemented in surveying and it meets the MIA (2011) accuracy requirements. However, in order to be able to make that assumption, first of all, outliers must be detected in the RTN GNSS measurements. For this purpose, the Delft theory may be used, or other methods based on the adjustment model with the parameters estimated by the method of least squares. These methods include: Baarda, Pope, Cross-Price, Ding-Coleman, coexistence levels and Ethrog (Prószyński and Kwaśniak, 2002). Having detected the outliers, they should be eliminated prior to the next process of using the RTN GNSS measurement results.

The issues related to statistical analysis of observations and the results of calculations have already been described in (Krzyżek, 2015) and (Krzyżek, 2014a). Due to the limited volume of the article and the previous results of the author's experiments, as well as basing on the conclusions drawn from (Blesch, 2006) and (Uznański, 2008) research studies, this article focuses only on proving the above-mentioned thesis in the form of a mathematical analysis of the subject algorithm.

2. Mathematical analysis of the algorithm in the modernized method for measurements of building structures

2.1 Characteristics of the algorithm of vector translation

In order to carry out the mathematical analysis of the scheme of ***vector translation***, it is necessary to remind the basic assumptions of this innovative solution. The main idea of the discussed algorithm is to adopt theoretical horizontal angles of a building as right angles. The resulting differences in the angular values are used to calculate the new X, Y coordinates of the corners by the ***half-angle*** method, (Krzyżek, 2015), in the main and counter directions, to obtain vector lengths dL_i (Fig. 2 and 3) between the newly calculated points and the values of the point coordinates obtained from the classical method of *line-line intersection*. Therefore, it can be concluded that the ***half-angle*** method is one of the components of the ***vector translation*** model. For each newly calculated corner of the building, independently for the main direction (Nr') and for the counter direction (Nr'') translation is performed by the length of the vector dL_i obtained from the previous point to give, once again, a new position of the corner: for the main direction (Nr'T) and for the counter direction (Nr''T). In further steps, the subsequent location of some corners of the building is determined, as points of intersection of the straight lines of the wall faces, determined from the newly calculated points from the main (Nr'T) and counter (Nr”T) directions, and so taking into account translation by the length of the vector dL_i. Then, again, horizontal angles of the building are calculated on all newly determined points and, based on the obtained deviations (relative to the right angles), the weights are calculated to the newly calculated coordinates of the points. Having calculated the weighted averages of the coordinates of the corners, their position is adjusted using the method of *vector addition* (the method studied in detail by the author)

2.2 The first adjustment of the horizontal angles of the building

To prove the validity of the assumptions regarding the algorithm of ***vector translation***, a hypothetical drawing of an L-shaped building structure, often met in the field, was used. To simplify the analysis, the building was located at the origin of the X, Y coordinates (Fig. 1). Theoretical coordinate values were adopted, corresponding to the full meters (5m or 10m) of the wall length. Figure 1 illustrates the position of the object assumed as error-free (blue) and the building resulting from the classical method of *line-line intersection* (red).

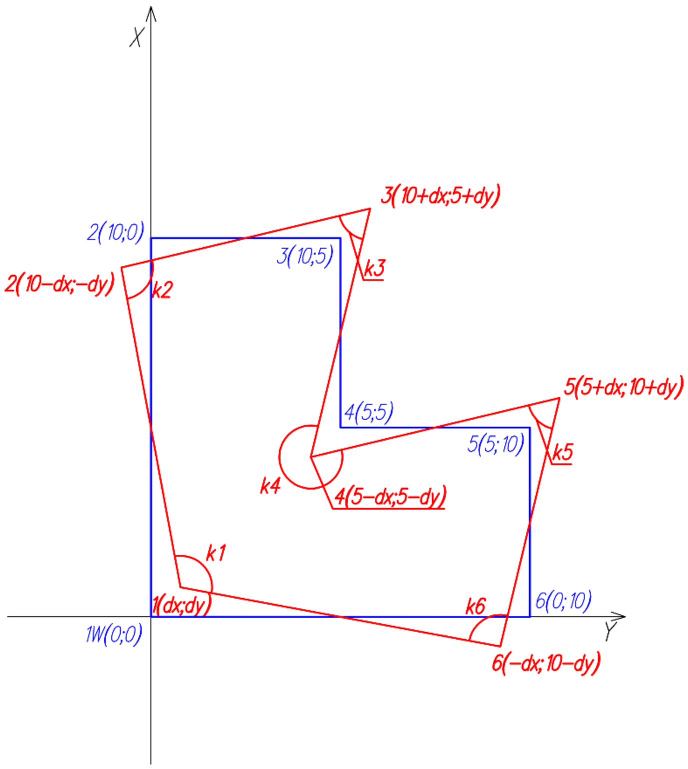


Figure 1: Example of the relationship between the position of a theoretical structure and after RTN GNSS measurement using the classical method of *line-line intersection*.

The first step of the mathematical analysis is to calculate the horizontal angles k_1-k_6 of the building, based on the coordinates (from the difference in azimuths) of the points (corners of the building) measured from the *line-line intersection*. Already at this stage, another assumption was adopted regarding identical values of displacements of coordinate components dx and dy between the theoretical horizontal angles and those captured from the measurement – these displacements were denoted as dX parameter and they refer to the whole construction.

$$k_1 = arctg\left(\frac{dX-5}{dX}\right) - arctg\left(\frac{dX}{dX-5}\right) + 180^0 \quad (1)$$

$$k_2 = arctg\left(\frac{dX}{dX-5}\right) - arctg\left(\frac{2dX+5}{2dX}\right) + 180^0 \quad (2)$$

$$k_3 = arctg\left(\frac{2dX+5}{2dX}\right) - arctg\left(\frac{-2dX}{-2dX-5}\right) \quad (3)$$

$$k_4 = arctg\left(\frac{-2dX}{-2dX-5}\right) - arctg\left(\frac{2dX+5}{2dX}\right) + 360^0 \quad (4)$$

$$k_5 = arctg\left(\frac{2dX+5}{2dX}\right) - arctg\left(\frac{-2dX}{-2dX-5}\right) \quad (5)$$

$$k_6 = arctg\left(\frac{-2dX}{-2dX-5}\right) - arctg\left(\frac{dX-5}{dX}\right) + 360^0 \quad (6)$$

For further analysis, to the formulas (1) – (6), auxiliary designations were introduced:

$$m = \left(\frac{dX-5}{dX}\right)$$

$$n = \left(\frac{dX}{dX-5}\right)$$

$$t = \left(\frac{2dX+5}{2dX}\right)$$

$$s = \left(\frac{-2dX}{-2dX-5}\right)$$

The next step is to determine the adjustments v_{ki} to the horizontal angles k_1-k_6 (Krzyżek, 2015):

$$v_{k1} = \frac{arctg(n)-arctg(m)-90^0}{2} \quad (7)$$

$$v_{k2} = \frac{arctg(t)-arctg(n)-90^0}{2} \quad (8)$$

$$v_{k3} = \frac{arctg(s)-arctg(t)+90^0}{2} \quad (9)$$

$$v_{k4} = \frac{arctg(t)-arctg(s)-90^0}{2} \quad (10)$$

$$v_{k5} = \frac{arctg(s)-arctg(t)+90^0}{2} \quad (11)$$

$$v_{k6} = \frac{arctg(m)-arctg(s)+90^0}{2} \quad (12)$$

2.3 Determining the position of corners of the building in the main and counter directions

The next step after the adjustment of horizontal angles (calculating the adjustments v_{ki} is calculating X, Y coordinates of corners of buildings in the main direction $X(Y)_{Nr'}$ (Fig. 2 and 3-7) and in the counter direction $X(Y)_{Nr''}$ (Fig. 8 and 9-13). In the formulas presented below (13) –

(22), the designation of $R_{i\text{-}j}$ was adopted as a tie distance of the building measured directly in the field.

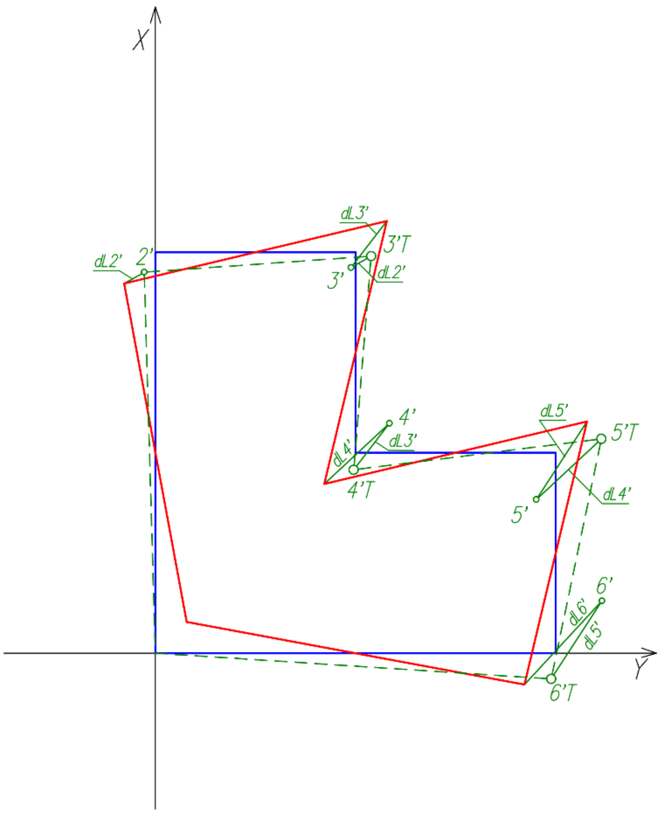


Figure 2: Exemplary presentation of the position of corners of a building determined in the main direction.

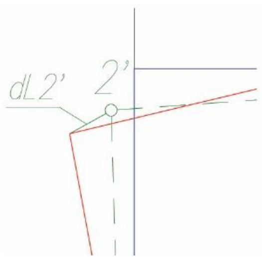


Figure 3: Corner No. 2

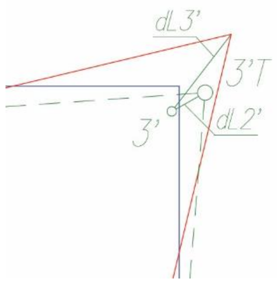


Figure 4: Corner No. 3

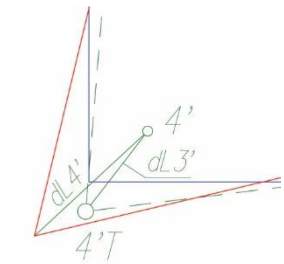


Figure 5: Corner No. 4

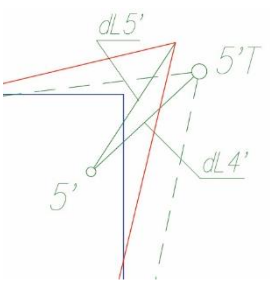


Figure 6: Corner No. 5

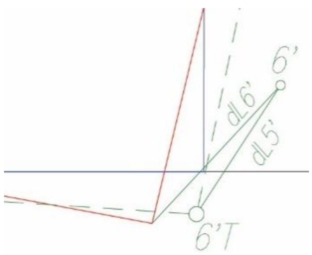


Figure 7: Corner No. 6

- **main direction (Fig. 2 and 3-7)**

Point 2'

$$X_{2'} = X_1 + R_{1-2} \cdot cos(A_{1-2} + v_{k1}) = dX + 10 \cdot cos\left(\frac{3arctg(n) - arctg(m) + 630^0}{2}\right)$$
$$Y_{2'} = Y_1 + R_{1-2} \cdot sin(A_{1-2} + v_{k1}) = dX + 10 \cdot sin\left(\frac{3arctg(n) - arctg(m) + 630^0}{2}\right) \quad (13)$$

Point 3'

$$X_{3'} = X_2 + R_{2-3} \cdot cos(A_{2-3} + v_{k2}) = 10 - dX + 5 \cdot cos\left(\frac{3arctg(t) - arctg(n) - 90^0}{2}\right)$$
$$Y_{3'} = Y_2 + R_{2-3} \cdot sin(A_{2-3} + v_{k2}) = -dX + 5 \cdot sin\left(\frac{3arctg(t) - arctg(n) - 90^0}{2}\right) \quad (14)$$

Point 4'

$$X_{4'} = X_3 + R_{3-4} \cdot cos(A_{3-4} - v_{k3}) = 10 + dX + 5 \cdot cos\left(\frac{arctg(s) + arctg(t) + 270^0}{2}\right)$$
$$Y_{4'} = Y_3 + R_{3-4} \cdot sin(A_{3-4} - v_{k3}) = 5 + dX + 5 \cdot sin\left(\frac{arctg(s) + arctg(t) + 270^0}{2}\right) \quad (15)$$

Point 5'

$$X_{5'} = X_4 + R_{4-5} \cdot cos(A_{4-5} + v_{k4}) = 5 - dX + 5 \cdot cos\left(\frac{3arctg(t) - arctg(s) - 90^0}{2}\right)$$
$$Y_{5'} = Y_4 + R_{4-5} \cdot sin(A_{4-5} + v_{k4}) = 5 - dX + 5 \cdot sin\left(\frac{3arctg(t) - arctg(s) - 90^0}{2}\right) \quad (16)$$

Point 6'

$$X_{6'} = X_5 + R_{5-6} \cdot cos(A_{5-6} - v_{k5}) = 5 + dX + 5 \cdot cos\left(\frac{arctg(s) + arctg(t) + 270^0}{2}\right)$$
$$Y_{6'} = Y_5 + R_{5-6} \cdot sin(A_{5-6} - v_{k5}) = 10 + dX + 5 \cdot sin\left(\frac{arctg(s) + arctg(t) + 270^0}{2}\right) \quad (17)$$

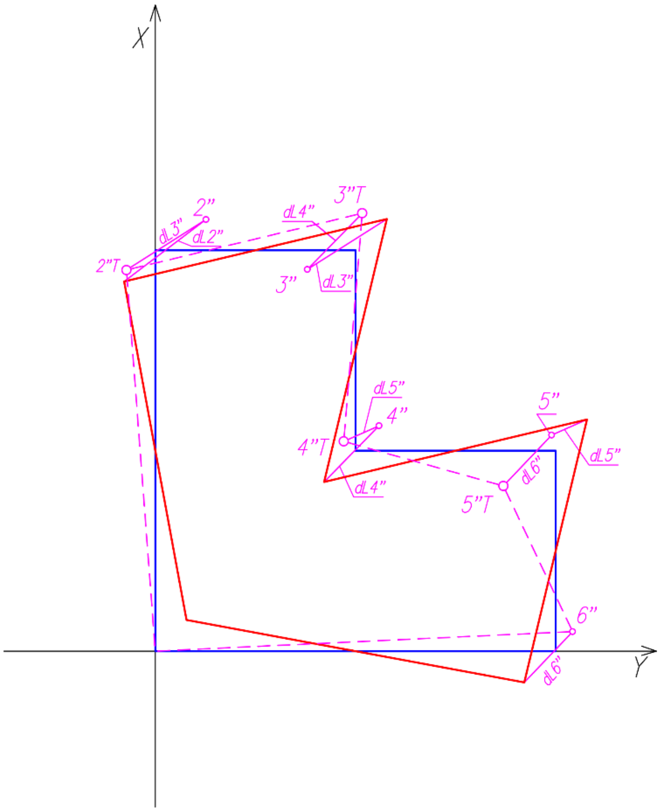
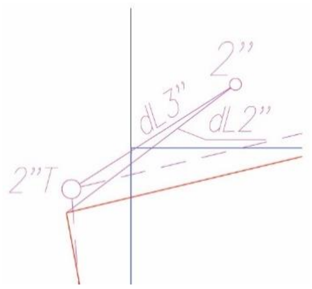
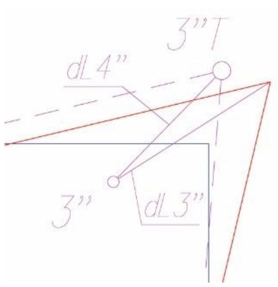


Figure 8: Exemplary presentation of the position of corners of a building determined in the counter direction.





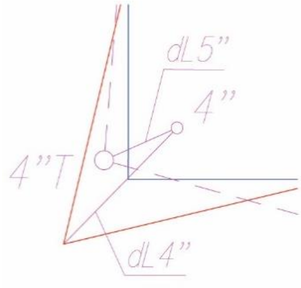
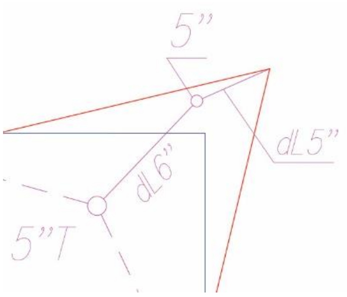


Figure 9: Corner No. 2 **Figure 10:** Corner No. 3 **Figure 11:** Corner No. 4



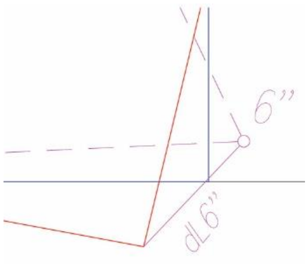


Figure 12: Corner No. 5 **Figure 13:** Corner No. 6

- **counter direction (Fig. 8 and 9-13)**

Point 6''

$$X_{6''} = X_1 + R_{1-6} \cdot cos(A_{6-1} + 180^0 - v_{k1}) = dX + 10 \cdot cos\left(\frac{3arctg(m) - arctg(n) + 1170^0}{2}\right)$$

$$Y_{6''} = Y_1 + R_{1-6} \cdot sin(A_{6-1} + 180^0 - v_{k1}) = dX + 10 \cdot sin\left(\frac{3arctg(m) - arctg(n) + 1170^0}{2}\right) \quad (18)$$

Point 5''

$$X_{5''} = X_6 + R_{6-5} \cdot cos(A_{5-6} + 180^0 - v_{k6}) = -dX + 5 \cdot cos\left(\frac{3arctg(s) - arctg(m) + 630^0}{2}\right)$$

$$Y_{5''} = Y_6 + R_{6-5} \cdot sin(A_{5-6} + 180^0 - v_{k6}) = 10 - dX + 5 \cdot sin\left(\frac{3arctg(s) - arctg(m) + 630^0}{2}\right) \quad (19)$$

Point 4''

$$X_{4''} = X_5 + R_{5-4} \cdot cos(A_{4-5} + 180^0 + v_{k5}) = 5 + dX + 5 \cdot cos\left(\frac{arctg(s) + arctg(t) + 450^0}{2}\right)$$

$$Y_{4''} = Y_5 + R_{5-4} \cdot sin(A_{4-5} + 180^0 + v_{k5}) = 10 + dX + 5 \cdot sin\left(\frac{arctg(s) + arctg(t) + 450^0}{2}\right) \quad (20)$$

Point 3''

$$X_{3''} = X_4 + R_{4-3} \cdot cos(A_{3-4} + 180^0 - v_{k4}) = 5 - dX + 5 \cdot cos\left(\frac{3arctg(s) - arctg(t) + 810^0}{2}\right)$$

$$Y_{3''} = Y_4 + R_{4-3} \cdot sin(A_{3-4} + 180^0 - v_{k4}) = 5 - dX + 5 \cdot sin\left(\frac{3arctg(s) - arctg(t) + 810^0}{2}\right) \quad (21)$$

Point 2''

$$X_{2''} = X_3 + R_{3-2} \cdot cos(A_{2-3} + 180^0 + v_{k3}) = 10 + dX + 5 \cdot cos\left(\frac{arctg(s) + arctg(t) + 450^0}{2}\right)$$

$$Y_{2''} = Y_3 + R_{3-2} \cdot sin(A_{2-3} + 180^0 + v_{k3}) = 5 + dX + 5 \cdot sin\left(\frac{arctg(s) + arctg(t) + 450^0}{2}\right) \quad (22)$$

To the formulas above (13) – (22), for the points in the main and counter directions, the following auxiliary designations were introduced:

$$\left(\frac{3arctg(n) - arctg(m) + 630^0}{2}\right) = (A)$$

$$\left(\frac{3arctg(t) - arctg(n) - 90^0}{2}\right) = (B)$$

$$\left(\frac{arctg(s) + arctg(t) + 270^0}{2}\right) = (C)$$

$$\left(\frac{3arctg(t) - arctg(s) - 90^0}{2}\right) = (D)$$

$$\left(\frac{3arctg(m) - arctg(n) + 1170^0}{2}\right) = (E)$$

$$\left(\frac{3arctg(s) - arctg(m) + 630^0}{2}\right) = (F)$$

$$\left(\frac{arctg(s) + arctg(t) + 450^0}{2}\right) = (G)$$

$$\left(\frac{3arctg(s) - arctg(t) + 810^0}{2}\right) = (H)$$

Additional designations *(A)* - *(H)* were used in the next step, consisting in using newly created vectors dL_i in subsequent calculations to obtain simplified formulas (23) - (30) for the next position of the corners, denoted as $X(Y)_{Nr'T}$ in the main direction (Fig. 2 and 3-7) and $X(Y)_{Nr''T}$ in the counter direction (Fig. 8 and 9-13):

- **main direction (Fig. 2 and 4-7):**

In the main direction, the position of the point 2'T is not determined, because the vector of the point which was previous to them, or the fixed point *1W*, has been adopted at zero.

Point 3'T

$$X_{3'T} = X_{3'} + \Delta x_{2-2'} = dX + 5cos(B) + 10cos(A)$$
$$Y_{3'T} = Y_{3'} + \Delta y_{2-2'} = dX + 5sin(B) + 10sin(A) \tag{23}$$

Point 4'T

$$X_{4'T} = X_{4'} + \Delta x_{3-3'} = -dX + 10 + 5cos(C) + 5cos(B)$$
$$Y_{4'T} = Y_{4'} + \Delta y_{3-3'} = -dX + 5sin(C) + 5sin(B) \tag{24}$$

Point 5'T

$$X_{5'T} = X_{5'} + \Delta x_{4-4'} = dX + 10 + 5cos(D) + 5cos(C)$$
$$Y_{5'T} = Y_{5'} + \Delta y_{4-4'} = dX + 5 + 5sin(D) + 5sin(C) \tag{25}$$

Point 6'T

$$X_{6'T} = X_{6'} + \Delta x_{5-5'} = -dX + 5 + 5cos(C) + 5cos(D)$$
$$Y_{6'T} = Y_{6'} + \Delta y_{5-5'} = -dX + 5 + 5sin(C) + 5sin(D) \tag{26}$$

- **counter direction (Fig. 8 and 9-12):**

In the counter direction, the position of the point 6"T is not determined, because the vector of the point which was previous to them, or the fixed point *1W,* has been adopted at zero.

Point 5’’T

$$X_{5''T} = X_{5''} + \Delta x_{6-6''} = dX + 5cos(F) + 10cos(E)$$
$$Y_{5''T} = Y_{5''} + \Delta y_{6-6''} = dX + 5sin(F) + 10sin(E) \quad (27)$$

Point 4’’T

$$X_{4''T} = X_{4''} + \Delta x_{5-5''} = -dX + 5cos(G) + 5cos(F)$$
$$Y_{4''T} = Y_{4''} + \Delta y_{5-5''} = -dX + 10 + 5sin(G) + 5sin(F) \quad (28)$$

Point 3’’T

$$X_{3''T} = X_{3''} + \Delta x_{4-4''} = dX + 5 + 5cos(H) + 5cos(G)$$
$$Y_{3''T} = Y_{3''} + \Delta y_{4-4''} = dX + 10 + 5sin(H) + 5sin(G) \quad (29)$$

Point 2’’T

$$X_{2''T} = X_{2''} + \Delta x_{3-3''} = -dX + 5 + 5cos(G) + 5cos(H)$$
$$Y_{2''T} = Y_{2''} + \Delta y_{3-3''} = -dX + 5 + 5sin(G) + 5sin(H) \quad (30)$$

2.4 The second adjustment of the horizontal angles of the building

Due to the large volume of evidence material, the algorithm was further analyzed with respect to a single point only - the point No. 2. For the remaining points No. 3, 5, 6, the analysis would be carried out in a manner identical to the proof described below. For point No. 4, however, an additional element of calculation would be the point of intersection of the lines of the wall faces. This attribute introduces an additional element of the formula to calculate the X and Y coordinates, which is weighed according to the same principles as other points.

Therefore, the calculations of the horizontal angles were performed again for the main direction k', k'_T and for the counter direction k'', k''_T.

- **main direction (Fig. 2 and 3)**

Point 2’

$$k_{2'} = A_{1W-2'} - A_{2'-3'T} + 180^0 \quad (31)$$

where:

$$A_{1W-2'} = arctg\left(\frac{dX + 10sin(A)}{dX + 10cos(A)}\right) + 360^0$$

$$A_{2'-3'T} = (B)$$

therefore:

$$k_{2'} = arctg\left(\frac{dX+10sin(A)}{dX+10cos(A)}\right) - (B) + 180^0 \quad (32)$$

- **counter direction (Fig. 8 and 9):**

Point 2''T

$$k_{2''T} = A_{1W-2''T} - A_{2''T-3''T} + 180^0 \quad (33)$$

where:

$$A_{1W-2''T} = arctg\left(\frac{-dX + 5 + 5sin(G) + 5sin(H)}{-dX + 5 + 5cos(G) + 5cos(H)}\right) + 360^0$$

$$A_{2''T-3''T} = arctg(t)$$

therefore:

$$k_{2''T} = arctg\left(\frac{-dX+5sin(H)}{-dX+5+5cos(H)}\right) - arctg(t) + 180^0 \quad (34)$$

2.5 Determining the weighted average of the position of a corner of a building

Having determined the horizontal angles again, the weights pi representing half of the angular deviation value should be calculated:

Point 2'

$$p_{2'} = \frac{1}{f_{k2'}^2} \quad (35)$$

where:

$$f_{k2'} = \frac{90^0 - k_{2'}}{2} = \frac{(B) - arctg\left(\frac{dX + 10sin(A)}{dX + 10cos(A)}\right) - 90^0}{2}$$

therefore:

$$p_{2'} = \frac{4}{\left[(B) - arctg\left(\frac{dX+10sin(A)}{dX+10cos(A)}\right) - 90^0\right]^2} \quad (36)$$

Point 2''T

$$p_{2''T} = \frac{1}{f_{k2''T}^2} \quad (37)$$

where:

$$f_{k2''T} = \frac{90^0 - k_{2''T}}{2} = \frac{arctg(t) - arctg\left(\frac{-dX+5sin(H)}{-dX+5+5cos(H)}\right) - 90^0}{2}$$

therefore:

$$p_{2'} = \frac{4}{\left[arctg(t) - arctg\left(\frac{-dX+5sin(H)}{-dX+5+5cos(H)}\right) - 90^0\right]^2} \tag{38}$$

With the calculated weights for a twice-determined corner No. 2, the weighted average of the Cartesian coordinates X and Y was calculated from the following formula:

$$X_{2W} = \frac{\frac{4(dX+10\cdot cosA)}{\left[B - arctg\left(\frac{dX+10\cdot sinA}{dX+10\cdot cosA}\right) - 90\right]^2} + \frac{4(-dX+5+5\cdot cosG+5\cdot cosH)}{\left[arctg(t) - arctg\left(\frac{-dX+5\cdot sinH}{-dX+5\cdot cosH}\right) - 90\right]^2}}{\frac{4}{\left[B - arctg\left(\frac{dX+10\cdot sinA}{dX+10\cdot cosA}\right) - 90\right]^2} + \frac{4}{\left[arctg(t) - arctg\left(\frac{-dX+5\cdot sinH}{-dX+5\cdot cosH}\right) - 90\right]^2}}$$

(39)

$$Y_{2W} = \frac{\frac{4(dX+10\cdot sinA)}{\left[B - arctg\left(\frac{dX+10\cdot sinA}{dX+10\cdot cosA}\right) - 90\right]^2} + \frac{4(-dX+5+5\cdot sinG+5\cdot sinH)}{\left[arctg(t) - arctg\left(\frac{-dX+5\cdot sinH}{-dX+5\cdot cosH}\right) - 90\right]^2}}{\frac{4}{\left[B - arctg\left(\frac{dX+10\cdot sinA}{dX+10\cdot cosA}\right) - 90\right]^2} + \frac{4}{\left[arctg(t) - arctg\left(\frac{-dX+5\cdot sinH}{-dX+5\cdot cosH}\right) - 90\right]^2}}$$

Taking into consideration the first adjusted coordinate X, which is denoted by X_W, and adopting the following denotations:

$$a = \left[B - arctg\left(\frac{dX+10\cdot sinA}{dX+10\cdot cosA}\right) - 90\right]^2 \tag{40}$$

$$b = \left[arctg(t) - arctg\left(\frac{-dX+5\cdot sinH}{-dX+5\cdot cosH}\right) - 90\right]^2 \tag{41}$$

we obtain:

$$X_W = \frac{\frac{4(dX+10\cdot cosA)}{a} + \frac{4(-dX+5+5\cdot cosG+5\cdot cosH)}{b}}{\frac{4}{a}+\frac{4}{b}} \tag{42}$$

Reducing the expression to a common denominator and making simple transformations, the following expression was obtained:

$$X_W = \frac{b(dX+10\cdot cosA)+a(-dX+5+5\cdot cosG+5\cdot cosH)}{a+b} \tag{43}$$

and then, after substituting the relationships (40) and (41) to the formula (43), the following formula was obtained:

$$X_W = \frac{b}{a+b}\cdot(dX+10\cdot cosA) + \frac{a}{a+b}\cdot(-dX+5+5\cdot cosG+5\cdot cosH) \tag{44}$$

which, after transformations takes the form:

$$X_W = dX\left(\frac{b-a}{a+b}\right) + \frac{b}{a+b}10\cdot cosA + \frac{a}{a+b}(5+5\cdot cosG+5\cdot cosH) \tag{45}$$

Then, after the introduction of additional denotations:

$$q = \left(\frac{b-a}{a+b}\right) \tag{46}$$

$$r = \frac{b}{a+b} 10 \cdot cosA + \frac{a}{a+b}(5 + 5 \cdot cosG + 5 \cdot cosH) \quad (47)$$

where:

$a > 0, \quad b > 0, \quad q < 1$

the following formula is obtained:

$$X_W = dX \cdot q + r \quad (48)$$

2.6 Substantiating the assumptions of the algorithm of vector translation

The next step is to prove that the absolute value of the difference between the coordinates of the theoretical (error-free) value X_T and the adjusted value X_W is less than the absolute value of the difference between the coordinates of the theoretical (error-free) value X_T and the measured value X_P (the value X_P is determined from the *line-line intersection* used in RTN GNSS technology). Thus, the following inequality must be proved:

$$|X_W - X_T| < |X_P - X_T| \quad (49)$$

Substituting the assumptions regarding the coordinates defined in Fig. 1, and the relationship (48), to the equation (49), we obtain:

$$|qdX + r - 10| < |-dX| \quad (50)$$

At this point, it should be noted that the entire analysis is based on a hypothetical Figure 1, for which it was assumed that $dX \geq 0$. Therefore, further steps include transformation of the equation (50) to the formula:

$$|q| \left|\left(dX - \frac{10-r}{q}\right)\right| < dX/:|q| \quad (51)$$

and then bringing it to the form:

$$\left|\left(dX - \frac{10-r}{q}\right)\right| < \frac{dX}{|q|} \quad (52)$$

If:

$$\left(dX - \frac{10-r}{q}\right) \geq 0$$

the inequality (52) takes the form

$$dX > \frac{10-r}{q} \cdot \frac{|q|}{|q|-1} \quad (53)$$

the solution of which depends on the parameter *q*:

- for $q < 0$

$$dX > \frac{10-r}{-|q|+1} \quad (54)$$

- for $q > 0$

$$dX > \frac{10-r}{|q|-1} \tag{55}$$

Given that $q = \frac{b-a}{a+b}$, it is necessary to assume two possible relations of the parameters *a* and *b*:

- **option 1:**

$a > b$ for $q < 0$ and

- **option 2:**

$b > a$ for $q > 0$.

The first option should definitely be rejected, as such a case in the field is rather unreal. To come up with such a relationship, parameter *a* must be greater than parameter *b*, which in turn would force an error of setting out a point on the line of the building wall at the level of approximately 5 meters. For example, even for a setting out error at the level of 2m, parameter *a* is much smaller than parameter *b*.

On the other hand, in the second option, the value of the denominator in the formula (55) is always negative. Therefore, while proving the inequality (55), it is sufficient to demonstrate that:

$$10 - r > 0 \tag{56}$$

Therefore, substituting the relationship (47) for *r*, it is necessary to prove that:

$$10 - \frac{b}{a+b} 10 \cdot cosA - \frac{a}{a+b}(5 + 5 \cdot cosG + 5 \cdot cosH) > 0 \tag{57}$$

If we assume all the parameters of the formula as error-free (theoretical), namely:

$A=H=0\ [^0]$

$G=270\ [^0]$

$\sqrt{a} = \sqrt{b} = 0[^0]$

then the formula (57) takes the following form:

$$10 > 0 \tag{58}$$

and thus the inequality (57) would be proven. However, if it is known that in fact these parameters are subject to errors, the inequality (57) is as follows:

- parameters *A* and *H* are adjusted azimuths of the sides, respectively *1W-2* and *4-3* and thus of the values close to $0[^0]$, that is $cosA \cong cosH \cong 0.9 \ldots$,

- parameter *G* is the adjusted azimuth of the side *5-4* and thus of the value close to $270[^0]$, or $cosG \cong 0$ (for this analysis $cosG = 0$)

- parameters $\frac{b}{a+b}$ and $\frac{a}{a+b}$ will always be a fractional value with interdependence, that is, when one of them will be close to *1*, the second will be around *0*. Most frequently, parameter $\frac{b}{a+b}$ will be equivalent to the values going to *1* and the parameter $\frac{a}{a+b}$ will oscillate around the value of *0*.

By adopting such assumptions, the components of the formula (57) will be as follows:

- $\left(\frac{b}{a+b} 10 \cdot cosA\right) < 10, np. 9.23 \ldots$

- $\left[\frac{a}{a+b}(5+5\cdot cosG+5\cdot cosH)\right]\cong 0.01\ ...$

Accordingly, the parameter *r*:

$$\frac{b}{a+b}10\cdot cosA+\frac{a}{a+b}(5+5\cdot cosG+5\cdot cosH)<10 \tag{59}$$

which, in turn, proves the validity of the formula (56) that $10-r>0$.

In order to prove the inequality (52), the case when $\left(dX-\frac{10-r}{q}\right)<0$ must still be considered. However, looking closely at this inequality, it can be concluded that for small values of the parameter dX, of a few or a dozen or so centimeters, this inequality does not occur. Only with large values of the setting-out error, at the level of dX = 0.90m, this inequality is satisfied. But such inaccuracy of fitting a point (the corner of the building) on the line of the face of the building wall with a length of 10 m is very unlikely, and even unrealistic and it was given only as a theoretical value, which should be detected and eliminated as an outlier (by the Delft or other method) at the initial stage of preparing the results.

The proof carried out for the coordinate Y of the same point has been performed in the same way as for the parameter X. The results of the analysis performed for Y lead to the same conclusions as for the X coordinate, which have been presented above.

Due to the large volume of evidence material, the article does not include proving the formulas for the X, Y coordinates of the point No. 4, where there is yet another location of the corner of a building with the additional weight, resulting from the intersection of the wall faces. These additional parameters have a far more beneficial effect on the reliability of the determined coordinates, and thus they further confirm the presented mathematical analysis.

The last step in the method of ***vector translation*** is the use of the method of *vector addition*.

3. Summary and conclusions

The main aim of the performed mathematical analysis was to confirm the results of the research studies that the author has conducted so far and which have been proven empirically in (Krzyżek, 2015). It referred to three innovative methods – ***vector addition, half-angle and vector translation*** - leading to more accurate determination of coordinates of the corners of buildings captured from measurements in the RTN GNSS mode using indirect measurement method of *line-line intersection*. The ***half-angle method*** is a component of the method of ***vector translation***, which brought the mathematical analysis to the last-mentioned method. The total volume of the evidence material that would be used for a general analysis of the formulas is so extensive that it was necessary to make a number of assumptions, or simplifications, which would not disqualify the validity of the presented arguments at the same time. The most important argument to prove was the assumption that using the method of *vector translation* in measurements of buildings in the RTN GNSS mode will result in the absolute value of the difference between the coordinates of the theoretical (error-free) value X_T and the adjusted value X_W being less than the absolute value of the difference between the coordinates of the theoretical (error-free) value X_T and the measured value X_P. The value of the coordinate X_P has been determined from the method of *line-line intersection* used in RTN GNSS technology for measurements of buildings. Basing on the research currently being carried out by the author, it may be assumed that failure to use the method of ***vector translation***, or using a different method, for example *vector addition* only, will not always prove the thesis formulated above.

In the method of *line-line intersection*, the most important factor of accuracy is precision of setting out of the base point on the extension of the wall face. A very detailed accuracy analysis of the indirect measurement methods used in real time measurements have been presented in (Beluch and Krzyżek, 2005). The authors analyzed geometric structures in which a significant factor was taking into account the error of setting out the base point or the corner of a building. Basing on the research (Beluch and Krzyżek, 2005) and the research experiments performed by the author, it may be concluded that in practical implementations of setting out base points on the extension of the wall faces of a building, the most common errors which occur, are at the level of a few or a dozen or so centimeters. The analysis of this article has proven that even for the values of about one meter (description of version 1 for the formula 46), the innovative solution called ***vector translation*** significantly increases the reliability of the determined coordinates of the corners of a building. An important implementation argument of these innovative solutions is to create a proper application, which would be fully compatible with GNSS receivers from different manufacturers. Achieving this goal would definitely facilitate surveying work, while contributing to the balancing of the quality of measurement results captured from classical measurements (tacheometry) of building s, and those performed in the RTN GNSS mode.

ACKNOWLEDGEMENT

This work was carried out within the statutory studies of the AGH University of Science and Technology, Faculty of Mining Surveying and Environmental Engineering No. 11.11.150.006.

REFERENCES

Angrisano, A., Gaglione S., and Gioia C. “Performance assessment of GPS/GLONASS single point positioning in an urban environment.” *Acta Geodaetica et Geophysica* 48 (2013): 149-161. Accessed June, 2013. doi: 10.1007/s40328-012-0010-4.

Bakuła, M. “Study of Reliable Rapid and Ultrarapid Static GNSS Surveying for Determination of the Coordinates of Control Points in Obstructed Conditions.” *Journal of Surveying Engineering* 139(4) (2013): 188-193. Accessed November, 2013. doi: 10.1061/(ASCE)SU.1943-5428.0000109.

Beluch, J., and Krzyżek R. “GPS RTK technology used for indirect methods of topographical surveys.” *In Technical Sciences Supplement,* edited by Mieszkalski L., Cenkowski S., Chrzanowski A., Laskowski J., Tipałow W. N., and Wasilewski A., 47–60. Olsztyn: University of Warmia and Mazury, 2005.

Blesch, K. W. “Testing theory and the minimal detectable bias.” Journal of Spatial Science 51 (2006): 7-19. Accessed August 13, 2010. doi:10.1080/14498596.2006.9635061.

Krzyżek, R. "Reliability analysis of the results of RTN GNSS surveys of building structures using indirect methods of measurement." *Geodesy and Cartography* 63(2) (2014a): 161-181. Accessed December, 2014. doi: 10.2478/geocart-2014-0012.

Krzyżek, R., Adamczyk T., Bieda A., Dąbrowski J., Dąbrowska T., Florek R., Grynishak M., Hanus P., Jasińska E., Kwartnik-Pruc A., Preweda E., Skulich M., Szafarczyk A., Markiewicz B., and Florek-Paszkowski R. "Analysis and verification of the accuracy of RTK GPS technology in the implementation of control points in detailed measurements." *In Chosen issues of geodetic science,* edited by Dąbrowski J., 87-106. Jarosław: 2014b.

Krzyżek, R. "Modernization of the method of line-line intersection using RTN GNSS technology for determining the position of corners of buildings." *Artyficial Satellites Journal of Planetary Geodesy* 50(1) (2015): 41-57. Accessed March, 2015. doi: 10.1515/arsa-2015-0004.

MIA. "Regulation of Minister of Interior and Administration - in case of technical standards of performing detailed surveys and working out and sending results of these surveys to National Geodetic and Cartographic Database (in Polish)." *Journal of Laws* 263(entry 1572). Warsaw: Government Legislation Centre, 2011.

Naus, K., and Nowak A. "Time Network corrections availability on the Gulf of Gdansk area during GNSS/RTN measurements, in the context of ship's movement parameters determination." *Logistyka – nauka* 6 (2014): 7913-7922.

Pelc-Mieczkowska, R. "Analysis of GPS/RTK positioning reliability in hard observational conditions." *ZeszytyNaukowe Politechniki Rzeszowskiej* 59(1/II) (2012): 217-226.

Pirti, A., Yucel M., and Gumus K. "Testing Real Time Kinematis GNSS (GPS and GPS/GLONASS) methods in obstructed and unobstructed sites." *Geodetski Vestnik* 57(3) (2013): 498-512. Accessed 2013. doi: 10.15292/geodetski-vestnik.2013.03.498-512.

Prószyński, W., and Kwaśniak S. *Reliability of geodetic network*. Warsaw: Warsaw University of Technology, 2002. oai:polona.pl:4671016.

Uznański, A. "Quality Control of Geodetic Networks at Leica Geo Office." Geomatics and *Environmental Engineering* 2(1) (2008): 77-84.

Uznański, A. "Analysis of RTN Measurement Results Referring to ASG-EUPOS Network." *Geomatics and Environmental Engineering* 4(1/1) (2010): 153-161.

Recebido em Maio de 2015.

Aceito em Agosto de 2015.