THE FAST FOURIER TRANSFORM AND ITS APPLICATION TO TIDAL OSCILLATIONS *

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SYNOPSIS

This paper proposes a new way of tidal spectral analysis based on the Cooley-Tukey algorithm, known as the Fast Fourier Transform. The Fast Fourier Transform analysis is used to compute both the harmonic constants of the tide and the power spectrum.The latter is obtained by means of a weighted sum. A new way is also derived to obtain the formula giving the number of the degrees of freedom,on which is based the confidence interval corresponding to the noise spectrum.

1 - INTRODUCTION

Old methods of tidal analysis were developed in order to permit manual calculations with desk calculators. Such analyses can now be considerably improved with the use of electronic computers.

Computers have also facilitated the use of time series analysis of mean sea level fluctuations as influenced by tidal oscillations.

Horn's (1960) least square method and the very similar one developed by Cartwright and Catton (1963), based on discrete Fourier analysis, are among the earliest methods of tidal analysis entirely dependent upon electronic computers. Both of these

* This is the reprint with minor corrections and improved programs. The former reprint is obsolete.

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methods included approximately the same number of constituents used in Doodson's and other methods of analysis (61 constituents), for which computer programs have been prepared. However Doodson himself (1957) recognized that many more constituents would be necessary to improve predictions, as is observed in his method of computing corrections for predictions based on the classical list of constituents. The corrections were found through analysis of the differences in time and height of recorded and predicted curves. All these methods are lengthy (30 to 60 minutes computer time) and not diagnostic.

A decade ago spectral analysis came into common usage as a diagnostic tool but only quite recently found an application in tidal analysis. Subsequently, it was confirmed that many more shallow water constituents were necessary to represent tidal curves accurately. This was Zetler and Cummings'(1967) conclusion from a study on the port of Anchorage (Alaska), and Lennon and Rossiter's conclusion from the port of London (Lennon,1969). Both researches showed, independently, that some 50 additional shallow water constituents were necessary to improve predictions.

Munk and Cartwright (1966) provided a completely different approach to the problem considering the "response" of a tidal basin to the driving forces of the equilibrium tide. In addition, shallow water terms are included to take into account the non-linear response of the tidal basin to the principal constituents. The number of such terms, required for an accurate prediction is very much smaller than the ones necessary in the harmonic prediction method. The method, a generalization of the old Laplace method (Franco, 1967), is a neat and thorough approach to tidal prediction. Munk and Cartwright give no quantitative estimation of the computational effort involved; but it is thought to be considerable.

If the harmonic method is preferred, accurate predictions are conditioned by the inclusion of shallow water constituents, which differ from port to port. Unfortunately, work already done shows that results are individual and cannot be extended or transferred from area to area. An example can be found in the comparison between the results from Zetler-Cummings (op.cit.) and Lennon-Rossiter (Lennon, 1969) researches and the figures calculated by Rock (personal communication). However, the diagnosis of the shallow water effects may be considerably simplified if a method is devised to permit the full use of the Fast Fourier Transform (FFT) (Cooley and Tukey, 1965). The procedure for spectral analysis can be completely changed if means are available to correct, according to the natural angular frequencies of the tidal harmonics, the various Fourier coefficients obtained from the FFT. To show this is one of the aims of this paper.

To give an idea of the advantages of the method, let us examine a flow diagram of the operations involved (Fig. 1). One of the main characteristics of the method is the possibility of using the Fourier coefficients to obtain the power spectrum and the harmonic constants. One can use the tidal values of $a_j = R_j \cos r_j$; and $b_j = R_j \sin r_j$ computed from the Fourier coefficients to correct these coefficients for the tidal effect, and thus isolate the noise contribution to the Fourier series. This is carried out in the tidal frequency bands. The power spectrum obtained through these corrected Fourier coefficients will then be the noise spectrum plus the tidal oscillations not considered in the harmonic analysis.

Although the use of the Cooley-Tukey algorithm (FFT), as a means of simplifying calculations in spectral analysis, was suggested by Zetler (1969), the authors are not aware of any publication on its use as applied to tides. Thus we believe that our method is original.

In order to compare the proposed method with Lennon-Rossiter's a flow diagram is presented for the latter (Fig. 2). The procedure involves a Doodson harmonic analysis, a prediction of hourly heights which is in itself a long task even for electronic computers, and a very lengthy high resolution Fourier analysis for a whole number of lunations. Besides the length of the initial calculation, an additional drawback is that the whole procedure must be repeated to obtain the final spectrum.

2 - FOURIER ANALYSIS*

Classical Fourier analysis is one of the methods for determining tidal harmonic constants. In addition to the above

* See list of mathematical symbols annexed.

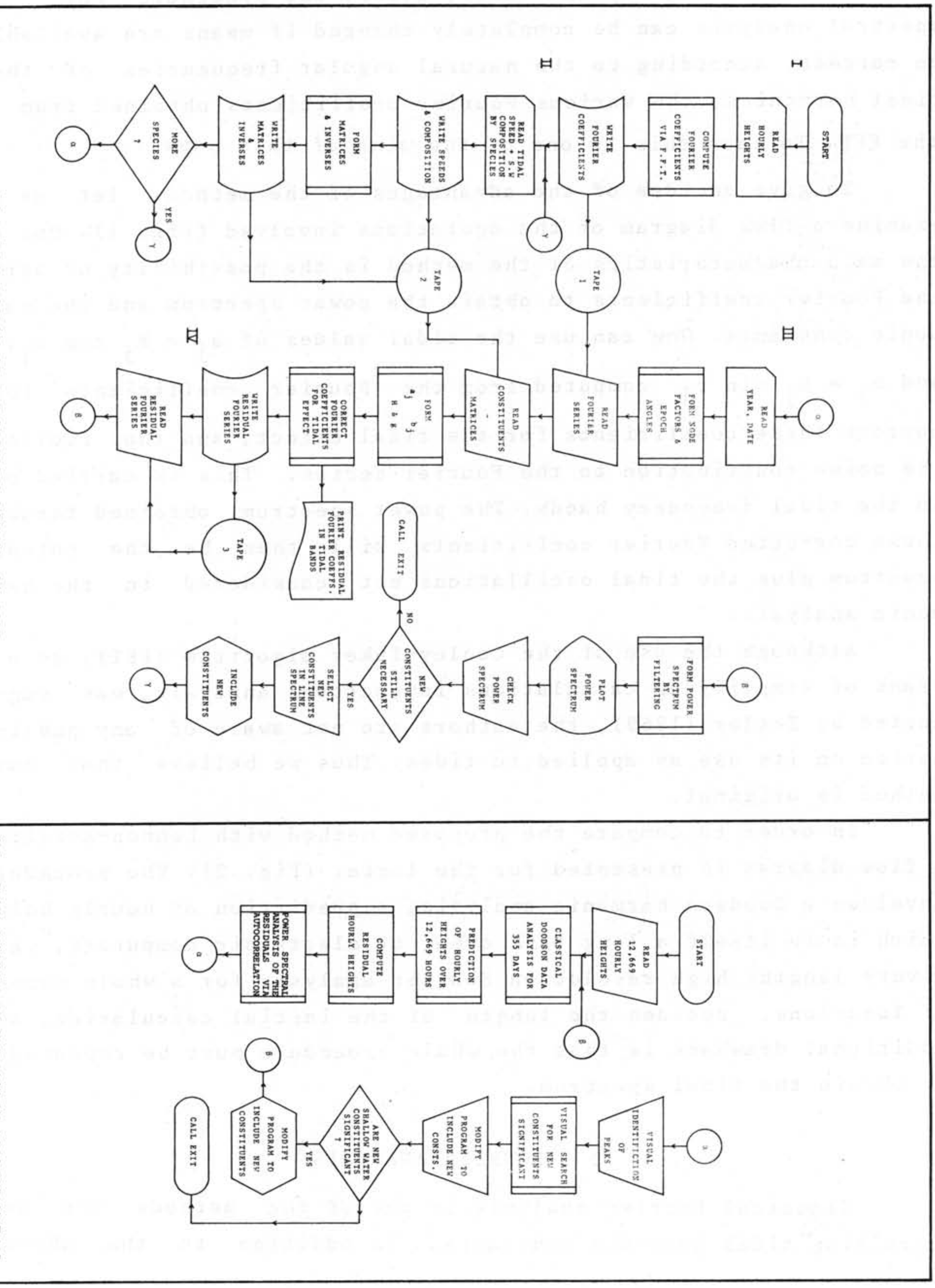

FIG I-FLOW DIAGRAM OF THE PROPOSED METHOD-ISO CODE

FIG 2-FLOW DIAGRAM OF THE LENNON-ROSSITER METHOD mentioned Catton-Cartwright method, the Miyasaki (1958) method is well known. However, these studies were attached to a whole number of lunations and their analysis cannot take advantage of the Cooley-Tukey algorithm based on sampling at $N = 2^{\gamma}$ points. To show how the harmonic analysis of a tidal curve can be undertaken via the Cooley-Tukey algorithm is the aim of this section.

Suppose that the tidal height at instant *t* is given by

$$y(t) = R_o + \sum_{j=1}^{Q} R_j \cos (q_j t - r_j) + v(t)$$

where v(t) is a "gaussian noise with *zero mean*". In complex notation we have

$$y(t) = R_o + \frac{1}{2} \sum_{j=1}^{Q} R_j \left[e^{i(q_j t - r_j)} + e^{-i(q_j t - r_j)} \right] + v(t)$$

$$= R_o + \frac{1}{2} \sum_{j=1}^{Q} \left[R_j e^{-ir_j} e^{iq_j t} + R_j e^{ir_j} e^{-iq_j t} \right] + v(t)$$

Now if we admit that r_j and q_j are negative for negative values of j, and that $r_j = q_j = 0$ for j=0 it can be written

$$y(t) = \frac{1}{2} R_o + \frac{1}{2} \sum_{j=-Q}^{Q} R_j e^{-ir_j} e^{iq_j t} + v(t)$$

Finally, if we put

$$\begin{cases} \frac{1}{2} R_j e^{-ir_j} = c_j & j \neq 0 \\ R_j = c_j & j = 0 \end{cases} \qquad (2a)$$

it results

$$y(t) = \sum_{j=-Q}^{Q} c_j e^{iq_j t} + v(t) \qquad (2b)$$

If we analyse this curve by using the Fourier technique, it is not possible to adopt the exact angular frequencies q_j. All the dis - crete angular frequencies of the Fourier analysis are given by

$$q_n = 2\pi n / N\tau \qquad (n = 0,1,2, \ldots \frac{N}{2} - 1) \qquad (2c)$$

where τ is the sampling interval (usually 1 hour). For the exact number of cycles in the same time interval corresponding to the angular frequencies q_j, we have

$$F_j = N\tau q_j / 2\pi \qquad (2d)$$

Now the complex Fourier coefficients for curve (2b) are

$$c_n = \frac{1}{N} \sum_{t=0}^{N-1} y(t)\, e^{-iq_n t} \qquad (n = 0,1,2, \ldots . N-1)$$

But if we call ε_n the complex Fourier coefficient obtained from the analysis of v(t), it follows that

$$\varepsilon_n = \frac{1}{N} \sum_{t=0}^{N-1} v(t)\, e^{-iq_n t} \tag{2e}$$

Then we can write

$$c_n - \varepsilon_n = \frac{1}{N} \sum_{t=0}^{N-1} \left[y(t) - v(t) \right] e^{-iq_n t} \tag{2f}$$

However expression (2c) shows that, for τ=1 hour

$$q_n N = 2\pi n$$

hence

$$e^{-iq_n N} = e^{-i2\pi n} = 1$$

Thus (2f) can be modified as follows:

$$c_n - \varepsilon_n = \frac{1}{N} \sum_{t=0}^{N} \left[y(t) - v(t) \right] e^{-iq_n t} - \frac{1}{N}\left[y(N) - v(N) \right]$$

But v(N) is usually small as compared to N, consequently v(N)/N can be neglected. In addition we can replace y(t) - v(t) by its value taken from (2b). Consequently, the last expression can be changed into

$$c_n + \frac{1}{N} y\,(N) - \varepsilon_n = \frac{1}{N} \sum_{j=-Q}^{Q} c_j \sum_{t=0}^{N} e^{-i(q_n - q_j)t} \tag{2g}$$

Now

$$\sum_{t=0}^{N} e^{i\omega t} = \frac{e^{i\omega(N+1)} - 1}{e^{i\omega} - 1} = \frac{e^{i\omega(N+1)/2}}{e^{i\omega/2}} \cdot \frac{e^{i\omega(N+1)/2} - e^{-i\omega(N+1)/2}}{e^{i\omega/2} - e^{-i\omega/2}}$$

but, since

$$e^{ix} - e^{-ix} = 2i \sin x$$

and

$$\omega = -(q_n - q_j)$$

it results

$$\sum_{t=0}^{N} e^{-i(q_n - q_j)} = e^{-i(q_n - q_j)N/2} \frac{\sin\left[(q_n - q_j)(N+1)/2\right]}{\sin\left[(q_n - q_j)/2\right]}$$

But

$$e^{-i(q_n-q_j)N/2} = e^{-iq_n N/2} \cdot e^{iq_j N/2}$$

or, according to (2c), for τ=1 hour,

$$e^{-i(q_n-q_j)N/2} = e^{-i\pi n} \cdot e^{iq_j N/2} = (-1)^n e^{iq_j N/2}$$

Thus

$$\sum_{t=0}^{N} e^{-i(q_n-q)t} = e^{iq_j N/2} \frac{\sin\left[(q_n-q_j)\ (N+1)/2\right]}{\sin\left[(q_n-q_j)/2\right]} (-1)^n$$

Hence expression (2g) can be changed into

$$c_n + \frac{1}{N} y(N) - \varepsilon_n = \frac{1}{N} \sum_{j=-Q}^{Q} c_j \ e^{iq_j N/2} \frac{\sin\left[(q_n-q_j)\ (N+1)/2\right]}{\sin\left[(q_n-q_j)/2\right]} (-1)^n$$

If only positive values of j are considered, according to (2a) we have

$$c_n + \frac{1}{N} y(N) - \varepsilon_n = \frac{c_o}{N} + \sum_{j=1}^{Q} \left\{ c_j e^{iq_j N/2} \frac{\sin\left[(q_n-q_j)\ (N+1)/2\right]}{N \sin\left[(q_n-q_j)/2\right]} \right.$$

$$\left. + \ c_{-j} e^{-iq_j N/2} \frac{\sin\left[(q_n+q_j)\ (N+1)/2\right]}{N \sin\left[(q_n+q_j)/2\right]} \right\} (-1)^n \qquad (2h)$$

Consequently, if we call

$$(-1)^n \frac{\sin\left[(q_n-q_j)\ (N+1)/2\right]}{N \sin\left[(q_n-q_j)/2\right]} = A_{nj} \qquad (2i)$$

and

$$(-1)^n \frac{\sin\left[(q_n+q_j)\ (N+1)/2\right]}{N \sin\left[(q_n+q_j)/2\right]} = B_{nj} \qquad (2j)$$

and recall that $r_j<0$ for $j<0$, then we have from (2a) and (2h) to (2j)

$$c_n + \frac{1}{N}\left[y(N) - c_o\right] - \varepsilon_n = \frac{1}{2}\sum_{j=1}^{Q} R_j\left[e^{-i(r_j - q_j N/2)} A_{nj} + e^{i(r_j - q_j N/2)} B_{nj}\right] \quad (2k)$$

Now it is well known that the pair of trigonometric Fourier coefficients (a_n, b_n), for the total oscillation, and (ξ_n, η_n) for the noise oscillation, are related to the complex Fourier coefficients, respectively by

$$c_n = (a_n - ib_n)/2 \quad , \quad c_o = a_o$$

and (2l)

$$\varepsilon_n = (\xi_n - i\eta_m)/2 \ , \ \varepsilon_o = 0$$

Thus we can develop the exponentials of (2k) and write

$$a_n + \frac{2}{N}\left[y(N) - a_o\right] - \xi_n - i(b_n - \eta_n) = \sum_{j=1}^{Q}\left[R_j \cos (r_j - q_j N/2)(A_{nj} + B_{nj}) - iR_j \sin (r_j - q_j N/2)\ (A_{nj} - B_{nj})\right]$$

Now, equating the real and the imaginary parts and considering the N/2 values of a_n and b_n obtained from the Fourier analysis, the following independent systems can be written:

$$\{a_n + \frac{2}{N}\left[y(N) - a_o\right] - \xi_n\} = ||A_{nj} + B_{nj}||\{a_j\} \quad (2m)$$

and

$$\{b_n - \eta_n\} = ||A_{nj} - B_{nj}||\{b_j\} \quad (2n)$$

where

$$a_j = R_j \cos (r_j - q_j N/2)$$

and (2o)

$$b_j = R_j \sin (r_j - q_j N/2)$$

Since $Q<N/2$ the systems (2m) and (2n) are redundant and can be solved by the least square method according to the conditions

$$\Sigma\ \xi_n^2 = \text{minimum}$$

and

$$\Sigma\ \eta_n^2 = \text{minimum}$$

respectively. But this can be simplified because constituents of different species practically do not contaminate each other. Consequently, the systems (2m) and (2n) may be split into sub-systems, one pair for each species. The frequency band containing the diurnal constituents, for example, is limited by $290 \leq n \leq 380$ cycles per 8192 hours (2^{13}), and therefore, about 90 equations exist for computing about 20 values of both a_j and b_j. These redundant systems are of the form:

$$M\{X\} = \{L\}$$

The corresponding normal equations are

$$M^T M\{X\} = M^T\{L\} \qquad (2p)$$

where T indicates transposition of matrix M. These systems can be solved by inverting the square matrix $M^T M$ which gives $(M^T M)^{-1}$ and pre-multiplying both members of (2p) by $(M^T M)^{-1}$:

$$\{X\} = (M^T M)^{-1} M^T\{L\}$$

If unknowns are to be determined directly in terms of L then the matrix

$$(M^T M)^{-1} M^T = M'$$

must be found in order to give

$$\{X\} = M'\{L\} \qquad (2q)$$

It must be pointed out that the separation of unknowns a_j and b_j into two independent systems results from adding the correction $\frac{2}{N}\left[y(N)-a_o\right]$ to all values of a_n. Although no fixed central time has been established before hand, it is interesting to note that expressions (2o) contain the phase correction $q_j N/2$, which is an adjustment of r_j to the central time N/2.

In order to give a clear idea of the results to be expected with an analysis of 8192 samples, the inverse of the normal matrices $(M^T M)^{-1}$ is presented in Table 2-I, for the semidiurnal constituents. The dominant diagonal indicates that good results may be obtained.

Once known a_j and b_j, the Fourier coefficients ξ_n and η_n, corresponding to the noise, can be found by taking from (2m) and

TABLE 2-I

Inverse Matrix for Computing $R_j \cos(q_j N/2 - r_j)$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27
	0,997	0,026	0,008	-0,013	-0,006	0,001	-0,006	0,012	0,009	0,006	-0,003	0,005	0,007	-0,007	-0,006	-0,000	0,002	0,003	-0,004	0,005	-0,005	0,002	0,000	0,002	0,005	-0,003	-0,002
		0,993	-0,176	0,034	0,025	0,015	-0,002	0,014	-0,012	-0,011	-0,002	-0,003	-0,006	-0,008	0,008	0,003	0,000	-0,002	0,003	-0,004	0,006	-0,004	-0,002	-0,001	0,006	0,006	0,004
1	1,006		0,994	0,018	0,023	0,022	-0,011	-0,001	-0,004	-0,006	-0,006	0,003	0,001	-0,001	0,003	0,004	-0,002	-0,002	0,002	0,001	0,000	0,003	-0,002	0,001	-0,002	0,003	0,002
2	-0,034	1,045		0,999	-0,068	-0,061	0,022	0,014	-0,004	0,001	0,009	-0,007	-0,006	-0,004	-0,001	-0,004	0,004	-0,004	0,004	-0,003	0,003	0,002	0,002	-0,002	-0,001	-0,001	-0,001
3	-0,035	0,188	1,042		1,000	-0,067	0,022	0,024	-0,010	-0,005	0,008	-0,008	-0,009	-0,008	0,005	-0,002	0,003	-0,004	0,005	-0,005	0,005	-0,001	0,001	-0,002	-0,004	0,002	0,000
4	0,014	-0,043	-0,030	1,014		1,000	0,016	0,032	-0,017	-0,012	0,004	-0,008	-0,010	-0,010	0,008	0,001	0,002	-0,004	0,005	-0,006	0,006	-0,003	-0,000	-0,002	-0,005	0,004	0,002
5	0,008	-0,034	-0,033	0,076	1,014		0,998	-0,172	0,032	0,026	0,002	0,006	0,011	0,013	-0,011	-0,004	-0,001	0,003	-0,004	0,006	-0,007	-0,005	0,002	0,001	0,007	-0,006	-0,004
6	0,001	0,023	-0,030	0,070	0,077	1,013		0,996	0,018	0,024	0,022	-0,008	-0,002	0,004	-0,007	-0,008	0,004	-0,003	0,002	-0,000	0,001	0,005	0,004	-0,002	0,004	-0,005	-0,004
7	0,008	0,002	0,014	-0,030	-0,032	-0,028	1,038		0,998	-0,069	-0,088	0,022	0,014	0,005	0,002	0,010	-0,007	0,006	-0,005	0,005	-0,003	-0,004	-0,005	0,003	-0,000	0,003	0,003
8	0,014	-0,013	0,001	-0,023	-0,033	-0,040	0,183	1,040		1,000	-0,168	0,029	0,023	0,015	-0,004	0,008	-0,007	0,008	-0,008	0,007	-0,006	-0,002	-0,004	0,004	0,003	0,001	0,002
9	-0,010	0,015	0,007	0,005	0,012	0,019	-0,041	-0,032	1,022		0,999	0,017	0,024	0,026	-0,016	-0,005	-0,002	0,004	-0,006	0,008	-0,009	-0,005	0,002	0,002	0,008	-0,006	-0,004
10	-0,007	0,015	0,010	-0,002	0,006	0,013	-0,036	-0,038	0,093	1,043		0,998	-0,071	-0,065	0,026	0,016	-0,003	-0,000	0,003	-0,006	0,008	-0,008	-0,005	-0,000	-0,008	0,008	0,006
11	0,001	0,007	0,010	-0,010	-0,007	-0,002	-0,014	-0,031	0,107	0,187	1,046		0,997	-0,071	0,025	0,023	-0,008	0,005	-0,002	-0,001	0,004	-0,009	-0,007	0,002	-0,006	0,008	0,006
12	-0,006	0,003	-0,004	0,010	0,011	0,010	-0,006	0,007	-0,028	-0,039	-0,032	1,018		0,997	0,018	0,029	-0,013	0,010	-0,008	0,005	-0,002	-0,009	-0,008	0,004	-0,003	0,007	0,006
13	-0,008	0,007	-0,001	0,009	0,012	0,013	-0,013	-0,000	-0,020	-0,033	-0,037	0,082	1,020		0,998	-0,168	0,028	-0,025	0,022	-0,018	0,014	0,008	0,010	-0,007	-0,001	-0,004	-0,005
14	-0,009	0,009	0,002	0,007	0,010	0,013	-0,016	-0,006	-0,011	-0,024	-0,036	0,076	0,083	1,018		1,000	0,015	-0,020	0,023	-0,024	0,025	-0,006	0,000	-0,005	-0,010	0,006	0,003
15	0,007	-0,010	-0,005	-0,002	-0,006	-0,009	0,005	0,011	-0,003	0,006	0,020	-0,035	-0,036	-0,031	1,039		0,999	0,071	-0,070	0,018	-0,065	0,017	0,007	0,002	0,014	-0,010	-0,007
16	0,002	-0,007	-0,006	0,003	0,001	-0,002	0,009	0,012	-0,011	-0,007	0,006	-0,025	-0,033	-0,039	0,181	1,037		0,998	0,072	-0,071	0,009	-0,021	-0,011	-0,000	-0,014	0,012	0,008
17	0,003	-0,000	0,003	-0,006	-0,005	-0,004	0,001	-0,005	0,010	0,011	0,006	0,004	0,011	0,018	-0,042	-0,033	1,031		0,997	0,073	0,072	0,025	0,015	-0,002	0,013	-0,012	-0,009
18	-0,005	0,002	-0,002	0,006	0,006	0,006	-0,003	0,003	-0,010	-0,012	-0,009	-0,000	-0,007	-0,004	0,040	0,037	-0,097	1,033		0,996	0,074	-0,028	-0,019	0,005	-0,011	0,013	0,010
19	0,006	-0,004	0,001	-0,006	-0,007	-0,007	0,005	-0,002	0,009	0,013	0,011	-0,004	0,003	0,011	-0,037	-0,030	0,096	-0,099	1,035		0,996	0,031	0,023	-0,007	-0,009	-0,012	-0,010
20	-0,006	0,005	-0,000	0,006	0,008	0,008	-0,007	-0,000	-0,008	-0,012	-0,013	0,007	0,000	-0,008	0,032	0,040	-0,094	0,098	-0,101	1,036		1,000	-0,007	0,022	0,024	-0,005	0,000
21	0,007	-0,006	-0,001	-0,005	-0,008	-0,009	0,009	0,002	0,007	0,012	0,014	-0,010	-0,004	0,014	-0,028	-0,039	0,091	-0,096	0,099	-0,101	1,035		1,000	0,016	0,032	-0,012	-0,006
22	-0,003	0,006	0,004	-0,000	0,002	0,004	-0,007	-0,007	0,003	-0,000	-0,007	0,011	0,011	0,010	-0,006	0,007	-0,027	0,032	-0,036	0,040	-0,043	1,011		0,997	-0,174	0,029	0,018
23	-0,001	0,003	0,003	-0,002	0,000	0,001	-0,004	-0,005	0,004	0,003	-0,002	0,007	0,008	0,009	-0,009	-0,000	-0,014	0,018	-0,023	0,027	-0,031	0,073	1,009		0,993	0,029	0,026
24	-0,003	0,002	-0,001	0,003	0,004	0,004	-0,012	0,001	-0,004	-0,005	-0,005	0,001	-0,001	-0,004	0,009	0,009	-0,005	0,003	-0,000	0,003	0,006	-0,029	-0,025	1,038		0,993	-0,070
25	-0,006	0,007	0,003	0,003	0,006	0,007	-0,009	-0,005	-0,002	-0,006	-0,010	0,010	0,007	0,003	0,006	0,014	-0,020	0,019	-0,018	0,016	-0,013	-0,029	-0,038	0,185	1,046		1,000
26	-0,005	-0,008	-0,014	-0,000	-0,003	-0,006	0,008	0,007	-0,002	0,001	0,008	-0,010	-0,000	0,008	0,002	-0,008	0,007	-0,018	0,019	-0,019	0,019	0,005	0,014	-0,038	-0,040	1,017	
27	0,003	-0,005	-0,014	0,000	-0,012	-0,003	0,006	0,006	-0,003	-0,000	0,005	-0,008	-0,008	-0,007	0,003	-0,004	0,012	-0,013	0,014	-0,015	0,015	-0,000	0,007	-0,026	-0,034	0,074	1,008

Inverse Matrix for Computing $R_j \sin(q_j N/2 - r_j)$

1	2MNS2	4	OQ2	7	2N2	10	N2	13	M2	16	L2	19	S2	22	MSN2	25	2MS2N2
2	2NS2	5	MNS2	8	(MU2)	11	(NU2)	14	MSK2	17	2SK2	20	R2	23	KJ2	26	2SN2
3	3M2S2	6	2MK2S2	9	SNK2	12	OP2	15	(LAM2)	18	T2	21	K2	24	2SM2	27	SKN2

(2n), respectively

$$\{\xi_n\} = \| A_{nj} + B_{nj} \| \{a_j\} - \{a_n + \frac{2}{N}\left[y(N) - a_o\right]\} \quad (2r)$$

and

$$\{\eta_n\} = \| A_{nj} - B_{nj} \| \{b_j\} - \{b_n\} \quad (2s)$$

Appendices II and III, respectively, contain the computer programs to solve systems (2q) and compute the residuals ξ_n and η_n.

3 - SPECTRAL ANALYSIS

Let us take a *normalized* oscillation expressed by

$$x(t) = \sum_{j=-Q}^{Q} c_j e^{iq_j t} \quad (3a)$$

where

$$\begin{cases} c_j = \frac{1}{2} R_j e^{-ir_j} & \text{for } j \neq 0 \\ c_j = 0 & \text{for } j = 0 \end{cases} \quad (3b)$$

Suppose that another normalized oscillation has harmonic terms with the same angular frequencies but with different amplitudes and phases. Such an oscillation can be expressed by

$$y(t) = \sum_{k=-Q}^{Q} c'_k \; e^{iq_k t} \quad (3c)$$

where

$$\begin{cases} c'_k = \frac{1}{2} R'_k e^{-ir'_k} & \text{for } k \neq 0 \\ c'_k = 0 & \text{for } k = 0 \end{cases} \quad (3d)$$

For a time $(t-\Theta)$ where Θ is any time *lag* counted from t, formula (3c) can be changed into

$$y(t-\Theta) = \sum_{k=-Q}^{Q} c'_k \; e^{iq_k(t-\Theta)}$$

The product of this expression by (3a) gives

$$x(t)y(t-\Theta) = \sum_{j=-Q}^{Q} \sum_{k=-Q}^{Q} c_j c'_k \; e^{-iq_k\Theta} \; e^{i(q_j+q_k)t}$$

or, if we call

$$c_j c'_k e^{iq_k\Theta} = c_{jk} \qquad (3e)$$

and

$$q_j + q_k = \omega_{jk} \qquad (3f)$$

$$x(t)y(t-\Theta) = \sum_{j=-Q}^{Q} \sum_{k=-Q}^{Q} c_{jk} \; e^{i\omega_{jk}t}$$

The mean value of this product over $-T/2<t<T/2$, is

$$K(\Theta) = \langle x(t)y(t-\Theta)\rangle = \sum_{j=-Q}^{Q} \sum_{k=-Q}^{Q} c_{jk} \left[\frac{1}{T} \int_{-T/2}^{T/2} e^{i\omega_{jk}t} dt \right]$$

$$= \sum_{j=-Q}^{Q} \sum_{k=-Q}^{Q} c_{jk} \frac{e^{i\omega_{jk}T/2} - e^{-i\omega_{jk}T/2}}{i\omega_{jk}T}$$

$$= \sum_{j=-Q}^{Q} \sum_{k=-Q}^{Q} c_{jk} \frac{\sin \omega_{jk}T/2}{\omega_{jk}T/2}$$

Now, since $q_j = -q_{-j}$ we conclude from expression (3f) that $\omega_{jk}=0$ for $k=-j$. Consequently, since $(\sin x)/x=1$ for $x=0$, it follows that

$$K(\Theta) = \underbrace{\sum_{j=-Q}^{Q} c_{j(-j)}}_{k=-j} + \underbrace{\sum_{j=-Q}^{Q} \sum_{k=-Q}^{Q} c_{jk} \frac{\sin \omega_{jk}T/2}{\omega_{jk}T/2}}_{k\neq -j} \qquad (3g)$$

But the function $(\sin x)/x$ decays very quickly when x increases. Thus $(\sin \omega_{jk}T/2)/(\omega_{jk}T/2)$ will be small for large values of $\omega_{jk}T/2$. Hence, for the usual values of ω_{jk} the second term of the above expression is negligible when T is large. Consequently, if we take (3e) into account, we have as a good approximation of (3g)

$$K(\Theta) = \sum_{j=-Q}^{Q} c_j c'_{-j} \; e^{-iq_j\Theta}.$$

However, expressions (3b) and (3d) show that we can write:

$$\begin{cases} c_j c'_{-j} = \frac{1}{4} R_j R'_j \; e^{i(r'_j - r_j)} = \gamma_j & (j \neq 0) \\ c_j c'_{-j} = 0 & (j = 0) \end{cases} \tag{3h}$$

Thus

$$K(\Theta) = \sum_{j=-Q}^{Q} \gamma_j \; e^{-iq_j\Theta} \tag{3i}$$

From (3h) and (3i) we can derive the trigonometrical form of (3i):

$$K(\Theta) = \sum_{j=1}^{Q} \frac{1}{2} R_j R'_j \cos \; (q_j\theta + r'_j - r_j)$$

If a Fourier analysis exists for discrete values of x(t) and y(t) with N hourly heights, then, a good estimate of $K(\Theta)$ can be expressed by

$$\hat{K}(\Theta) = \sum_{n=0}^{N/2-1} \frac{1}{2} R_n R'_n \cos \; (q_n\Theta + r'_n - r_n)$$

or, in complex form

$$\hat{K}(\Theta) = \sum_{-N/2+1}^{N/2-1} \gamma_n \; e^{iq_n\Theta} \tag{3j}$$

where

$$\begin{cases} \gamma_n = \frac{1}{4} R_n R'_n \; e^{i(r'_n - r_n)} & (n \neq 0) \\ \gamma_n = 0 & (n = 0) \end{cases} \tag{3k}$$

If values of $K(\Theta)$ are known by averaging the products $x(t)y(t-\Theta)$ for continuous values of Θ, the Fourier analysis of $K(\Theta)$ will be the usual way of obtaining an *estimate* of amplitudes $\frac{1}{2} R_j R'_j$ and phases $(r'_j - r_j)$; this will give for $-m<\Theta<m$:

$$c_s = \frac{1}{2m} \int_{-m}^{m} K(\Theta) \; e^{-iq_s\Theta} \; d\Theta \tag{3l}$$

where

$$q_s = 2\pi s/2m = \pi s/m \tag{3m}$$

But according to (3j) a good approximation is given by

$$c''_s = \frac{1}{2m} \sum_{-N/2+1}^{N/2-1} \gamma_n \int_{-m}^{m} e^{-i(q_s-q_n)\hat{\Theta}} d\Theta$$

$$= \frac{1}{2m} \sum_{-N/2+1}^{N/2-1} \gamma_n \left[\frac{e^{-i(q_s-q_n)\Theta}}{-i(q_s-q_n)} \right]_{-m}^{m}$$

or

$$c''_s = \sum_{-N/2+1}^{N/2-1} \gamma_n \frac{\sin\left[(q_s-q_n)m\right]}{(q_s-q_n)m} \qquad (3n)$$

However, the Fourier analysis of x(t) and y(t) give, respectively,

$$\begin{cases} a_n = R_n \cos r_n \\ b_n = R_n \sin r_n \end{cases} \quad \text{and} \quad \begin{cases} a'_n = R'_n \cos r'_n \\ b'_n = R'_n \sin r'_n \end{cases}$$

Thus,

$$c_n = (a_n - ib_n)/2 = \frac{1}{2} R_n \, e^{-ir_n}$$

$$c'_n = (a'_n - ib'_n)/2 = \frac{1}{2} R'_n \, e^{-ir'_n}$$

and, if we designate the complex conjugate of c'_n by c'^*_n, it follows that

$$c_n c'^*_n = \frac{1}{4} R_n R'_n \, e^{i(r'_n - r_n)}$$

or, according to (3k)

$$c_n c'^*_n = \gamma_n$$

Thus

$$c''_s = \sum_{-N/2+1}^{N/2-1} c_n c'^*_n \frac{\sin\left[(q_s-q_n)m\right]}{(q_s-q_n)m} \qquad (3.o)$$

Since $c_{-n} = c^*_n$ we can avoid the summation through negative values of *n* by changing (3.o) into

$$c''_s = \sum_{s\ n=0}^{N/2-1} \left\{ c_n c'^*_n \frac{\sin\left[(q_s-q_n)m\right]}{(q_s-q_n)m} + c^*_n c'_n \frac{\sin\left[(q_s+q_n)m\right]}{(q_s+q_n)m} \right\}$$

Hence, the cross spectrum c''_s is a weighted sum of the values of $c_n c'^*_n$ and $c^*_n c'_n$, the maximum weight corresponding to $q_n = q_s$.

Since c_n and c'_n are the complex Fourier coefficients resulting from the analysis of N values of x(t) and y(t), respectively, and with sampling interval equal to τ the angular frequency will be $q_n = 2\pi n/N\tau$. Thus, according to (3m) we have:

$$q_s \pm q_n = 2\pi(sN\tau/2m \pm n)/N\tau$$

or, if we call

$$sN\tau/2m = p$$

so that *p* is an integer, then

$$q_s \pm q_p = 2\pi(p \pm n)/N\tau$$

Consequently, if we replace this value of $q_s \pm q_p$ in (3o) and put $c''_s = \hat{S}_{xy}(p)$, this will be the *cross spectrum estimate* centered at *p*:

$$\hat{S}_{xy}(p) = \sum_{n=0}^{N/2-1} \left\{ c_n c'_n{}^* \frac{\sin\left[2\pi(p-n)m/N\tau\right]}{2\pi(p-n)m/N\tau} + c_n{}^* c'_n \frac{\sin\left[2\pi(p+n)m/N\tau\right]}{2\pi(p+n)m/N\tau} \right\} \quad (3p)$$

Function $(\sin\pi x)/\pi x$ decays very quickly when $x>2$. Thus the limits of summation can be conveniently reduced according to the condition

$$(p-n)2m/N\tau = \pm 2$$

and

$$(p+n)2m/N\tau = 2$$

In the first case we derive

$$n = \begin{cases} n'' = p + N\tau/m \\ n' = p - N\tau/m \end{cases} \quad (3q)$$

and in the second

$$n = \begin{cases} n_1 = N\tau/m - p > 0 \\ 0 \end{cases} \quad (3r)$$

thus

$$\hat{S}_{xy}(p) = \left\{ \sum_{n=n'}^{n''} c_n c'^*_n \frac{\sin\left[\pi(p-n)2m/N\tau\right]}{\pi(p-n)2m/N\tau} + \sum_{n=0}^{n_1} c_n^* c'_n \frac{\sin\left[\pi(p+n)2m/N\tau\right]}{\pi(p+n)2m/N\tau} \right\} \quad (3s)$$

The term in (p+n) of this formula can be neglected for values of p greater than $N\tau/m$. Since the practical values of $N\tau/m$ are not usually high, then the second term must only be used for very low frequencies. Consequently it is possible to discuss the behaviour of only the (p-n) term. The graph of Fig. 3 represented by the dashed line is the curve of equation:

$$\Phi(p-n) = \frac{\sin\left[\pi(p-n)2m/N\tau\right]}{\pi(p-n)2m/N\tau}$$

for $2m/N\tau = 1$.

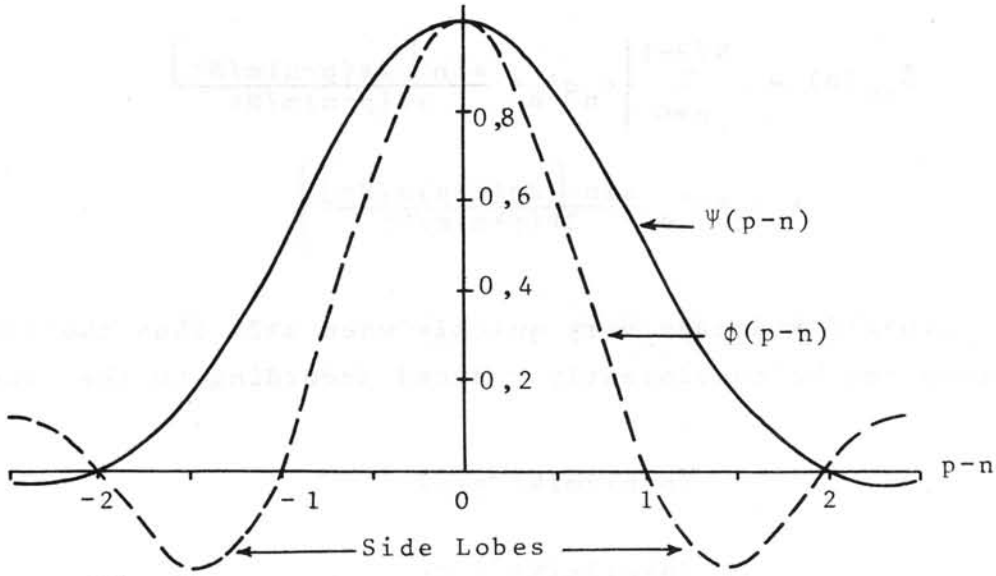

Fig.3 - Functions $\phi(p-n)$ and $\Psi(p-n)$ for continuous values of p-n.

The figure shows two undesirable side lobes. If we remember that $\hat{S}_{xy}(p)$ is only an *estimate* these side lobes can be eliminated by multiplying the cross correlation function $\hat{R}(\Theta)$ by the fading function $(1 + \cos \pi\Theta/m)$. Thus according to (3j) we have

$$\hat{K}(\Theta)(1 + \cos \pi\Theta/m) = \sum_{-N/2+1}^{N/2-1} \frac{1}{2} (2+e^{i\pi\Theta/m} + e^{-i\pi\Theta/m}) \gamma_n e^{iq_n\Theta}$$

Now we replace the value of $K(\Theta)$ in (3 1) by the second member of the above expression and integrate, in the same way we obtained (3n). Then we use the same arguments with which we arrived at (3s) and find

$$\hat{S}_{xy}(p) \simeq \left[\sum_{n=n'}^{n''} c_n c'^*_n \Psi(p-n) + \sum_{n=0}^{n_1} c_n^* c'_n \psi(p+n) \right] \qquad (3t)$$

where

$$\psi(p\pm n) = \frac{\sin \pi(p\pm n)2m/N\tau}{\left[\pi(p\pm n)2m/N\tau\right] \{1- \left[(p\pm n)2m/N\tau\right]^2\}} \qquad (3u)$$

This function is represented in Fig. 1 by the solid line. It is seen that this function is negligible for arguments greater than 2.

According to (3q) the "filter" $\psi(p\pm n)$ covers $2N\tau/m$ harmonics. It will be shown later that this quantity has an important statistical meaning.

The *power spectrum density* is obtained as a particular case of the cross spectrum. In fact if we establish the correlation for the same function, with a lag Θ we have from (3h) for $R'_j=R_j$ and $r'_j=r_j$

$$\gamma_j = \frac{1}{4} R_j^2 = c_j c_j^* = |c_j|^2$$

and expression (3i) takes the form

$$A(\Theta) = \langle y(t)y(t-\Theta)\rangle = \sum_{j=-Q}^{Q} \frac{1}{4} R_j^2 e^{-iq_j\Theta}$$

or, in trigonometric form,

$$A(\Theta) = \frac{1}{2} \sum_{j=0}^{Q} R_j^2 \cos q_j\Theta, \qquad (3v)$$

which is the *autocorrelation* function. Its Fourier analysis will give an estimate of the power spectral density $\frac{1}{2} R^2(p)$ centered in the harmonic of order p. Such an estimate can be derived immediately from (3u) by making $c_n = c'_n$, which gives

$$\hat{S}_{yy}(p) = \left\{ \sum_{n=n'}^{n''} |c_n|^2 \psi(p-m) + \sum_{n=0}^{n_1} |c_n|^2 \psi(p+n) \right\} \quad (3w)$$

where

$$|c_n|^2 = (a_n^2 + b_n^2)/4 \quad (3x)$$

A computer program for finding $\hat{S}_{yy}(p)$ is given in Appendix IV.

In order to establish the filter's maximum width and still avoid mixing different species of tides, the following method is suggested. Let q_s and q_{s+1}, respectively, be the largest angular frequency of species s and the smallest angular frequency of species s+1. Thus, if n_s and n_{s+1} are the respective frequencies in units of the fundamental frequency $1/N\tau$, then we can write

$$q_s = 360^o n_s / N\tau$$

$$q_{s+1} = 360^o n_{s+1} / N\tau$$

consequently

$$q_{s+1} - q_s = 360^o (n_{s+1} - n_s) / N\tau$$

and

$$n_{s+1} - n_s = (q_{s+1} - q_s) N\tau / 360^o$$

From an extended table of tidal constituents as derived by Zetler-Cummings or Lennon-Rossiter it can be concluded that $q_{s+1} - q_s$ is about 11^o which gives for the maximum filter width

$$n_{s+1} - n_s = 0{,}015 N\tau \quad (3y)$$

4 - THE NOISE SPECTRUM

It was shown in section 2 that ε_n is the Fourier complex coefficient of the noise analysis. Thus, according to (3w), the noise power spectrum is given by

$$\hat{S}_{vv}(p) = \left[\sum_{n=n'}^{n''} |\varepsilon_n|^2 \psi(p-n) + \sum_{n=0}^{n_1} |\varepsilon_n|^2 \psi(p+n) \right] \quad (4a)$$

where, according to (3w) and(21)

$$|\varepsilon_n|^2 = (\xi_n{}^2 + \eta_n{}^2)/4 \tag{4b}$$

Values of ξ_n and η_n can be found for the tidal frequency bands through (2r) and (2s). Outside these bands it can be assumed that coefficients a_n and b_n given the FFT do not have tidal contributions. Thus the values of $|\xi_n|^2$ to be introduced into (4a) are those found through (2r) and (2s) for the tidal frequency bands and $|c_n|^2$ given by the FFT, outside these bands. This procedure corresponds to the usual "prewhitening" which consists in obtaining the power spectrum of the residuals equal to the difference between the actual and predicted tides.

Since the *known* tidal effect has been eliminated before the energy density power spectrum has been determined, any spike of such a spectrum may be understood as the effect of a tidal constituent not included in the matrices.However, if no spike appears in the spectrum we can admit that only gaussian noise is present.

In order to establish the "confidence interval" of $\hat{S}_{vv}(p)$ when v(t) is a gaussian noise, we begin by simplifying (4a). In fact, the term in p+n only applies to very low frequencies and thus expressions (4a) and (4b) give

$$4\hat{S}_{vv}(p) = \sum_{n=n'}^{n''} (\xi_n^2 + \eta_n^2)\ \psi\ (p-n) \tag{4c}$$

Now, since v(t) is gaussian and it is linked to ξ_n and η_n through a linear equation, then ξ_n and η_n are also gaussian. Thus, it is reasonable to assume that the mean values of ξ_n^2 and η_n^2 are nearly equal to the same quantity, say, μ_o^2. Consequently, we have from (4c) the following approximation:

$$4\hat{S}_{vv}(p) \simeq \mu_o^2 \sum_{n=n'}^{n''} 2\psi\ (p-n)$$

or, if we call

$$2 \sum_{n=n'}^{n''} \psi\ (p-n) = \nu \tag{4d}$$

then

$$4\hat{S}_{vv}(p) \simeq \mu_o^2 \nu \tag{4e}$$

But a better approximation will be reached if we return from μ_o to the individual values of ξ_n^2 and η_n^2. In this case we must have $\nu/2$ values of ξ_n^2 and $\nu/2$ values of η_n^2, hence

$$4\hat{S}_{vv}(p) \simeq \sum_{n=p-\nu/2}^{p+\nu/2} (\xi_n^2 + \eta_n^2)$$

which is a chi-squared distribution with ν degrees of freedom. In order to find ν let us recall that the approximate area of the curve $\psi(p-n)$ is given by

$$\text{Area} = \sum_{n=n'}^{n''} \psi(p-n)\ \Delta n \tag{4f}$$

where $\Delta n=1$. But a close approximation of the curve of $\psi(z)$ between the limits shown in Fig.4 is

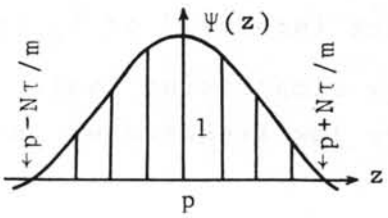

Fig.4 - Function $\Psi(z)$

$$\psi(z) = \tfrac{1}{2}\left[(1-\alpha^2 z^2)^3 + (1-\alpha^2 z^2)^2\right]$$

where

$$\alpha = m/N\tau \tag{4g}$$

Hence, since we assume that $\psi(z)=0$ for $|z|>N\tau/m$, the area of the curve is

$$\text{Area} = \int_{-1/2}^{1/2} \frac{1}{2}\left|(1-\alpha^2 z^2)^3 + (1-\alpha^2 z^2)^2\right| dz = \frac{0{,}99}{\alpha} \simeq 1/\alpha$$

or, according to (4d), (4f) and (4g)

$$\nu = 2N\tau/m \tag{4h}$$

This is the result which is found in classical books by following a much more complicated procedure.

Since ν is known it is possible to determine the "confidence" interval" of $4\hat{S}_{vv}(p)$. This is an interval where the value of $4\hat{S}_{vv}(p)$ will have 95% probability to be included. The extreme values of $4\hat{S}_{vv}(p)$ are found with the aid of the coefficients taken from Table 4-I (Munk, Snodgrass & Tucker, 1959) which must be multiplied by $4\hat{S}_{vv}(p)$ to give the extreme values. Thus it is evident that the confidence limits of $\hat{S}_{vv}(p)$ are 1/4 of those of $4\hat{S}_{vv}(p)$. Consequently, these limits are the products of $\hat{S}_{vv}(p)$ multiplied by the coefficients.

TABLE 4-I

Confidence limits

ν	Coef.	ν	Coef.	ν	Coef.	ν	Coef.	ν	Coef.
1	0.2 -1000	4	0.36-8.3	8	0.46-3.8	20	0.59-2.1	150	0.81-1.27
2	0.21- 40	5	0.39-6.0	10	0.49-3.1	50	0.69-1.55	200	0.83-1.23
3	0.32- 14	6	0.42-4.8	15	0.55-2.4	100	0.78-1.35	300	0.86-1.18

Expression (4h) shows that if *m* is increased the value of ν is reduced for a constant value of the parameter $2N\tau$. Consequently, the accuracy of the analysis, from the statistical point of view, is greater for large values of ν. However, expression (3o) shows that the larger *m* is, the larger is the filter width and the less is the resolution. Hence *m* must be fixed according to the purpose of the analysis.

It was seen that formula (4g) gives only an approximate value of ν. But a more accurate value can be obtained from the analysis itself. In fact, if we know $4\hat{S}_{vv}(p)$, expression (4e) can be considered an actual equality where ν is not known and is given by

$$\nu = \frac{4S_{vv}(p)}{\mu_o^2} \qquad (4i)$$

where

$$\mu_o^2 = \frac{\sum_{n=0}^{N/2-1} (\xi_n^2 + \eta_n^2)}{N} \qquad (4j)$$

is the mean value of the noise energy.

5 - CONCLUSION

It is interesting to quote the following statement made by Franco (1970):"

"We are now in a position to foresee a new development of this subject so far as tidal analysis is concerned. In actual fact, it is not usual to take advantage of the Fourier analysis used to obtain the power spectrum to compute tidal harmonic constants. However, we believe that either the Myazaki or the Cartwright-Catton method may be used to "adjust" the Fourier analysis to the angular frequency of the astronomical constituents in order to find these constants. If so, we will be able to "fish" the needed harmonic terms from among those given by the Fourier analysis. The only objection is that the span is tied to a power of 2 and not to a classical multiple of one lunation. We know, however, that some least square analyses have been effected with no regard paid to the conventional spans and that the results were shown to be correct. Hence we hope to find an economical solution for avoiding heavy supplementary computations in order to arrive at the harmonic constants from the Fourier analysis itself, such as it is used to obtain the power spectrum."

Thus the present work confirms the above statement. The Cartwright-Catton method in fact has been extended by the addition of a number of redundant equations which adjust the Fourier coefficients to the known constituent terms. Consequently, the method is formally similar to the least square method and table 2-I shows how the inverse matrices to obtain R cos r and R sin r are well conditioned. Each matrix is inverted by species, the band of Fourier frequencies slightly exceeding the known tidal frequency band.

The central time used in the Cartwright-Catton method does not permit the use of the FFT algorithm; but the difficulty can be overcome by adding a small correction to the Fourier coefficient of the cosine term.

The fact that tidal analysis is not necessarily tied to a whole number of semi-lunations was demonstrated by Munk and Hasselman (1964). They made it clear that a good separation of the constituents with frequencies f_1, f_2 depends only on the length record T and the signal / noise ratio. They proved, that for the *usual noise level*, resolution can be better than 1/T if

$$|f_2-f_1|>1/T\sqrt{signal/noise}$$

Godin (1970) studied very recently the effect of background noise on the resolution of the tidal constituents. His conclusion is that for constituents with very different frequencies such an effect can be disregarded even for very short spans. "However the noise does disturb drastically the resolution of close constituents and actually prohibits the attempt of resolving constituents whose relative phase difference is less than a given minimum value". In addition he says that in his personal experience of tidal analysis, components with close frequencies can be resolved if the phase shift is about 288°. Since his approach is through the least square analysis we believe that the same results can be reached with the procedure here described.

If Godin's criterion is adopted to select new constituents we are able to search on the line spectrum of the Fourier analysis, for the new constituents which can be considered in the analysis of the residuals. In order to give an idea of the work involved in such a selection, let us take the example of the semidiurnal tide. Table 5-I shows amplitudes $\Delta R_n = \sqrt{\xi_n^2 + \eta_n^2}$ of the residuals corresponding to the angular frequencies q_n. Resolution of the Fourier analysis is about 0.04 degrees per hour, which corresponds to 328° in 8192 hours. However, if Godin's criterion is adopted, the difference between the hourly speeds of the old and new constituents must be $\Delta q_n > 288/8192 = 0{,}035$. The hourly speeds of the new constituents are obtained, as usually, through the combinations of the hourly speeds of the main constituents.

By finding the residual amplitudes

$$\Delta R_n = \sqrt{\xi_n^2 + \eta_n^2}$$

and the hourly speeds

$$q_n = 360n/N$$

TABLE 5-I

n	621	622	623	624	625	626	627
QN	27.2900391	27.3339844	27.3779297	27.4218750	27.4658203	27.5097656	27.5537109
DELTA-R	0.4471E-01	0.2181E 00	0.1448E 00	0.2505E 00	0.3616E 00	0.1207E 00	0.3564E 00
n	628	629	630	631	632	633	634
QN	27.5976562	27.6416016	27.6855469	27.7294922	27.7734375	27.8173828	27.8613281
DELTA-R	0.2074E 00	0.2033E 00	0.2984E 00	0.1559E 00	0.1662E 00	0.1232E 00	0.1142E 00
n	635	636	637	638	639	640	641
QN	27.9052734	27.9492187	27.9931641	28.0371094	28.0810547	28.1250000	28.2128906
DELTA-R	0.3610E 00	0.5817E 00	0.2461E 00	0.4086E 00	0.2312E 00	0.1478E 00	0.2281E 00
n	642	643	644	645	646	647	648
QN	28.2128906	28.2568359	28.3447266	28.3447266	28.3886719	28.4326172	28.4765625
DELTA-R	0.2179E 00	0.3487E 00	0.4825E 00	0.3743E 00	0.6989E-01	0.4159E 00	0.1171E 00
n	649	650	651	652	653	654	655
QN	28.5205078	28.5644531	28.6083984	28.6523437	28.6962891	28.7402344	28.7841797
DELTA-R	0.4149E 00	0.9233E 00	0.1424E 00	0.1972E-01	0.3373E 00	0.3330E 00	0.3879E 00
n	656	657	658	659	660	661	662
QN	28.8281250	28.8720703	28.9160156	28.9599609	29.0039063	29.0478516	29.0917969
DELTA-R	0.5407E 00	0.6445E 00	0.1801E 01	0.2683E 01	0.6070E 00	0.1019E 01	0.2220E 01
n	663	664	665	666	667	668	669
QN	29.1357422	29.1796875	29.2236328	29.2675781	29.3115234	29.3554687	29.3994141
DELTA-R	0.2669E 01	0.1354E 01	0.5602E 00	0.9082E-01	0.4256E 00	0.3400E 00	0.3711E 00
n	670	671	672	673	674	675	676
QN	29.4433594	29.4873047	29.5312500	29.5751953	29.6191406	29.6630859	29.7070312
DELTA-R	0.3613E 00	0.1693E 01	0.3195E 00	0.4396E 00	0.5707E 00	0.2817E 00	0.1964E 00
n	677	678	679	680	681	682	683
QN	29.7509766	29.7949219	29.8388672	29.8828125	29.9267578	29.9707031	30.0146484
DELTA-R	0.2248E 00	0.3305E 00	0.3419E 00	0.2017E 00	0.2761E 00	0.2323E 00	0.6955E 00
n	684	685	686	687	688	689	690
QN	30.0585937	30.1025391	30.1464844	30.1904297	30.2343750	30.2783203	30.3222656
DELTA-R	0.7626E 00	0.1104E 00	0.1268E 01	0.8020E 00	0.1733E 00	0.1733E 00	0.5018E 00
n	691	692	693	694	695	696	697
QN	30.3662109	30.4101562	30.4980469	30.4980469	30.5419922	30.5859375	30.6298828
DELTA-R	0.5249E 00	0.2115E 00	0.3089E 00	0.7356E-01	0.4150E 00	0.6047E 00	0.1636E 00
n	698	699	700	701	702	703	704
QN	30.6738281	30.7177734	30.7617188	30.8056641	30.8496094	30.8935547	30.9375000
DELTA-R	0.3494E 00	0.2256E 00	0.2913E 00	0.2913E 00	0.3248E 00	0.4283E-01	0.1848E 00
n	705	706	707	708	709		
QN	30.9814453	31.0253906	31.0693359	31.1132812	31.1572266		
DELTA-R	0.1135E 00	0.4012E 00	0.2218E 00	0.3515E 00	0.1278E 00		

for the values of *n* (cycles per period N), it is possible to organize tables for the frequency bands of the tidal oscillation, e.g. table 5-I where one can see the q_n values corresponding to important values of ΔR_n. Thus, it is possible to search for new shallow-water constituents having hourly speeds near the tabulated values of q_n. Such a selection is based upon the same operations indicated by Doodson (1928). These operations are very delicate, and difficulties appear in the selection of the constituents. For instance, there exists the possibility of obtaining the same hourly speed with different combinations. In such case, one should select the combination, the amplitude constituents of which give the largest product. Since the nodal factors of the compound constituents are given by the product of the individual node factors, only a very long period analysis (18.67 years) will show the appropriate combination.

Table 5-I shows that residual amplitudes resulting from cleaning the tidal spectrum from the 18 classical (Doodson) semi-diurnal constituents are very small indeed. According to Zetler - Cummings (1967) these residuals in the semidiurnal band do not justify the extra work of searching for new constituents.*

The above mentioned choice of new constituents is sufficient inside the frequency bands. However, only a more elaborate spectral analysis will show all the frequency bands to which the research must be extended. In addition, such an analysis will show the statistical accuracy of the results.

Appendices II and III are the programs to compute the values for a_j and b_j for any number of tidal constituents.

It remains to draw some most important conclusions about the search for new constituents. Fig. 5 shows that some energy is due to the fifth diurnal tide which is not represented by the classical 61 constituents. It is obvious that such a peak would persist in the residual spectrum resulting from the removal of the 61 constituents. Fig. 6 shows that peak (solid line). In the figure the interrupted line represents the power spectrum of the residuals resulting from a 147 constituents tidal analysis. These constituents, extended up to the 12^{th} diurnal species, except for the 3^{rd} and 5^{th} diurnal species, did not show any improvement on the spectrum of the residuals. In fact the use of 29 semidiurnal constituents, instead of the classical 18 constituents, increased the residual energy of the power spectrum. The same can be repeated for the 4^{th} diurnal species. Thus, it was decided to use the classical constituents only for the LP, D, SD and 6^{th} diurnal species and new shallow water constituents to represent the 3^{rd}, 4^{th} and 5^{th} diurnal species. The final result can be seen in Fig. 7.

* Existence of tidal cusps according to Munk, Zetler & Groves(1965) were considered.

The total number of constituents corresponding to the final result is 82.

One of the most important steps in the analysis, that of generating new shallow water constituents to explain observed peaks, is shown in the flow diagram as a manual process. A computer program has been written to perform this function, but it is not included herein for the sake of simplicity.

The implications of the method in the field of tidal analysis are many and varied, but perhaps the most important is that an increase in the number of points analysed (in this case 8192 hourly readings) in order to increase the resolution of the analysis does not occasion a disproportionate increase in computer processing time.

Although the entire analysis was carried out on an IBM/360/44, a high powered computer is not essential, since the FFT and the matrix operations can be carried out in a series of separate stages. Thus with certain program modifications a 16K word memory with suitable high-speed input/output facilities should be sufficient.

RESUMO

Êste trabalho propõe um novo caminho para a análise espectral da maré baseada no algorítmo de Cooley-Tukey. A análise através da "Transformação Rápida de Fourier" (Fast Fourier Transform - FFT) é empregada tanto para calcular as constantes harmônicas da maré quanto para a obtenção do espectro de energia. Êste é calculado por meio de uma soma ponderada. Também é dada uma nova dedução da fórmula que exprime o número de graus de liberdade em que se baseia o intervalo de confiança correspondente ao espectro do ruído.

O trabalho foi redigido em inglês a fim de facilitar o intercâmbio de informações.

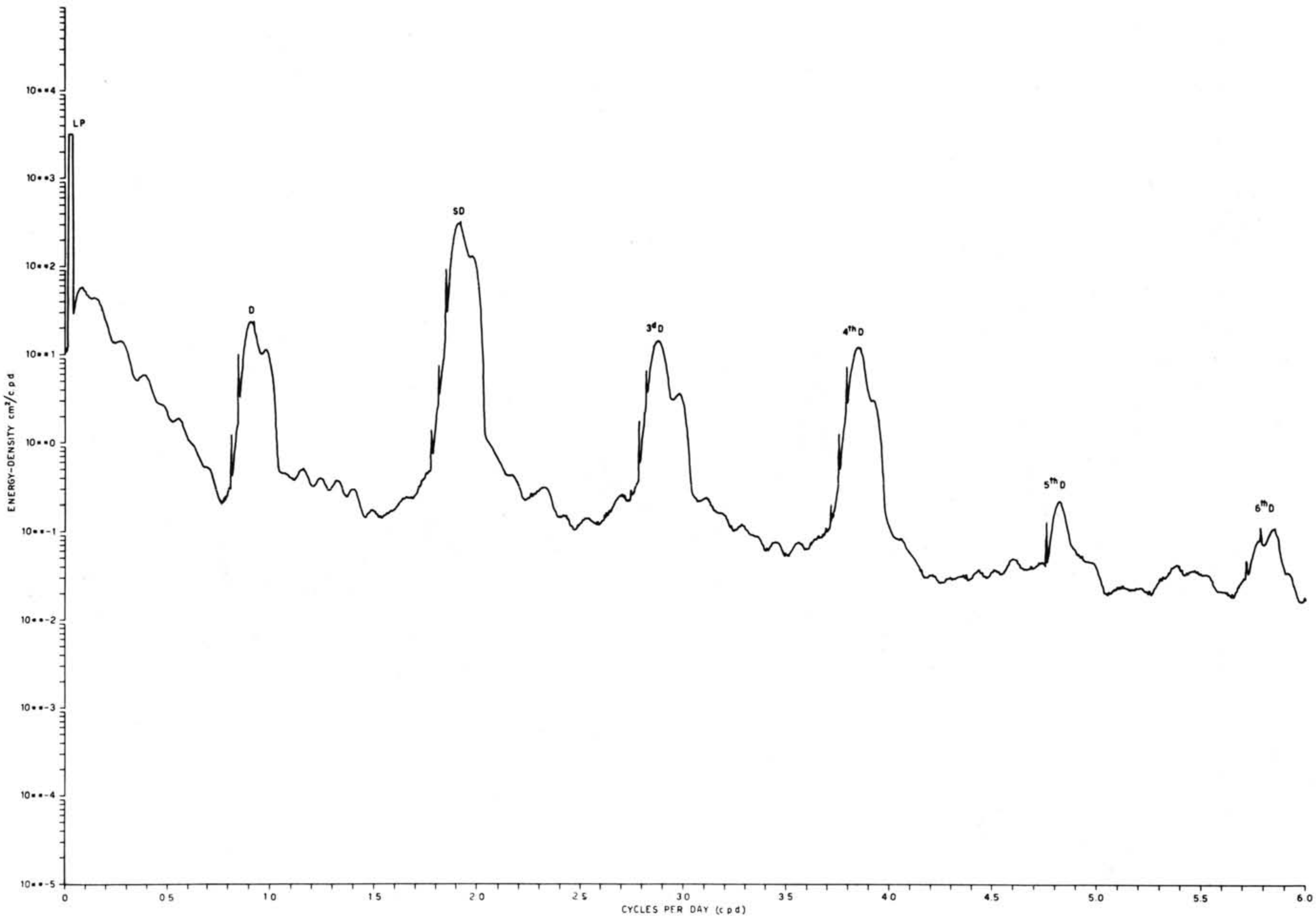

Fig 5 - TIDAL SPECTRUM - CANANEIA

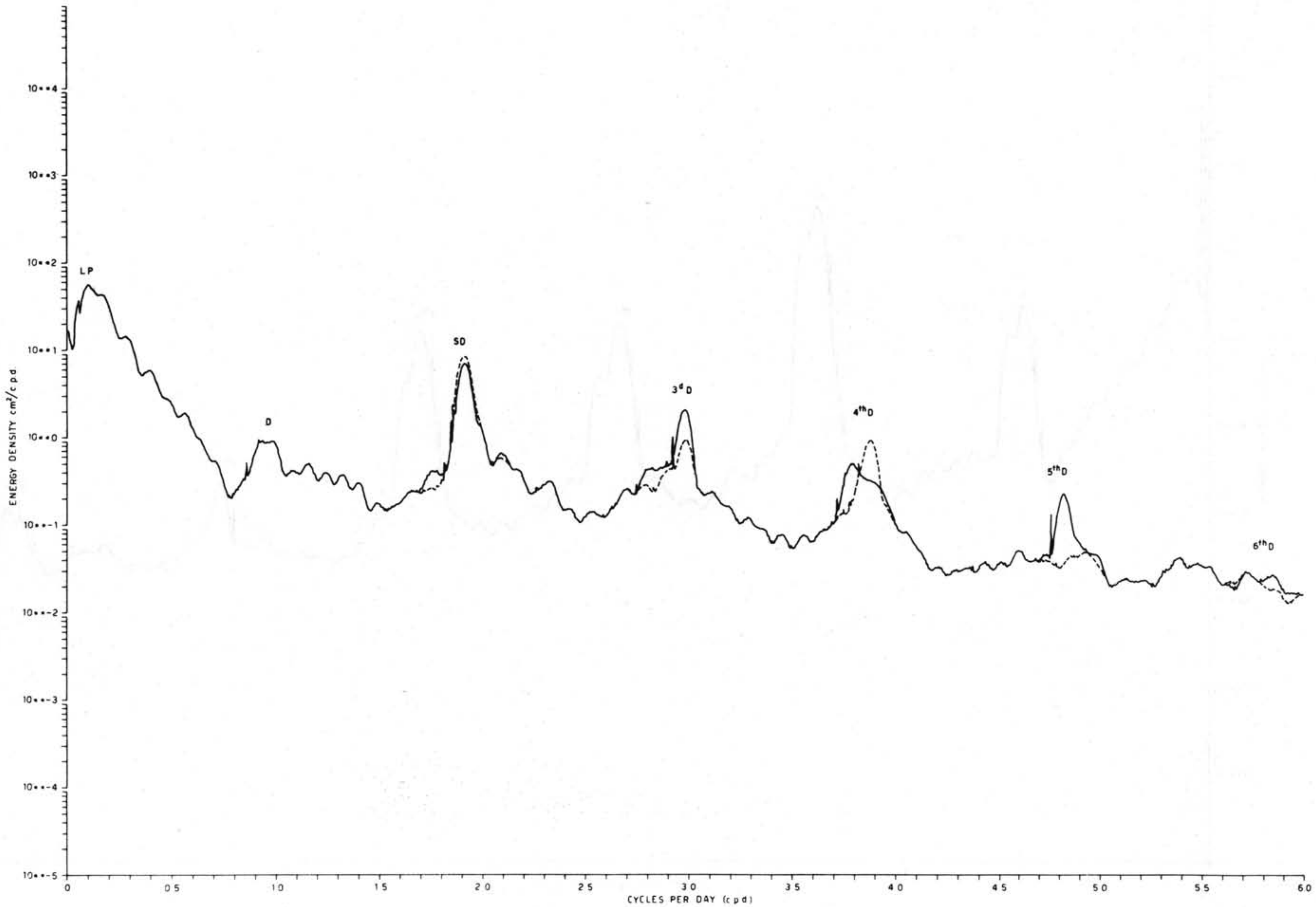

Fig 6 - RESIDUAL SPECTRA (—) 61 constituents (– –) 147 constituents

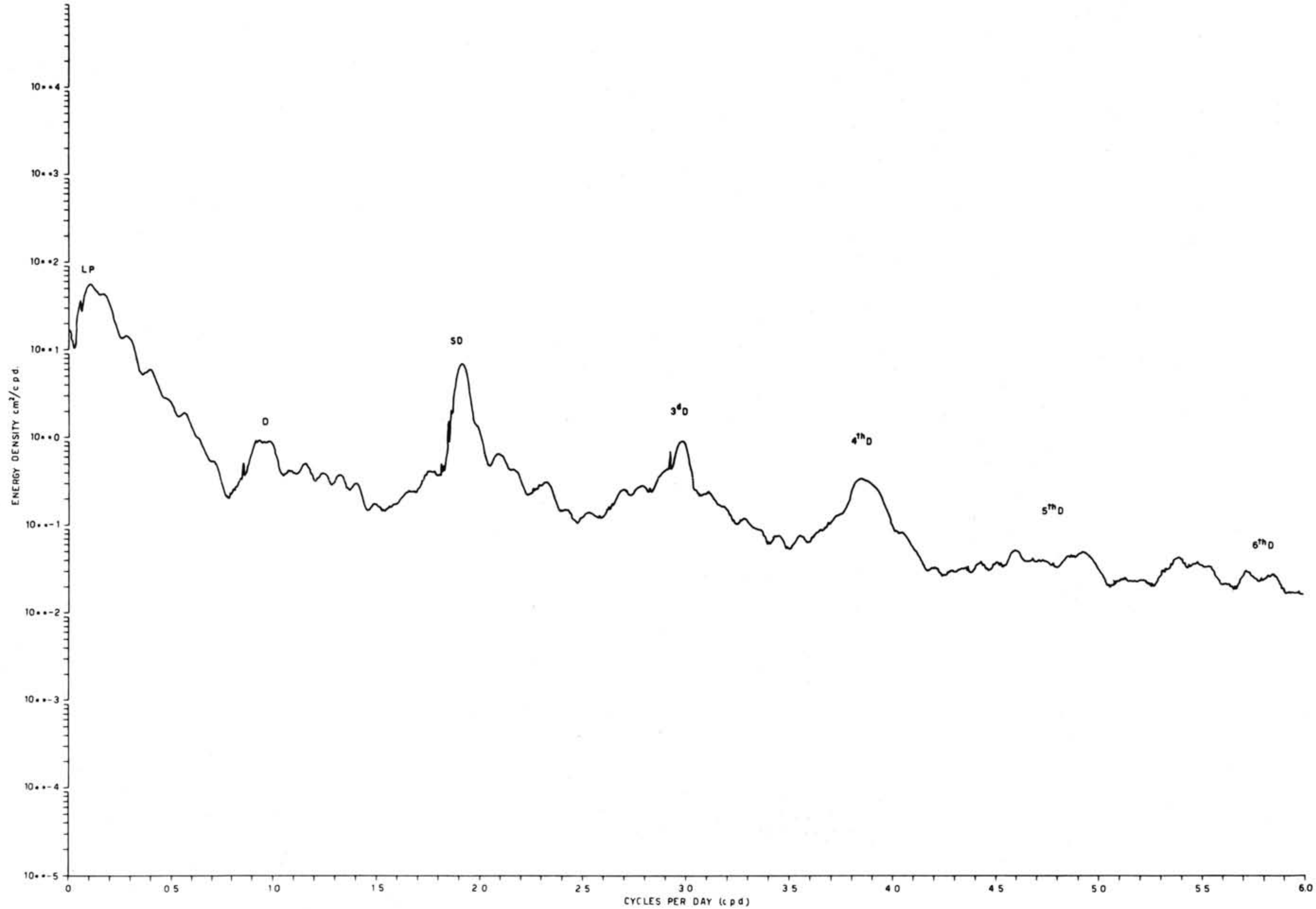

Fig 7 - FINAL RESIDUAL SPECTRUM: 82 constituents

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LIST OF SYMBOLS USED

a,b	cosine/sine component
A,B	cosine/sine matrices
$A(\Theta)$	auto-correlation function for time lag Θ
c	vector denoting phase and amplitude of oscillation
C_s	spectral estimate at frequency $s.\Delta f$ (Δf = fundamental frequency)
F_j	nearest Fourier frequency to j^{th} tidal constituent
j	subscript denoting tidal constituent
$K(\Theta)$	cross-correlation function for time lag Θ
m	maximum number of lags
n	subscript denoting Fourier number
N	number of values in Fourier series
p	index denoting Fourier number
q	angular frequency (speed number)
Q	number of tidal constituents
r	phase lag reckoned from the time origin
R	amplitude of tidal constituent
s	index denoting discrete frequency of spectral estimate
$\hat{S}_{xy}$	cross-spectral estimate between series x and y
t	time
$v(t)$	gaussian noise as a function of time
$x(t)$	time series
$y(t)$	tidal heights as a function of time
z	index for values of discrete weighting function
α	damping coefficient for weighting function
γ	raw Fourier spectral estimate
ε	estimate vector denoting phase and amplitude of random oscillation
Θ	time lag
ϕ,ψ	weighting functions
μ_o	r.m.s. amplitude of white noise
ν	degrees of freedom
τ	sampling interval
ω	angular speed difference/sum
ξ,η	cosine/sine of residual noise

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APPENDICES

SET OF PROGRAMS FOR TIDAL ANALYSIS

by the

"Instituto Oceanográfico" Method

N.B. The programs are presented separately to provide greater flexibility to the user. In practice, however, programs in Appendices I, II, III, IV, are inter-linked and executed sequentially by the computer.

INPUT OF DATA

I - F.F.T. PROGRAM -

	FORMAT
HEADER CARD	
GAMA = Power of two ($N=2^{GAMA}$)	I2
MAIN DECK	
Y(I) = Tidal heights every hour	24F3.0

II - MATRIX GENERATION -

HEADER CARD	
N = No. of values in series ($N=2^{GAMA}$)	I4
NBLOC = No. of tidal species present	I4
TIDAL CARDS	
SPEED = Angular speed of constituent in degrees per solar hour	F10.7
SHALLOW WATER	
COMPOSITION FACTORS = ICOM (I), Positive or negative integers indicating the composition of the shallow water constituent in terms of 30 fundamental constituents. (Sa, Ssa excepted)	30I2
PRINCIPAL CONSTITUENT NO. = Number denoting one of thirty-two principal constituents (see list); if not then =0	I2
CON Symbolic name of constituent	A8

III - H & g CALCULATION -

TITLE CARD	
YEAR CARD	
YEAR = Year of the start of the series	F5.0
N = No. of values in tidal series	I5
NDAY = First day of the series according to the Julian calendar	I5

HEADER CARD

N = No. of values in tidal series — I4

NBLOC = No. of species to be resolved — I4

YN = Last value (n^{th}) of tidal series — F4.0

NTAPE = 1 if matrices are written on same tape as Fourier series, else = 2 — I4

MTAPE = Symbolic tape unit for output of residual Fourier series — I4

IV - AUTO-SPECTRUM

HEADER CARD

N = No. of values in tidal series

NO = No. of series to be processed

MDIV = Arbitrary divider to increase the resolution from a predetermine minimum (>5)

L = When MDIV is not specified the resolution can be controlled by specifying the half-filter width L

KTAPE = If non-zero then auto-spectra are written on Magnetic Tape (2), else print-out and punch-out of auto-spectra are effected.

LIST OF 32 PRINCIPAL CONSTITUENTS -

Sa, Ssa, Mm, Mf, 2Q, σ_1, Q_1, ρ_1, O_1, M_1, χ_1, π_1, P_1, S_1, K_1, ψ_1, ϕ_1, θ_1, J_1, OO_1, $2N_2$, μ_2, N_2, ν_2, M_2, λ_2, L_2, T_2, S_2, R_2, K_2, M_3.

APPENDIX I

FAST FOURIER TRANSFORM

```
      COMPLEX YA(8192),WBK,YF2,AUX
      INTEGER*2BETA(2048),GAMA,COEF1,COEF2
      REAL*8Y(8192)
      DIMENSION  S(2049)
      EQUIVALENCE (Y(1),YA(1))
      READ(5,1000) GAMA
      REWIND 4
      N=2**GAMA
      XN=N
      FACT=6.28318/XN
      FCT=XN/360.
      ND8=N/8
      ND4=ND8&ND8
      ND42=ND4&2
      ND41=ND4&1
      ND2=ND4&ND4
      ND22=ND2&2
      ND82=ND8&2
C    GENERATING THESINE SERIES
      ANG=FACT*ND8
      S(ND8&1)=SIN(ANG)
      S(1)=0.
      S(ND41)=1.
      DO 100 I=ND82,ND4
      JJ=ND42-I
      ANG=ANG&FACT
      S(I)=SIN(ANG)
  100 S(JJ)=COS(ANG)
C    FORMING THE BIT-REVERSED COEFFICIENTS
      IE=GAMA-2
      NL=1
      BETA(1)=0
      DO 200 M=1,IE
      LL=NL
      NL=NL&NL
      DO 200 JJ=1,LL
      BETA(JJ)=BETA(JJ)*2
  200 BETA(JJ&LL)=BETA(JJ)&2
      READ(5,1200)(Y(I),I=1,N)
      YSUM=0.
      DO 105 I=1,N
  105 YSUM=Y(I)&YSUM
      YSUM=YSUM/N
      DO 106 I=1,N
      YX=Y(I)-YSUM
  106 YA(I)=CMPLX(YX,0.)
      YN=Y(N)
      MM2=N
      KK=1
C    THE HARD CORE
      DO 520 L=1,GAMA
      MM=MM2/2
      MMM=MM-1
      IB=1
C    SELECTING THE COMPLEX MULTIPLIERS
      J=1
      K=1
      KEND=KK
```

```
  220 MJ=BETA(K)&J
      IF(MJ.GT.ND41)GOTO 403
      NC=ND42-MJ
      WBK=CMPLX(S(NC),S(MJ))
      GOTO 410
  403 NS=ND22-MJ
      NC=ND42-NS
      WBK=CMPLX(-S(NC),S(NS))
  410 IEND=IB&MMM
      DO 500 COEF1=IB,IEND
      COEF2=COEF1&MM
      YF2=YA(COEF2)*WBK
      YA(COEF2)=YA(COEF1)-YF2
  500 YA(COEF1)=YA(COEF1)&YF2
C    DECIMATION IN TIME FOR THE FAST FOURIER TRANSFORM
C     THE DATA IS NATURALLY ORDERED ON ENTRY AND BIT REVERSED ON EXIT
      IB=IB&MM2
      K=K&1
      IF(K.LE.ND4) GOTO 515
      K=1
      J=2
      KEND=KEND-ND4
  515 IF(K.LE.KEND)GOTO 220
      KK=KK&KK
  520 MM2=MM
C   BIT-REVERSING THE SERIES IN TWO PARTS
      KK=ND4
      DO 600 K=1,ND4
      MM=BETA(K)
      MM=MM&MM&1
      MK=MM&2
      KK=KK&1
      IF(MK.LE.KK)GOTO 590
      AUX=YA(KK)
      YA(KK)=YA(MK)
      YA(MK)=AUX
  590 IF(MM.LE.K)GOTO 600
      AUX=YA(K)
      YA(K)=YA(MM)
      YA(MM)=AUX
  600 CONTINUE
      RI=1./ND2
      ND21=ND2&1
      DO 610 K=1,ND21
  610 YA(K)=YA(K)*RI
      YA(1)=YA(1)*0.5
C    NOTE THAT ONLY THE THE N/2 COEFFIENTS DESIRED ARE FULLY BIT-REVERSED
      WRITE(4)(YA(I),I=1,ND21)
      REWIND 4
 1000 FORMAT(I2)
 1200 FORMAT(24F3.0)
      CALL EXIT
      END
```

APPENDIX II

MATRIX GENERATION

```
      EXTERNAL MFSD
      DOUBLE PRECISION CONST(35),CONS(24)
      DIMENSION SPED(35),EPS(35),ARA(50 0),ARB(5000),RINV(900),RONE(35),
     1RFIM(5000),NSPEC(13),NFOUR(35),NMAT(13)
      DIMENSION ISP(24),JDUM(24)
      REWIND 2
      READ(2)
      READ(5,500)N,NBLOC,YN
  500 FORMAT(2I4,F4.0)
C  N IS THE NUMBER OF VALUES PER SERIES, NBLOC THE NUMBER OF TIDAL
C  SPECIES,  YN THE LAST VALUE OF THE SERIES
      XN=N
      RN=1./XN
      XN1=XN&1
      KSW=1
      FCT=XN/360.
      FACT=6.28318/XN
      FCT2=FACT*0.5
      KOUNT=0
    2 J=0
      KOUNT = KOUNT & 1
      JSW = 0
      NEND=0
      JJ=0
    5 IF(NEND.EQ.24) GOTO 7
      NBEG=NEND&1
      NEND=NEND&6
      JJ=JJ&1
      READ(5,5001)(ISP(J),JDUM(J),CONS(J),J=NBEG,NEND)
 5001 FORMAT(I10,I2,A8,2I2,A8,2I2,A8,2I2,A8,2I2,A8,2I2,A8)
C   THE USE OF EXTRA VARIABLES IS AN ARTIFICE TO WRITE FOUR WHOLE CARDS
C   ON MAGNETIC TAPE
C  THE PROGRAM USER SHOULD ASCERTAIN THAT HIS COMPUTER CAN HANDLE  AN
C  INTEGER NUMBER OF 10 DIGITS.  IF NOT THEN MAKE MAKE THE FORMAT 2I6,A8
C  AND COMBINE (ISP,JDUM)   TO OBTAIN ANGULAR SPEED
      SPED(JJ)=ISP(NBEG)*1.0E-07
      CONST(JJ)=CONS(NEND)
      IF(SPED(JJ).NE.0.) GOTO 5
      JSW=1
      NCON=JJ-1
    7 WRITE(2,5001)(ISP(J),JDUM(J),CONS(J),J=1,24)
      NEND=0
      IF(JSW.EQ.0) GOTO 5
      NCON1=NCON -1
      NSPEC(KOUNT)=NCON
      DO 100 J=1,NCON
      SPEED=SPED(J)*FCT
      IFOUR=(SPEED&0.5)
      NFOUR(J)=IFOUR
      EPSI=SPEED-IFOUR
      SPED(J)=SPEED
  100 EPS(J)=EPSI
      KK=0
      NST=NFOUR(1)-2
      IF(NST.LT.0)NST=0
      NFIN=NFOUR(NCON)&2
      IF(KOUNT.EQ.13) NFIN=N/2
      NOFOR =NFIN-NST&1
      NM=NCON*NOFOR
      NMAT(KOUNT)=NM
      DO 200 M=1,NCON
```

```
      SPEED=SPED(M)
      SIGN=1
      IF(MOD(NST,2).EQ.0) SIGN=-1
      DO 150 IFOUR=NST,NFIN
      SIGN=-SIGN
      KK=KK&1
      ARG=(-SPEED&IFOUR)*FCT2
      ANG=(SPEED&IFOUR)*FCT2
      PART=SIN(ANG*XN1)*RN/SIN(ANG)
      SENQ=XN1*RN
      IF(ARG.NE.0.)SENQ=SIN(ARG*XN1)*RN/SIN(ARG)
      ARA(KK)=(SENQ&PART)*SIGN
  150 ARB(KK)=(SENQ-PART)*SIGN
  200 CONTINUE
C    BLOCK TO FORM THE ORIGINAL  MATRIX
      CALL MATSYM(ARA,RINV,NCON,NOFOR)
      DELTA=0.0001
C    NORMALISING THE MATRIX TO SYMETRIC FORM VIA A SUBROUTINE
  485 CALL SINV(RINV,NCON,DELTA,IER)
C      INVERTING THE MATRIX VIA A SUBROUTINE
      IF(IER)699,205,202
  202 WRITE(6,6202)
 6202 FORMAT(//20X,'INSTABILITY AT STAGE NO. ',I4,' OF INVERSION')
  205 WRITE(6,659)
  659 FORMAT (/////10X,'INVERTED MATRIX')
      IE=0
      DO 355 K=1,NCON
      IB=IE&1
      IE=IE&K
  355 WRITE(6,646)(RINV(KK),KK=IB,IE)
  646 FORMAT(/13(1X,F8.5))
      IF(KSW.EQ.2) GOTO 465
      CALL SYMTPR(RINV,ARA,RONE,NCON,NOFOR,RFIM)
C    MULTIPLYING THE INVERSE BY THE TRANSPOSE MATRIX
      KSW=2
      GOTO 470
  465 CALL SYMTPR(RINV,ARB,RONE,NCON,NOFOR,RFIM)
      KSW=1
  470 WRITE(6,665)
  665 FORMAT(///'                  FINAL NORMALISED MATRIX')
      WRITE(6,630)(CONST(KK),KK=1,NCON)
      WRITE(6,631)(NFOUR(KK),KK=1,NCON)
      WRITE(6,632)(EPS(KK),KK=1,NCON)
      WRITE(2)(RFIM(K),K=1,NM)
      IF(KSW.EQ.2)GOTO  638
      WRITE(2)(ARB(K),K=1,NM)
      IF(KOUNT.NE.NBLOC)GOTO 2
      WRITE(6,690)(NSPEC(K),K=1,KOUNT)
  690 FORMAT(36X,'NUMBER  OF  CONSTITUENTS  PER  SPECIES'/'  LONG   ',
     1'DIURNAL   SEMI-     TER-      QUARTO-  QUINTO-  SEXTO- SETIMO- ',
     2'OITAVO-     NONO-   DECIMO-   DEC-PRIM  DEC-SEG'/12(3X,I3,3X))
      ENDFILE 2
      REWIND 2
      CALL EXIT
  638 WRITE(6,641)
      WRITE(2)(ARA(K),K=1,NM)
  641 FORMAT('1',10X,'NORMAL MATRIX SINE COEFFICIENTS')
      WRITE(6,630)(CONST(KK),KK=1,NCON)
      WRITE(6,631)(NFOUR(KK),KK=1,NCON)
      WRITE(6,632)(EPS(KK),KK=1,NCON)
      CALL MATSYM(ARB,RINV,NCON,NOFOR)
```

```
      GOTO 485
  630 FORMAT(10(2X,A8))
  631 FORMAT(10(2X,I4,4X))
  632 FORMAT(10(2X,F8.5))
  699 WRITE(6,6099)
 6099 FORMAT(//40X,'MATRIX INVERSION UNSUCESSFUL')
      END
      SUBROUTINE MATSYM(A,R,N,M)
      DIMENSION A(1),R(1)
C  SUBROUTINE TO MULTIPLY A GENERAL MATRIX BY THE TRANSPOSE OF ITSELF
C  RETURNING THE RESULT AS AN UPPER TRIANGULAR SYMMETRICAL MATRIX
C  A IS THE INPUT MATRIX STORED AS A TALL THIN MATRIX IN ONE DIMENSION
C  R IS THE UPPER TRIANGULAR SYMMETRICAL OUTPUT MATRIX
C  N IS THE NUMBER OF COLUMMS OF A (LESS THAN M)
C  M IS THE NUMBER OF ROWS OF A
      NEND = 0
      IR = 0
      DO 60 LJ = 1,N
      NBEG = NEND & 1
      NEND = NEND & M
      JJ = 0
      DO 55 LK = 1,LJ
      IR = IR & 1
      YSUM = 0.
      DO 50 LM = NBEG,NEND
      JJ = JJ & 1
   50 YSUM = A(LM)*A(JJ)&YSUM
   55 R(IR) = YSUM
   60 CONTINUE
      RETURN
      END
      SUBROUTINE SINV(A,N,EPS,IER)
C
C
      DIMENSION A(1)
      DOUBLE PRECISION DIN,WORK
C
C        FACTORIZE GIVEN MATRIX BY MEANS OF SUBROUTINE MFSD
C        A = TRANSPOSE(T) * T
      CALL MFSD(A,N,EPS,IER)
      IF(IER) 9,1,1
C
C        INVERT UPPER TRIANGULAR MATRIX T
C        PREPARE INVERSION-LOOP
    1 IPIV=N*(N+1)/2
      IND=IPIV
C
C        INITIALIZE INVERSION-LOOP
      DO 6 I=1,N
      DIN=1.DO/DBLE(A(IPIV))
      A(IPIV)=DIN
      MIN=N
      KEND=I-1
      LANF=N-KEND
      IF(KEND) 5,5,2
    2 J=IND
C
C        INITIALIZE ROW-LOOP
      DO 4 K=1,KEND
      WORK=0.DO
      MIN=MIN-1
```

```
      LHOR=IPIV
      LVER=J
C
C        SATRT INNER LOOP
      DO 3 L=LANF,MIN
      LVER=LVER+1
      LHOR=LHOR+L
    3 WORK=WORK+DBLE(A(LVER)*A(LHOR))
C        END OF INNER LOOP
C
      A(J)=-WORK*DIN
    4 J=J-MIN
C        END OF ROW-LOOP
C
    5 IPIV=IPIV-MIN
    6 IND=IND-1
C        END OF INVERSE-LOOP
C
C        CALCULATE INVERSE(A) BY MEANS OF INVERSE(T)
C        INVERSE(A) = INVERSE(T) * TRANSPOSE(INVERSE(T))
C        INITIALIZE MULTIPLICATION-LOOP
      DO 8 I=1,N
      IPIV=IPIV+I
      J=IPIV
C
C        INITIALIZE ROW-LOOP
      DO 8 K=I,N
      WORK=0.D0
      LHOR=J
C
C        START INNER LOOP
      DO 7 L=K,N
      LVER=LHOR+K-I
      WORK=WORK+DBLE(A(LHOR)*A(LVER))
    7 LHOR=LHOR+L
C        END OF INNER LOOP
C
      A(J)=WORK
    8 J=J+K
C        END OF ROW- AND MULTIPLICATION-LOOP
C
    9 RETURN
      END
      SUBROUTINE MFSD(A,N,EPS,IER)
C
C
      DIMENSION A(1)
      DOUBLE PRECISION DPIV,DSUM
C
C        TEST ON WRONG INPUT PARAMETER N
      IF(N-1) 12,1,1
    1 IER=0
C
C        INITIALIZE DIAGONAL-LOOP
      KPIV=0
      DO 11 K=1,N
      KPIV=KPIV+K
      IND=KPIV
      LEND=K-1
C
C        CALCULATE TOLERANCE
```

```
      TOL=ABS(EPS*A(KPIV))
C
C        START FACTORIZATION-LOOP OVER K-TH ROW
      DO 11 I=K,N
      DSUM=0.D0
      IF(LEND) 2,4,2
C
C        START INNER LOOP
    2 DO 3 L=1,LEND
      LANF=KPIV-L
      LIND=IND-L
    3 DSUM=DSUM+DBLE(A(LANF)*A(LIND))
C        END OF INNER LOOP
C
C        TRANSFORM ELEMENT A(IND)
    4 DSUM=DBLE(A(IND))-DSUM
      IF(I-K) 10,5,10
C
C        TEST FOR NEGATIVE PIVOT ELEMENT AND FOR LOSS OF SIGNIFICANCE
    5 IF(SNGL(DSUM)-TOL) 6,6,9
    6 IF(DSUM) 12,12,7
    7 IF(IER) 8,8,9
    8 IER=K-1
C
C        COMPUTE PIVOT ELEMENT
    9 DPIV=DSQRT(DSUM)
      A(KPIV)=DPIV
      DPIV=1.D0/DPIV
      GO TO 11
C
C        CALCULATE TERMS IN ROW
   10 A(IND)=DSUM*DPIV
   11 IND=IND+I
C
C        END OF DIAGONAL-LOOP
      RETURN
   12 IER=-1
      RETURN
      END
      SUBROUTINE SYMTPR(A,C,B,N,M,R)
C    MULTIPLICATION OF A SYMMETRIC UPPER TRIANGULAR MATRIX BY THE TRANSPOSE
C    OF A GENERAL MATRIX
C    A IS THE UPPER TRIANGULAR MATRIX WITH N ROWS
C    B IS A WORK VECTOR OF SIZE N
C    C IS THE INPUT MATRIX OF SIZE M BY N AND CANNOT OCCCUPY THE SAME
C    POSITION AS THE OUTPUT MATRIX R
C    R IS THE OUTPUT MATRIX OF SIZE N BY M AND CANNOT OCCUPY THE SAME
C    POSITION AS THE INPUT MATRIX
C     N IS THE NUMBERS OF ROWS
C     M IS THE NUMBER OF COLUMNS
      DIMENSION A(1),B(1),C(1),R(1)
      JST = 0
      IR = 0
      DO 50 L = 1,N
      JJ = JST
      JSTEP = 1
      MM = 0
   10 JJ = JJ & JSTEP
      MM = MM & 1
      B(MM) = A(JJ)
      IF(MM-L)10,15,15
```

APPENDIX III

CALCULATION OF TIDAL COMPONENTES a_j AND b_j, HARMONIC CONSTANTS H&g AND CORRECTION OF FOURIER COEFFICIENTS FOR TIDAL EFFECTS TO OBTAIN RESIDUALS ξ_n AND η_n

```
      DOUBLE PRECISION CONS(35),CON(4)
C     ANALYSIS AND CALCULATION OF THE RESIDUAL FOURIER ENERGY
C     EXTRACTION OF THE H AND G VALUES FOR THE TIDE FROM A GIVEN FOURIER
      DIMENSION SPED(35),VZU(35),FN(35),ICOM(33,4),ISP(4)
      DIMENSION A(35),B(35),X(4097),Y(4097),VECT(4000),AMAR(200),
     1 BMAR(200),VV(13),IQ(10)
      DIMENSION V(32),F(12),U(12),SPEED(32),VMU(32),IFU(32),BCOS(3),
     1 BSEN(3),FF(35), Z(128),AU(32),AF(32),AM(12),AAM(4)
      DATA Z/4*0.,5*270.,2*90.,192.,270.,180.,90.,168.,4*90.,5*0.,2*180.
     1,282.,0.,258.,0.,180.,
     22*0.,1.,2.,-4.,-4.,-3.,-3.,-2.,-1.,-1.,6*0.,2*1.,2.,-4.,-4.,-3.,
     3-3.,-2.,-1.,-1.,4*0.,-3.,
     41.,2.,2*0.,1.,3.,1.,3.,2*1.,3.,-2.,-1.,0.,1.,2.,3.,-1.,2*1.,2.,4.,
     52.,4.,2.,0.,2.,-1.,0.,1.,2.,3.,
     62*0.,-1.,0.,2.,0.,1.,-1.,2*0.,-1.,6*0.,1.,-1.,0.,2.,0.,1.,-1.,0.,
     71.,-1.,5*0./
      DATA AU/9*0.,-23.74,10.80,-8.86,-12.94,-36.68,-2.14,-17.74,
     1 0.,2.68,-1.34,0.68,1.34,4.02,0.,0.68,
     2 0.,-0.38,0.19,-0.07,-0.19,-0.57,0.,-0.04/
      DATA AF/1.0000,1.0429,1.0089,1.0050,1.0129,1.1027,1.0004,1.0241,
     1-0.1300,0.4135,0.1871,0.1150,0.1676,0.6504,-0.0373,0.2863,
     2 0.0013,-0.0040,-0.0147,-0.0088,-0.0170,0.0317,0.0002,0.0083,
     3 2*0.,0.0014,0.0006,0.0016,-0.0014,0.,-0.0015/
      DATA AM/277.025,280.190,334.385,259.157,129.38481,-0.23872,
     1 40.66249,-19.32818,13.17640,0.98565,0.11140,-0.05295/
      DATA IFU/ 2*1,2,3,5*4,11,6,3*1,5,2*1,2*6,7,6*8,10,3*1,9,12/
      DATA SPEED/0.0410686,0.0821373,0.5443747,1.0980331,12.8542862,
     112.9271398,13.3986609,13.4715145,13.9430356,14.4920521,14.5695476,
     214.9178647,14.9589314,15.0000000,15.0410686,15.0821353,15.1232059,
     315.5125897,15.5854433,16.1391017,27.8953548,27.9682084,28.4397295,
     428.5125831,28.9841042,29.4556253,29.5284789,29.9589333,30.0000000,
     530.0410667,30.0821373,43.4761563/
      READ(5,570)
  570 FORMAT(72H
     1                )
      READ(5,502)YEAR,N,NDAY
  502 FORMAT(F5.0,2I5)
      WRITE(6,630)YEAR,N
  630 FORMAT(20X,'NODE FACTORS AND EPOCH ANGLES FOR ',F6.0,'  CENTRED',
     1' ON A  ',I4,' HR. INTERVAL'//)
      XD2=N/2.
      JN=N/48
      XN=JN
      RESTO=XD2-XN*24.
      XN=XN&NDAY-1.
      L=(YEAR-1901.)/4.
      XL=L
      B(1)=YEAR-1900.
      B(2)=XN&XL
      DO 10 I=1,4
      KK=I
      XSUM=AM(KK)
      DO 9 J=1,2
      KK=KK&4
    9 XSUM=B(J)*AM(KK) &XSUM
   10 AAM(I)=REDUZ(XSUM)
      WRITE(6,600)(AAM(I),I=1,4)
  600 FORMAT('  S=',F9.5,'  H=',F9.5,'  P=' ,F9.5,'  N=',F9.5///)
      AN=AAM(4)
      DO 20 I=1,32
      KK=I
```

```
      XSUM=Z(KK)
      DO 19 J=1,3
      KK=KK&32
   19 XSUM=AAM(J)*Z(KK) &XSUM
   20 V(I)=XSUM
C     CALCULATION OF THE V  ANGLES
C      CALCULATION OF THE BASIC ANGLES
      PIF=6.28318/360.
      ANOD=AN*PIF
      ANG=ANOD
      DO 25 I=1,3
      BCOS(I)=COS(ANG)
      BSEN(I)=SIN(ANG)
   25 ANG=ANG&ANOD
C     CALCULATION OF THE NODE FACTORS F AND THE LUNAR ANGLES U
      F(1)=1.000
      U(1)=0.
      DO 30 J=2,9
      KK=J-1
      YSUM=0.
      XSUM=AF(KK)
      DO 29 I=1,3
      KK=KK&8
      XSUM= AF(KK)*BCOS(I) & XSUM
   29 YSUM=AU(KK)*BSEN(I) & YSUM
      F(J)=XSUM
   30 U(J)=YSUM
      P=AAM(3)*PIF
      PP=P&P
      PN=P-ANOD
      PPNN=PN&PN
      PPN=PP-ANOD
      FCOSUL=1.-COS(PP)*0.2505-COS(PPN)*0.1102-COS(PPNN)*0.0156 -BCOS(1)
     1 *0.0370
      FSENUL=(-SIN(PP)*0.2505)-SIN(PPN)*0.1102-SIN(PPNN)*0.0156-BSEN(1)*
     10.0370
      FCOSUM=COS(P)*2. &COS(PN)*0.4
      FSENUM=SIN(PN)*0.2 &SIN(P)
      F(10)=SQRT(FCOSUL*FCOSUL &FSENUL*FSENUL)
      U(10)=ATAN(FSENUL/FCOSUL)
      F(11)=SQRT(FCOSUM*FCOSUM &FSENUM*FSENUM)
      U(11)=ATAN(FSENUM/FCOSUM)
      F(12)=F(8)**1.5
      U(12)=U(8)*1.5
C     SEPARATE CALCULATIONS OF F AND U FOR L2,M1,M3 RESPECTIVELY
      DO 40 J=1,32
      KK=IFU(J)
C     DUMMY SUBSTITUTIONS
      VMU(J)=REDUZ(SPEED(J)*RESTO &V(J) &U(KK))
   40 FF(J)=F(KK)
      M=0
      FACT=180./3.14159
    1 READ(5,500)N,NBLOC,YN,NTAPE,NPUN,MTAPE
  500 FORMAT(2I4,F4.0,3I4)
C     N IS THE EXTENT OF ORIGINAL SERIES
C     NBLOC IS THE NUMBER OF SPECIES FOR PROCESSING
C     YN IS THE LAST VALUE OF ORIGINAL SERIES
C     NTAPE = 1 IF THE MATRICES ARE WRITTEN ON SAME TAPE AS FOURIER SERIES,
C     ELSE NTAPE = 2
C     NPUN = 0 IF SUPRESSION OF PUNCHING H & G FOR TIDAL CONSTITUENTS IS
C     REQUIRED, ELSE ANY NON-ZERO VALUE
```

```
C  MTAPE IS THE SYMBOLIC TAPE UNIT FOR THE OUTPUT OF THE RESIDUAL SERIES
C   IF MTAPE =0 THEN NO OUTPUT IS MADE
      XN=N
      ND2=N/2
      ND21=ND2&1
      FCT=XN/360.
      REWIND 1
      REWIND NTAPE
      IF(MTAPE.NE.0)REWIND MTAPE
C    NOTE IT IS POSSIBLE FOR MTAPE TO BE EITHER TAPE 1 OR 2
      READ(1)(X(I),Y(I),I=1,ND21)
      FCT=XN/360.
      SPINC=360./XN
      CORR=(YN-X(1))/ND2
      X(1)=0.
      KOUNT=0
    2 J=0
      KOUNT=KOUNT&1
      WRITE(6,6001)
 6001 FORMAT(///10X,'SPEED READ',10X,'SPEED CALCD',10X,'NODE FACTOR',9X,
     1'EPOCH ANGLE',9X,'CONSTITUENT'///)
C    BLOCK TO READ NAMES,SPEEDS AND COMPOSITION FACTORS FROM TAPE
    5 READ(NTAPE,2100)(ISP(JK),(ICOM(I,JK),I=3,33),CON(JK),JK=1,4)
 2100 FORMAT(I10,31I2,A8)
      DO 8 JK=1,4
      J = J & 1
      IF(ISP(JK).EQ.0 ) GOTO 120
      SPED(J)=ISP(JK)*1.E-07
      CONS(J)=CON(JK)
      IF(ICOM(33,JK).EQ.0) GOTO 6
      JS=ICOM(33,JK)
      FN(J)=FF(JS)
      VZU(J) = VMU(JS)
      VELOC = SPEED(JS)
      GOTO 8
    6 ANG=0.
      FNO = 1.
      VELOC = 0.
      DO 7 KS=3,32
      IC=ICOM(KS,JK)
      IF(IC.EQ.0) GOTO 7
      ANG = VMU(KS)*IC & ANG
      FNO = FF(KS)**IABS(IC)*FNO
      VELOC = SPEED(KS)*IC & VELOC
    7 CONTINUE
      FN (J) = FNO
      VZU (J) = REDUZ (ANG)
    8 WRITE(6,6002)SPED(J),VELOC,FN(J),VZU(J),CONS(J)
 6002 FORMAT(10X,F11.7,9X,F11.7,9X,F10.5,10X,F10.2,10X,A8)
      GOTO 5
  120 NCON = J - 1
      NST=SPED(1)*FCT-0.5
      IF(KOUNT.EQ.1)NST=1
      NFIM=SPED(NCON)*FCT&3.5
C     KOUNT INDEXES THE SPECIES NUMBER
      IF(KOUNT.EQ.13)NFIM=ND21
C     TO INDEX THE MATRIX DIRECTLY THE FOURIER SPEED NUMBER IS
C     INCREASED BY ONE - - - THE SPAN OF THE MATRIX IS
C     INCREASED BY REDUCING NST BY 2 AND AUGMENTING
C     NFIM BY 2
      NOFOR =NFIM-NST&1
```

```
      NMAT=NOFOR*NCON
      DO 36 K=NST,NFIM
   36 X(K)=X(K)&CORR
C     CORRECTION OF THE COSINE TERM
      READ(NTAPE)(VECT(K),K=1,NMAT)
C     THE INVERSE COSINE MATRIX IS READ AND THE COSINE CONSTITUENT OF
C    THE TIDE CALCULATED
      IM=0
      DO 50 L=1,NCON
      RSUM=0.
      DO 38 K=NST,NFIM
      IM=IM&1
   38 RSUM=VECT(IM)*X(K)&RSUM
   50 A(L)=RSUM
      READ(NTAPE)(VECT(K),K=1,NMAT)
C     THE NORMAL COSINE MATRIX IS READ AND THE COSINE
C     COMPONENT OF THE TIDAL FOURIER SERIES CALCULATED.
      KK=0
      DO 58 K=NST,NFIM
      XSUM=0.
      KK=KK&1
      IM=KK
      DO 55 L=1,NCON
      XSUM=VECT(IM)*A(L)&XSUM
   55 IM=IM&NOFOR
      AMAR(KK)=XSUM
   58 X(K)=X(K)-XSUM
C     THE PROCESS IS REPEATED FOR THE SINES
      READ(NTAPE)(VECT(K),K=1,NMAT)
      IM=0
      DO 70 L=1,NCON
      RSUM=0.
      DO 60 K=NST,NFIM
      IM=IM&1
   60 RSUM=VECT(IM)*Y(K)&RSUM
   70 B(L)=RSUM
      KK=0
      READ(NTAPE)(VECT(K),K=1,NMAT)
      DO 85 K=NST,NFIM
      XSUM=0.
      KK=KK&1
      IM=KK
      DO 80 L=1,NCON
      XSUM=VECT(IM)*B(L)&XSUM
   80 IM=IM&NOFOR
      BMAR(KK)=XSUM
   85 Y(K)=Y(K)-XSUM
      NF1=NOFOR&1
      BMAR(NF1)=NFIM&0.5
      AMAR(NF1)=NST&0.5
      NF2=NF1&1
      DO 88 MA=NF2,200
   88 AMAR(MA)=0.
C     THE TIDAL FOURIER COEFFICIENTS ARE NOW
C     CONTAINED IN TWO BLOCKS OF 200(MAXIMUM) NUMBERS
C     AMAR & BMAR FOR FUTURE USE IF DESIRED
C     THE LAST TWO NUMBERS CONTAIN THE EXTENT
C     OF THE ARRAY FOR FUTURE MANIPULATION
      WRITE(6,6300)
 6300 FORMAT(5X,'COSINE',6X,'SINE',4X,'AMPLITUDE',4X,'PHASE',3X,'CONSTI'
     1,'TUENT',3X,'NUMBER'/)
```

```
      DO 180 L=1,NCON
      HT=SQRT(A(L)*A(L)&B(L)*B(L))
      FN(L)=HT/FN(L)
      IFOUR= SPED(L)*FCT&0.5
C     CALCULATION OF PHASE ANGLE
      FI=0.
      IF(A(L)-0.)90,92,91
   90 FI=180.
   91 THETA=ATAN(B(L)/A(L))*FACT&FI
      IF(THETA.LT.0.)THETA=THETA&360.
      GO TO 179
   92 THETA = 90.
      IF(B(L).LT.0.)THETA=270.
  179 VZU(L) = REDUZ(VZU(L) & THETA)
  700 FORMAT(F11.7,10X,F9.3,4X,F6.2,32X,A8)
  180 WRITE(6,640)A(L),B(L),HT,THETA,CONS(L),IFOUR
  640 FORMAT(4(4X,F7.2),4X,A8,5X,I4)
      WRITE(6,6010)
 6010 FORMAT(//40X,'VALUES OF H & G'/)
      DUM = 0.
      WRITE(6,6002)(SPED(L),DUM,FN(L),VZU(L),CONS(L),L=1,NCON)
      IF(NPUN.NE.0)PUNCH 700,(SPED(L),FN(L),VZU(L),CONS(L),L=1,NCON)
      WRITE(6,655)
  655 FORMAT(//20X,'SPEED  NUMBERS  AND  RESIDUAL  FOURIER  ENERGY IN ',
     1'THE  TIDAL  BAND'//10X,'THE FIRST ROW OF EACH BLOCK CONTAINS THE'
     2,'SPEED NUMBER IN DEGREES AND THE SECOND THE RESIDUAL FOURIER',
     3' AMPLITUDES'/)
      VARIA=0.
      IFOUR=NST-1
      SPEDE=SPINC*IFOUR
      JK=0
      DO 190 K=NST,NFIM
      JK=JK&1
      SPED(JK)=SPEDE
      IQ(JK)=IFOUR
      RESEN=X(K)*X(K)&Y(K)*Y(K)
      SPED(JK& 10)=SQRT(RESEN)
      SPEDE=SPEDE + SPINC
      IFOUR=IFOUR&1
C   THE SECOND PART OF SPED NOW CONTAINS RESIDUAL FOURIER AMPLITUDE
      IF(JK.NE.10)GOTO 190
      WRITE(6,659)(IQ(J),J=1,10),(SPED(J),J=1,10),(SPED(J),J=11,20)
  659 FORMAT(/10(4X,I4,4X)/10(1X,F11.7)/10(1X,E11.5))
      JK=0
  190 VARIA=VARIA&RESEN
      JK1=JK&1
      JK10=JK&10
      DO 195 J=JK1,10
      IQ(J)=0.
  195 SPED(J)=0.
      WRITE(6,659)(IQ(J),J=1,10),(SPED(J),J=1,10),(SPED(J),J=11,JK10)
C     STATEMENT TO WRITE END OF BLOCK
      VARIA = SQRT(VARIA/NOFOR)
      WRITE(6,645)VARIA
      VV(KOUNT) = VARIA
  645 FORMAT('   RESIDUAL ENERGY IN TIDAL BAND =',E12.4)
      IF(NBLOC.NE.KOUNT)GOTO 2
      XSX = 0.
      DO 205 J = 1,ND21
  205 XSX =X(J)*X(J)&Y(J)*Y(J)&XSX
      XSX = SQRT(XSX/ND21)
```

```
      WRITE(6,6005)XSX,(VV(KL),KL=1,NBLOC)
 6005 FORMAT(//40X,'TOTAL NOISE LEVEL =',E12.4//20X,'NOISE LEVEL IN',
     1'INDIVIDUAL TIDAL BANDS'/5(10X,E12.4))
      IF(MTAPE.NE.0)WRITE(MTAPE)(X(L),Y(L),L=1,ND21)
      CALL EXIT
      END
      FUNCTION REDUZ(ARG)
      ANG=ARG
    1 IF(ANG.GE.0.)GOTO 2
      ANG=ANG&360.
      GOTO 1
    2 IF(ANG.LT.360.)GOTO 3
      ANG=ANG-360
      GOTO 2
    3 REDUZ=ANG
      RETURN
      END
```

APPENDIX IV

POWER SPECTRAL ANALYSIS

```
      COMPLEX C(4097),CZ
      DIMENSION Z(4097),PEPSI(30),Y(30),X(30),V(30)
      COMMON RX,RY
      EQUIVALENCE (CZ,RX),(C(1),Z(1))
      READ (5,501)N,NO,MDIV,L,KTAPE
  501 FORMAT(5I4)
C     N IS THE NO. OF VALUES IN THE ORIGINAL SERIES (N=2**M)
C    NO= THE NO. OF AUTO-SPECTRA TO BE FORMED
C    MDIV IS AN ARBITRARY DIVIDER TO INCREASE THE RESOLUTIN  FROM A
C     PREDETERMINED  MINUMUM, IF MDIV=0 THEN THE HALF WIDTH OF THE FILTER IS
C      SPECIFIED  BY L
C     KTAPE .NE.0 IS A PARAMETER THAT SPECIFIES AUTO-SPECTRA OUTPUT IS TO
C     BE ON MAGNETIC TAPE,OTHERWIS  PRINT-OUT AND PUNCH-OUT ARE EFFECTED
      NI=N/2&1
      IF(L.EQ.0)L=0.015278*N
      IF(MDIV.NE.0.)L=L/MDIV
      SCAL=2./(L&1)
      PEPSI(1)=SCAL*0.5
      XADD=0.
      REWIND 1
      IF(KTAPE.NE.0)REWIND2
      DO 10 IM=2,L
      XADD=XADD&2.
      U=XADD/L
      IF(U.NE.1.)GOTO9
      PEPSI(IM)=SCAL*0.5
      GOTO 10
    9 PEPSI(IM)=SIN(U*3.141593)*SCAL*0.31831/((-U*U&1.)*U)
   10 CONTINUE
      KOUNT=0
      WRITE(6,610)(PEPSI(I),I=1,L)
    5 KOUNT=KOUNT&1
  610 FORMAT(10X,10(1X,F10.7))
      WRITE(6,611)KOUNT
  611 FORMAT(///10X,'AUTO-SPECTRUM NUMBER  ',I4)
      READ(1) (C(I),I=1,N)
      VSUM=0.
C     INITIALISATION AND LOW FREQUENCY CORRECTION
      DO 12 J=1,L
      CZ=C(J)
      ZZ=RX*RX&RY*RY
      Y(J)=ZZ
      VSUM=PEPSI(J)*ZZ&VSUM
      X(J)=0.
   12 V(J)=VSUM
      X(1)=Y(1)
      M=0
      L1=L&1
      JSW=1
      DO 30 J=L1,NI
      CZ=C(J)
      ZZ=RX*RX&RY*RY
      M=M&1
      IF(M-L)13,13,14
   13 VAD=VSUM-V(M)
      GOTO 14
   30 CONTINUE
      ZZ=0.
      JSW=2
   31 M=M&1
      IF(M.GT.NI)GOTO 45
```

```
 14 VA=VAD
    XU=Y(2)
    DO 15 IM=1,L
    XY=X(IM)
    X(IM)=XU
    XU=XY
    VA=(Y(IM)&XY)*PEPSI(IM)&VA
 15 Y(IM)=Y(IM&1)
    Y(L)=ZZ
    Z(M)=VA
    GOTO(30,31),JSW
 45 IF(KTAPE.NE.0)GOTO 60
    WRITE(6,620)(Z(I),I,I=1,NI)
620 FORMAT(8(1X,E9.3,I4))
    JE=0
    KCARD=0
    DO 131 I=1,NI,8
    JB=JE&1
    JE=JE&8
    KCARD=KCARD&1
131 PUNCH 1800,(Z(I),I=JB,JE),KOUNT,KCARD
1800 FORMAT(8E9.3,1X,I2,1X,I4)
    GOTO 61
 60 WRITE(2)(Z(I),I=1,NI)
 61 IF(KOUNT.NE.NO)GOTO 5
    CALL EXIT
    END
```